TWO MORE PERFECTLY NORMAL NON-METRIZABLE MANIFOLDS

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Abstract. We show that there is a perfectly normal non-metrizable manifold if there is a Luzin subset of the real line, and that there is a countably compact perfectly normal non-metrizable manifold in any model of set-theory obtained by adding Cohen reals to a model of $ZFC + \diamondsuit$.

1. Introduction

An old question of R. Wilder [W] asks: Is every perfectly normal generalized manifold metrizable? M.E. Rudin and P.L. Zenor [RZ] showed that a negative answer is consistent with ZFC by constructing a counterexample from the continuum hypothesis CH. Another one constructed by Rudin under $\diamondsuit$, an axiom stronger than CH, is countably compact. (This one also appears in [RZ], but with a brief hand-waving argument; see [N_1], Example 3.14, for more details.) Rudin[R] later showed that on the other hand, under Martin's Axiom plus the negation of CH, every perfectly normal manifold is metrizable[1].

There has been continued interest in the theory of non-metrizable manifolds. See [N_1] and [N_2] for surveys of results up to about 1993. More recently, perfectly normal manifolds with interesting dimension-theoretic properties have been constructed by V.V. Fedorchuk and others; see, e.g., [F_1],[F_2], [F_3], and [FC].

But as yet, the only examples of perfectly normal non-metrizable manifolds appearing in the literature assume CH or something stronger. In this paper we construct, in two very different ways, perfectly normal manifolds that exist in models where CH fails. More precisely, we show:

(1) If there is a Luzin subset of the real line, then there is a perfectly normal hereditarily separable non-metrizable manifold.
(2) Consider the countably compact, perfectly normal, hereditarily separable manifold constructed as in [RZ] or [N_1] in a model of $ZFC + \diamondsuit$. If any number of Cohen reals are added to this model, then in the extension the "same" manifold has all of the afore-mentioned properties.

1991 *Mathematics Subject Classification.* 54E20, 54C35.

The research in this paper was completed in the Fall of 1989, while the authors were visiting the University of Wisconsin. This article was submitted to Topology and its Applications the following year, and was accepted modulo some revisions which the authors never completed. When the first author passed away in June, 2002, the second author, not wishing the mathematics herein to be lost, resubmitted the paper in 2003. The statement in the introduction that as yet all examples of perfectly normal non-metrizable manifolds in the literature use CH or something stronger is still true.

[1] Rudin's MA result does not completely settle the question about Wilder's *generalized* manifolds, which need not be locally metrizable. See [N_2] for a discussion of this.

Recall that L is a *Luzin set* in the real line $\mathbb{R}$ if L is an uncountable subset of $\mathbb{R}$ which has a countable intersection with every nowhere dense subset of $\mathbb{R}$, and that CH implies the existence of Luzin sets. Also recall that Luzin sets exist in various models of $\neg CH$ (e.g., add uncountably many Cohen reals to any model of ZFC. See [M] for more background information on Luzin sets.) Our construction for (1) uses ideas from the Rudin-Zenor CH construction along with some Luzin set combinatorics due to Todorcevic [T]. For (2), we use Borel codes locally to define what "same" means, and show that a key combinatorial property of Rudin's manifold is preserved by the extension.

2. Manifold from Luzin set

The purpose of this section is to prove:

Theorem 2.1. *If there is a Luzin subset of the real line, then there is a perfectly normal non-metrizable manifold.*

First, let us recall a consequence of the existence of a Luzin set regarding unbounded families in ω^ω. If $f, g \in \omega^\omega$, let $f <^* g$ mean that $f(n) < g(n)$ for sufficiently large $n \in \omega$. A family $\mathcal{F} \subset \omega^\omega$ is $<^*$-*unbounded* if there is no $g \in \omega^\omega$ with $f <^* g$ for all $f \in \mathcal{F}$. The least cardinal of a $<^*$-unbounded family 1n ω^ω is denoted by $\mathfrak{b}$.

If X is a Luzin set and D is a countable dense subset of X, it is immediate from the definition of a Luzin set that every open set containing D contains all but countably many points of X; i.e., X is concentrated about D. The existence of an uncountable set of reals concentrated about a countable set is equivalent to the statement "$\mathfrak{b} = \omega_1$". (See [vD], Theorem 10.2). So the existence of a Luzin set implies $\mathfrak{b} = \omega_1$.

Other related facts that will be used later are:

(1) There is a $<^*$-unbounded $<^*$-increasing family $\{f_\alpha : \alpha < \mathfrak{b}\}$ of increasing functions in ω^ω;
(2) If $\mathcal{F} \subset \omega^\omega$ is a $<^*$-unbounded family of increasing functions in ω^ω, then for every infinite subset A of ω, $\{f \restriction A : f \in \mathcal{F}\}$ is $<^*$-unbounded on A (i.e., for each $g \in \omega^A$, there is $f \in \mathcal{F}$ with $f(n) \geq g(n)$ for infinitely many $n \in A$).

See [vD] for these and many other facts about $\mathfrak{b}$.

Now let us describe the rough idea of the construction. As in [RZ], the underlying set for the manifold will be $X = B \cup \bigcup_{\alpha<\omega_1} I_\alpha$, where B is the open unit disk in $\mathbb{R}^2$ centered at the origin, and the I_α's are disjoint copies of $[0, 1)$. At stage $\alpha + 1$, $B \cup \bigcup_{\beta<\alpha} I_\beta$ is homeomorphic to B, and we define the topology on $B \cup \bigcup_{\beta\leq\alpha} I_\beta$ so that it remains homeomorphic to B. Also, the index α is associated with a point in ∂B – let us just say $\omega_1 \subset \partial B$ – and the topology on X is defined so that the map $f : X \to B \cup \omega_1 \subset \mathbb{R}^2$ defined by $f \restriction B = id_B$ and $f(x) = \alpha$ if $x \in I_\alpha$ is continuous. Perfect normality essentially comes from this f and the following property built into the construction:

($*$) If $H \subset X$ is closed, then for sufficiently large α, either $\alpha \notin \overline{f(H)}$ or $I_\alpha \subset H$.

In [RZ] this is done by indexing all countable subsets of X in type ω_1 as $\{A_\alpha\}_{\alpha<\omega_1}$, and making sure that at stage $\alpha + 1$, if α is a limit point of some $f(A_\beta)$, $\beta < \alpha$, then every point of I_α is a limit point of A_β . Of course, there are too many countable subsets of X to do it this way without CH. Instead we choose at stage α a certain sequence of compact subsets of $B \cup \bigcup_{\beta \leq \alpha} I_\beta$ such that the image under f of this sequence converges to $\alpha \in \partial B$, and make each point of I_α the limit of some subsequence of this sequence of compact sets. The difficulty is to make sure that this sequence is "generic" enough so that property $(*)$ will hold. The following lemma should illustrate some of what we mean.

Lemma 2.2. *Let ω_1 be Luzin set in ∂B. Then one can assign to each $\alpha \in \omega_1$ a sequence $\{K_{\alpha,n}\}_{n\in\omega}$ of disjoint compact subsets of B such that*

(i) *$K_{\alpha,n} \to \alpha$ (i.e., every Euclidean neighborhood of α contains all but finitely many $K_{\alpha,n}$'s);*

(ii) *whenever $Z \subset B$ and $\overline{Z} \cap \omega_1$ is uncountable, then for sufficiently large $\alpha \in \overline{Z}$, $\{n : K_{\alpha,n} \cap Z \neq \emptyset\}$ is infinite.*

Proof. Since $\overline{B} \setminus \{p\}$, where $p \in \partial B$, is homeomorphic to $[0, 1) \times (0, 1)$, we need only prove the lemma with the unit square $S = (0, 1) \times (0, 1)$ in place of B and ω_1 a Luzin subset of $\{0\} \times (0, 1)$. We may assume $\omega_1 \cap (\{0\} \times \mathbb{Q}) = \emptyset$.

For each n, let $V_n = \{1/2^{n+1}\} \times (0, 1)$. For each n and k such that $k/2^n$ is reduced to lowest terms, let $H(n, k)$ be the horizontal line segment with endpoints $(0, k/2^n)$ and $(1/2^n, k/2^n)$. This divides S into countably many rectangles.

Let $\{R_n\}_{n\in\omega}$ be the collection of these rectangles, where R_n includes the left side and the bottom (if the bottom is in S) of the rectangle. Then $\{R_n\}_{n\in\omega}$ is a partition of S into σ-compact sets. Write R_n as the increasing union of compact sets $\{K_{n,m}\}_{m\in\omega}$.

For each $\alpha \in \omega_1$, let $P(\alpha)$ be the union of all the rectangles meeting the horizontal line through α. Let $P(\alpha, n) = P(\alpha) \cap ((0, 1/2^n) \times (0, 1))$. Let $S(\alpha) = \{n : R_n \subset P(\alpha)\}$.

Now consider a $<^*$-unbounded $<^*$-increasing family $\{f_\alpha\}_{\alpha<\omega_1} \subset \omega^\omega$ of increasing functions. Let $\{K_{n,f_\alpha(n)}\}_{n\in S(\alpha)}$ be the sequence of compact sets assigned to α. We claim that this works.

Suppose $Z \subset S$ and $\overline{Z} \cap \omega_1$ is uncountable. Define $g : \omega \to \omega$ so that if $Z \cap R_n \neq \emptyset$, then $K_{n,g(n)} \cap Z \neq \emptyset$.

Fact 1. Let $T(\alpha) = \{n \in S(\alpha) : R_n \cap Z \neq \emptyset\}$. Then for sufficiently large $\alpha \in \overline{Z}$, $T(\alpha)$ is infinite.

Proof of Fact 1. If not, then for uncountably many $\alpha \in \overline{Z}$, there is n_α such that $P(\alpha, n_\alpha) \cap Z = \emptyset$. For some m, $n_\alpha = m$ for uncountably many α. These α's are dense in some open interval I in $\{0\} \times (0, 1)$. But the union of $P(\alpha, m)$'s for α in a dense subset of I contains a Euclidean neighborhood of each point in $I \subset S$, which must therefore meet Z. This proves Fact 1.

Now if $T(\alpha)$ is infinite, there is $\beta(\alpha)$ such that $f_{\beta(\alpha)}(n) \geq g(n)$ for infinitely many $n \in T(\alpha)$. Let E be a countable dense subset of $\{\alpha : T(\alpha) \text{ is infinite}\}$. Choose $\gamma > \sup\{\beta(\alpha) : \alpha \in E\}$.

Fact 2. For sufficiently large $\delta \in \overline{Z}$, $f_\gamma(n) \geq g(n)$ for infinitely many $n \in T(\delta)$.

Proof of Fact 2. If not, then for uncountably many $\delta \in \overline{Z}$, there is n_δ such that if $R_n \subset P(\delta, n_\delta)$ and $R_n \cap Z \neq \emptyset$, then $f_\gamma(n) < g(n)$. As in the proof of Fact 1, there is an integer m and an interval $I \subset \{0\} \times (0, 1)$ such that the δ's with $n_\delta = m$ are dense in I. This implies that if $R_n \subset (0, 1/2^m) \times I$ and $R_n \cap Z \neq \emptyset$, then $f_\gamma(n) < g(n)$. Now choose $\alpha \in E \cap I$. Then $f_{\beta(\alpha)}(n) \geq g(n)$ for infinitely many $n \in T(\alpha)$; since $\beta(\alpha) < \gamma$, the same is true for f_γ. It follows that $f_\gamma(n) \geq g(n)$ for some $n \in T(\alpha)$ with $R_n \subset (0, 1/2^m) \times I$, a contradiction. So Fact 2 holds.

Now Lemma 2.2 follows from Fact 2, because if $f_\gamma(n) \geq g(n)$ for infinitely many $n \in T(\delta)$, then $f_\delta(n) \geq g(n)$ for infinitely many $n \in T(\delta)$ whenever $\delta \geq \gamma$. $\square$

Lemma 2.3. *Let* $\{I_\alpha\}_{\alpha<\omega_l}$ *be a collection of disjoint copies of* $[0,1)$ *, and let* ω_1 *be a Luzin set. For* $Z \subset \bigcup_{\alpha<\omega_1} I_\alpha$*, let* $f(Z) = \{\alpha : Z \cap I_\alpha \neq \emptyset\}$*, and for* $K \subset [0,1)$*, let* K_α *denote its copy in* I_α*. Then there are* $\{H^k(\alpha) : \alpha < \omega_1, k < \omega\}$ *and* $f^k_\alpha : H^k(a) \to \omega$ *such that*

(i) $H^k(\alpha) \subset \alpha$*;*
(ii) $\bigcup_{k<\omega} H^k(\alpha)$ *is either finite or a sequence converging to* α *(in the Luzin set topology on* ω_1*);*
(iii) $H^k(\alpha) \cap H^l(\alpha) = \emptyset$ *for* $k \neq l$*;*
(iv) *whenever* $Z \subset \bigcup_{\alpha<\omega_1} I_\alpha$ *and* $f(Z)$ *has uncountable closure in* ω_1*, then for sufficiently large* $\alpha \in \overline{f(Z)}$*, for all* $k < \omega$*,* $Z \cap (\bigcup_{\beta \in H^k(\alpha)} [0, 1 - 1/2^{f^k_\alpha(\beta)}]_\beta)$ *is infinite, where for* $A \subset [0,1)$*,* A_β *denotes its copy in* I_β*.*

Proof. Since a Luzin subset of $\mathbb{R}$ is zero-dimensional, it is homeomorphic to a Luzin subset of the irrationals ω^ω; so we may assume $\omega_1 \subset \omega^\omega$ is Luzin in ω^ω.

The following construction of the $H^k(\alpha)$'s, through the proof of Fact 2 below, is a modification of a construction of Todorcevic (see pg. 52 of [T]).

For $\alpha < \omega_1$, let $e_\alpha : \alpha \to \omega$ be one-to-one such that if $\beta \geq \alpha$, then $e_\alpha =^* e_\beta \restriction \alpha$ (i.e., $e_a(\gamma) \neq e_\beta(\gamma)$ for at most finitely many $\gamma < \alpha$. See, e.g., the construction of an Aronszajn tree in [K].) Let $\Delta(\alpha, \beta) = \min\{n : \alpha(n) \neq \beta(n)\}$. (Recall $\omega_1 \subset \omega^\omega$.) Let $\{A_k\}_{k<\omega}$ be a partition of ω into infinite sets. Now define

$$H^k(\alpha) = \{\beta < \alpha : e_\alpha(\beta) < \Delta(\beta, \alpha) \text{ and } \Delta(\beta, \alpha) \in A_k\}.$$

Clearly (i) and (iii) are satisfied, and (ii) follows from the fact that e_α is one-to-one. We need to prove (iv).

Fact 1. Suppose $F \subset \omega_1$ is uncountable and $D \subset F$ is countable and dense. Then there exists $d \in D$ such that for each $k < \omega$, there exists $\alpha_k \in F \setminus D$ with $d \in H^k(\alpha_k)$.

Proof of Fact 1. Pick $\alpha < \omega_1$ such that $D \subset \alpha$. Since $\{e_\gamma \restriction D : \gamma \in F \setminus \alpha\}$ is countable, there exists $e : D \to \omega$ and uncountable $F' \subset F \setminus \alpha$ with $e_\gamma \restriction D = e$ for each $\gamma \in F'$. Let $\sigma \in \omega^\omega$ be such that F' is dense in $[\sigma] = \{x \in \omega^\omega : \sigma \subset x\}$, and choose $d \in [\sigma] \cap D$.

Let $k \in \omega$, and choose $n \in A_k$, $n > e(d) + |\sigma|$. Choose $\alpha_k \in F' \cap [d \restriction n^\frown \langle d(n) + 1\rangle]$. Then $e_{\alpha_k}(d) = e(d) < n = \Delta(d, \alpha_k) \in A_k$, so $d \in H^k(\alpha_k)$.

Fact 2. If $D \subset \omega_1$ has uncountable closure, then for sufficiently large $\alpha \in \overline{D}$, $H^k(\alpha) \cap D$ is infinite for each $k < \omega$.

Proof of Fact 2. Suppose not. Choose $\delta_\alpha \in \overline{D} \setminus (\alpha \cup D)$ and $k(\alpha) < \omega$ such that $H^{k(\alpha)}(\delta_\alpha) \cap D$ is finite. There is an uncountable $A \subset \omega_1$, $k < \omega$, and finite $E \subset D$ such that $k(\alpha) = k$ and $H^{k(\alpha)}(\delta_\alpha) \cap D = E$ for all $\alpha \in A$. Let $F = (D \setminus E) \cup \{\delta_\alpha : \alpha \in A\}$. Then $D \setminus E$ is dense in F, and for each $x \in F \setminus (D \setminus E)$, we have $H^k(x) \cap (D \setminus E) = \emptyset$. This contradicts Fact 1.

Now we define the f^k_α's. Let $\{a_\alpha\}_{\alpha<\omega_1}$ be an unbounded $<^*$-increasing family of increasing functions in ω^ω . For $\beta \in H^k(\alpha)$, let $f^k_\alpha(\beta) = a_\alpha(e_\alpha(\beta))$. It remains to prove that condition (iv) holds with these definitions of $H^k(\alpha)$ and f^k_α.

Let $Z \subset \bigcup_{\alpha<\omega_1} I_\alpha$ be such that $D = f(Z)$ has uncountable closure in ω_1. We may assume Z is countable, so let $D \subset \gamma$. Define $g : D \to \omega$ so that $Z \cap [0, 1 - 1/2^{g(d)}]_d \neq \emptyset$ for each $d \in D$.

Fact 3. If $D = \{d \in D : a_\alpha(e_\alpha(d)) \geq g(d)\}$, then for sufficiently large α, $\overline{D_\alpha} \supset \overline{D} \setminus D$.

Proof of Fact 3. Since $e_\alpha \restriction D =^* e_\gamma \restriction D$ for $\alpha > \gamma$, and the a_α's are $<^*$-increasing, it follows that $D_\beta \subset^* D_\alpha$ for $\beta < \alpha$ (i.e., $D_\beta \setminus D_\alpha$ is finite).

Let $x \in \overline{D} \setminus D$. There is a sequence $S_x \subset D$ with $S_x \to x$. So there is α_x such that $a_{\alpha_x}(e_\gamma(d)) \geq g(d)$ infinitely often on S_x. Then $a_{\alpha_x}(e_{\alpha_x}(d)) \geq g(d)$ infinitely often on S_x, so $x \in \overline{D_{\alpha_x}}$. Choose α such that $\{x : \beta_x < \alpha\}$ is dense in $\overline{D} \setminus D$. Since $D_\beta \subset^* D_\alpha$ for $\beta < a$, it follows that $\overline{D_\alpha} \supset \overline{D} \setminus D$.

Now by Facts 2 and 3, and $D_\beta \subset^* D_\alpha$ for $\beta < \alpha$, for sufficiently large $\alpha \in \overline{D}$, $H^k(\alpha) \cap D_\alpha$ is infinite for all $k < \omega$. If $d \in H^k(\alpha) \cap D_\alpha$, then $f^k_\alpha(d) = a_\alpha(e_\alpha(d)) \geq g(d)$, so $[0, 1 - 1/2^{f^k_\alpha(d)}]_d \supset [0, 1 - 1/2^{g(d)}]_d$. Condition (iv) follows immediately. □

The next lemma is similar to Lemma 1 of [RZ].

Lemma 2.4. *Let $(U_n)_{n<\omega}$ be a decreasing nested sequence of connected open subsets of B such that $B \cap \bigcap_{n\in\omega} \overline{U_n} = \emptyset$. Let $K_n \subset U_n \setminus \overline{U}_{n+1}$ compact, and let $\{N_r\}_{r\in Q\cap[0,1)}$ be a partition of ω into infinite sets.*

Then there is a homeomorphism $g : B \to B \setminus ([0, 1) \times \{0\}$ such that

(a) *$[0, 1) \times \{0\} \cup g(U_n)$ is open in B;*
(b) *for each $r \in Q \cap [0, 1)$, every neighborhood of $(r, 0)$ contains all but finitely many members of $\{g(K_n) : n \in N_r\}$.*

Proof. As in Step 1 of the proof of Lemma 1 of [RZ], there is a homeomorphism $h : B \to (-1, 1) \times (0, 1) = D$ satisfying:

(i) $h(U_n) \supset \{(x, y) \in D : y < \frac{1}{2^{n+1}}\} = D_n$;
(ii) $h(K_n) \subset D_n \setminus \overline{D}_{n+1}$ for each $n < \omega$.

(Connectedness of the U_n's is used here; it is needed to get the existence of the such an h.)

Now let $q_n : \overline{D}_n \setminus D_{n+1} \to \overline{D}_n \setminus D_{n+1}$ be a homeomorphism which leaves the boundary fixed and moves $h(K_n)$ so that if $n \in N_r$, then $(r, y) \in h(K_n)$ for some y, and $\mathrm{diam}(h(K_n)) < 1/2^n$. Let $f : D \to D$ be the result of pasting these homeomorphisms together.

Let $D' = D \cup \{(x, 0) : -1 < x < 1\}$, and define $k : D' \to D'$ by $k \restriction D = id_{D'}$, and $k(x, 0) = (|x|, 0)$. Then $k(D')$ with quotient topology is homeomorphic to B by a homeomorphism $j : k(D') \to B$ such that $j \restriction [0, 1) \times \{0\}$ is the identity. Finally, let $g = j \circ k \circ f \circ h$. Clearly g satisfies the desired conditions. □

Proof of Theorem 2.1. Let B be the open unit ball in $\mathbb{R}^2$, and let $\{I_\alpha\}_{\alpha<\omega_1}$ be disjoint copies of $[0, 1)$ unrelated to B. Let $\omega_1 \subset \partial B$ be Luzin in ∂B. Let $H^k(\alpha)$ and $f^k_\alpha : H^k(\alpha) \to \omega$ be as in Lemma 2.3, and let $\{K_{\alpha,n} : a < \omega_1, n < \omega\}$ be as in Lemma 2.2.

Let $X_\alpha = B \cup \bigcup_{\beta<\alpha} I_\beta$ and $X = \bigcup_{\alpha<\omega_1} X_\alpha$. Define $f : X \to B \cup \omega_1$ by $f \restriction B = id_B$ and $f(I_\alpha) = \alpha$. Let $\{r_k\}_{k<\omega}$ index $Q \cap [0, 1)$ with $r_0 = 0$.

Inductively construct a topology τ_α on X_α such that

(i) (X_α, τ_α) is homeomorphic to B and $f \restriction X_\alpha$ is continuous;

further, for $\beta < \alpha$,

(ii) (X_β, τ_β) is dense open in (X_α, τ_α);
(iii) $K_{\beta,n} \to 0$ in I_β; and
(iv) $\{[0, 1 - 1/2^{f^k_\beta(\gamma)}]_\gamma\}_{\gamma\in H^k(\beta)} \to r_k \in I_\beta$ if $H^k(\beta)$ is infinite.

Let τ_0 be the Euclidean topology on $X_0 = B$, and for limit α, let $\tau_\alpha = \bigcup_{\beta<\alpha} \tau_\beta$. At stage $\alpha + 1$, we need to say how neighborhoods of $x \in I_\alpha$ reach into X_α. Since $\bigcup_{k<\omega} H^k(\alpha)$ is a sequence converging to α in ∂B, clearly there is a sequence $B_n(\alpha)$ of Euclidean open sets containing α such that

(a) $B_n(\alpha) \cap B$ is connected;
(b) $\partial B_n(\alpha) \cap \omega_1 = \emptyset$;
(c) $\overline{B_{n+1}(\alpha)} \subset B_n(\alpha)$ and $\operatorname{diam}(B_n(\alpha)) \to 0$ as $n \to \infty$;
(d) Each member of $\{K_{\alpha,n}\}_{n\in\omega} \cup \bigcup_{k\in\omega} H^k(\alpha)$ is contained in some $B_n(\alpha) \setminus \overline{B_{n+1}(\alpha)}$;
(e) $B_n(\alpha) \setminus \overline{B_{n+1}(\alpha)}$ contains at most one member of $\{K_{\alpha,n}\}_{n\in\omega} \cup \bigcup_{k\in\omega} H^k(\alpha)$.

Let $U_n = f^{-1}(B_n(\alpha) \cap (B \cup \alpha))$. Then $\{U_n\}_{n\in\omega}$ is a nested family of connected open subsets of X_α with $X_\alpha \cap \bigcap_{n\in\omega} \overline{U}_n = \emptyset$.

Let $N_0 = \{n \in \omega : U_n \setminus \overline{U_{n+1}}$ contains some $K_{\alpha,m}\}$, and for $k \geq 1$, let $N_{r_k} = (n \in \omega : (B_n(\alpha) \setminus \overline{B_{n+1}(\alpha)}) \cap H^k(\alpha) \neq \emptyset\}$. Note that the N_{r_k}'s are disjoint. If $n \in N_0$, let K_n be that $K_{\alpha,m}$ which is contained in $U_n \setminus \overline{U_{n+1}}$. If $n \in N_{r_k}$, $k \geq 1$, let $K_n = [0, 1 - 1/2^{f^k_\alpha(\beta)}]_\beta$, where $\beta \in (B_n(\alpha) \setminus \overline{B_{n+1}(\alpha)}) \cap H^k(\alpha)$. Note that $K_n \subset U_n \setminus \overline{U_{n+1}}$.

Let $g_\alpha : X_\alpha \to B \setminus ([0, 1) \times \{0\})$ be a homeomorphism satisfying the conditions of Lemma 2.4. (Since X_α is homeomorphic to B, such g_α exists.) Define $g^*_\alpha : X_{\alpha+1} \to B$ by $g^*_\alpha \restriction X_\alpha = g_\alpha$ and $g^*_\alpha(x) = (x, 0)$ for $x \in I_\alpha$. Let τ_α be the topology on $X_{\alpha+1}$ which makes g^* a homeomorphism.

Note that if $V_n = U_n \cup I_\alpha$, then the V_n's are a decreasing sequence of open (in $X_{\alpha+1}$) supersets of I_α closing down on I_α. This follows from by Lemma 2.4(a), which implies that the images of the V_n's in B under g_α close down on $[0, 1) \times \{0\}$. With this in mind, it is easily seen that (i)-(iv) above are satisfied.

Let $\tau = \bigcup_{\alpha<\omega_1} \tau_\alpha$. Then (X, τ) is a non-metrizable manifold, being separable but not Lindelöf.

It remains to prove (X, τ) is perfectly normal, or equivalently, every closed set is a regular G_δ-set (see, e.g., [E], Exercise 1.5.K). So let H be closed in X.

Fact 1. For sufficiently large $\alpha \in \omega_1$, either $\alpha \notin \overline{f(H)}$ or $I_\alpha \subset H$.

Proof of Fact 1. Let $Z_0 = H \cap B$ and $Z_1 = H \cap (\bigcup_{\alpha<\omega_1} I_\alpha)$. By Lemma 2.2 and the construction, there exists $a_0 < \omega_1$ such that $\alpha > \alpha_0$ implies either $\alpha \notin \mathrm{c1}_{\mathbb{R}^2}(Z_0)$ or the point 0 in I_α is a limit point of Z_0, and in this latter case $I_\alpha \cap Z_1 \neq \emptyset$. And by Lemma 2.3 and the construction, there is $\alpha_1 < \omega_1$ such that $\alpha > \alpha_1$ implies either $\alpha \notin \overline{f(Z_1)}$ or $\mathrm{cl}_\tau(Z_1) \supset I_\alpha$.

Suppose $\alpha > \alpha_0 + \alpha_1$ and $\alpha \in \overline{f(H)}$; we need to show $I_\alpha \subset H$. If $\alpha \in \mathrm{c1}_{\mathbb{R}^2}(Z_0)$, then since $\alpha > \alpha_0$, we have $Z_1 \cap I_\alpha \neq \emptyset$, so $\alpha \in f(Z_1)$ and hence, by $\alpha > \alpha_1$, $I_\alpha \subset \mathrm{cl}_\tau(Z_1) \subset H$. If on the other hand $\alpha \notin \mathrm{c1}_{\mathbb{R}^2}(Z_0)$, then $\alpha \in \overline{f(Z_1)}$ and again $I_\alpha \subset \mathrm{cl}_\tau(Z_1) \subset H$.

Fact 2. $f^{-1}(\overline{f(H)}) \setminus H \subset X_\alpha$ for some α.

Proof of Fact 2. Let α be such that $\beta \geq \alpha$ implies either $\beta \notin \overline{f(H)}$ or $I_\beta \subset H$. Suppose $p \in f^{-1}(\overline{f(H)}) \setminus H$. Then $f(p) \in \overline{f(H)} \cap \omega_1$. Since $p \notin H$, $I_{f(p)} \not\subset H$. Thus by Fact 1, $f(p) < \alpha$, and so $f^{-1}(\overline{f(H)}) \setminus H \subset X_\alpha$.

Now we can complete the proof that H is regular G_δ in X. Since $\overline{f(H)}$ is regular G_δ in $B \cup \omega_1$, $f^{-1}(\overline{f(H)})$ is regular G_δ in X. By Fact 2, $f^{-1}(\overline{f(H)}) \setminus H$ can be covered by countably many open sets whose closures miss H. It follows that H is a regular G_δ-set. □

3. Adding Cohen reals

The purpose of this section is to show that the existence of a countably compact, perfectly normal, hereditarily separable non-metrizab1e manifold is consistent with the negation of the continuum hypothesis. We do this by considering the manifold with these properties constructed by Rudin under $\diamondsuit$, and we show that if one adds any number of Cohen reals to this model of $\diamondsuit$, then in the extension the "same" manifold retains the relevant properties. In order to work out this idea, we must find a way of describing what "same" means. One can use Borel codes to define what "same" means with respect to Borel sets in $\mathbb{R}^n$ and homeomorphisms from $\mathbb{R}^n$ into $\mathbb{R}^n$ (more details later). So we start with a way to describe 2-manifolds of weight ω_1 via homeomorphisms of $\mathbb{R}^2$ into $\mathbb{R}^2$. We are thinking of our manifold M as an increasing union of open submanifo1ds M_α, $\alpha < \omega_1$, with each M_α homeomorphic to $\mathbb{R}^2$. Then the identity map of M_β into M_α can be coded by a homeomorphism of $\mathbb{R}^2$ into $\mathbb{R}^2$. With this in mind, we call a collection $\{\phi_{\beta\alpha} : \beta \leq \alpha < \omega_1\}$ of homeomorphisms of $\mathbb{R}^2$ into $\mathbb{R}^2$ an ω_1-*system* if

(i) $\phi_{\beta\alpha} \circ \phi_{\gamma\beta} = \phi_{\gamma\alpha}$ whenever $\gamma < \alpha < \omega_1$;
(ii) $\phi_{\beta\beta} = id_{\mathbb{R}^2}$ for all $\beta < \omega_1$.

Now define M to be the collection of all sequences $\vec{x} = \langle x_\delta \rangle_{\delta \geq \alpha}$, $\alpha < \omega_1$, such that

(a) $\phi_{\alpha\delta}(x_\alpha) = x_\delta$ for all $\alpha \leq \delta < \omega_1$;
(b) there is no $\beta < \alpha$ and $x \in \mathbb{R}^2$ with $\phi_{\beta\alpha}(x) = x_\alpha$.

Note that the coherence property (i) of the ω_1-system implies that for each $x \in \mathbb{R}^2$ and $\alpha < \omega_1$, there is a unique $\vec{x} \in M$ with $x_\alpha = x$.

For each $\vec{x} =< x_\delta >_{\delta \geq \alpha} \in M$, call α the *rank* of $\vec{x}$ and denote it by $\mathrm{rk}(\vec{x})$. Let $M_\alpha = \{\vec{x} \in M : \mathrm{rk}(\vec{x}) \leq \alpha\}$, and let $\phi_\alpha : M_\alpha \to \mathbb{R}^2$ be defined by $\phi_\alpha(\vec{x}) = x_\alpha$. Note that each ϕ_α is a bijection, and $\phi_{\beta\alpha} = \phi_\alpha \circ \phi_\beta^{-1}$ whenever $\beta \leq \alpha < \omega_1$. Let τ_α be the topology on M_α making ϕ_α a homeomorphism. Note that if $\beta < \alpha < \omega_1$, then (M_β, τ_β) is an open subspace of (M_α, τ_α) since $\phi_\alpha(M_\beta) = \phi_\alpha \circ \phi_\beta^{-1} \circ \phi_\beta(M_\beta) = \phi_{\beta\alpha}(\mathbb{R}^2)$ is open in $\mathbb{R}^2$. Equip M with the topology $\tau = \bigcup_{\alpha<\omega_1} \tau_\alpha$. We call $M = M(\{\phi_{\beta\alpha}\})$ with this topology the *inverse semi-limit* of $\{\phi_{\beta\alpha} : \beta \leq \alpha < \omega_1\}$.

Let $\mathcal{E}$ be all open balls in $\mathbb{R}^2$ with rational centers and radii, and let

$$\mathcal{B}_\alpha = \{\phi_\alpha^{-1}(E) : E \in \mathcal{E} \text{ and } \forall \beta < a(\phi_\alpha^{-1}(E) \not\subset M_\beta)\}.$$

Then $\mathcal{B}_\alpha$ is a basis for all points of $M_\alpha \setminus \bigcup_{\beta<\alpha} M_\beta$. Let $\mathcal{B} = \bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha$. We call $\mathcal{B}$ the *standard basis* for M. Note that M is non-metrizable if M is separable and $\mathcal{B}$ is uncountable (for this means $M_\alpha \setminus \bigcup_{\beta<\alpha} M_\beta \neq \emptyset$ for uncountably many $\alpha < \omega_1$, whence M is not Lindelöf).

Now, given an inverse semi-limit $M = M(\{\phi_{\beta\alpha}\})$ in a model V, we can use Borel codes for the $\phi_{\beta\alpha}$'s to say what is meant by the "same" manifold in an extension $V[G]$ of the universe V. For the benefit of the reader unfamiliar with Borel codes and absoluteness of Π^1_1-relations, we start with a brief intuitive description of the ideas.

In V, let B be an open ball in $\mathbb{R}^n$ with rational center and radius. Let B^* denote the same ball in $V[G]$, i.e., the ball in $(\mathbb{R}^n)^{V[G]} = (\mathbb{R}^n)^*$ with the same center and radius. Now any open set U in $\mathbb{R}^n$ (in V) is a countable union $\bigcup_{n\in\omega} B_n$ of balls with rational center and radii, so we can consider the set U^* in $V[G]$ built in the same way as U, i.e., $U^* = \bigcup_{n\in\omega} B_n^*$. Similarly, if F is the complement of U, let $F^* = (\mathbb{R}^n)^* \setminus \bigcup_{n\in\omega} B_n^*$. One can go on to associate to any Borel set A in $(\mathbb{R}^n)^V$ a set A^* in $(\mathbb{R}^n)^{V[G]}$ built from balls with rational centers and radii in the same way that A is. Using a definable enumeration of the balls with rational centers and radii, the way in which A is built can be coded by a function $c : \omega \to \omega$ in V; c is called a *Borel code* for A. Then A^* is the set in $(\mathbb{R}^n)^{V[G]}$ built from the B^*'s using the same code c. Here's the basic absoluteness result that we need. (See pp. 537-540 of [J] for a proof of the result for $\mathbb{R}$; the proof for $\mathbb{R}^n$ is entirely analogous.)

Lemma 3.1.

(a) *For any Borel set A in V, A^* does not depend on the choice of Borel code for A;*

(b) *Let A, B, and $\{A_n\}_{n<\omega}$ be Borel sets in V, where the enumeration of the A_n's is also in V. Then $(A\cap B)^* = A^*\cap B^*$, $(\bigcup_{n\in\omega} A_n)^* = \bigcup_{n\in\omega} A_n^*$, $(A\setminus B)^* = A^*\setminus B^*$, and $A\subset B \iff A^*\subset B^*$.*

We will prove the rest of what we need about the * operation from Lemma 3.1.

Lemma 3.2. *In V, let $F\subset \mathbb{R}^n$ be closed. Then $F^* = \overline{F}$, where the closure is taken in $(\mathbb{R}^n)^{V[G]}$. Further, if F is compact, so is F^*.*

Proof. In V, let $\mathbb{R}^n\setminus F = \bigcup_{m\in\omega} B_m$, where the B_m's are balls with rational centers and radii. Then $(\mathbb{R}^n\setminus F)^* = (\mathbb{R}^n)^*\setminus F^* = \bigcup_{n\in\omega} B_m^*$, whence F^* is closed. If $F^*\setminus\overline{F}\neq\emptyset$, then some B^* meets F^* but not F. Hence $\emptyset\neq B^*\cap F^* = (B\cap F)^* = \emptyset^* = \emptyset$, a contradiction.

It remains to prove that F^* is compact if F is. Note that in this case the set F in $V[G]$ is bounded, and hence its closure is too. So F^* is closed and bounded, hence compact. $\square$

Any continuous $\phi : \mathbb{R}^2\to\mathbb{R}^2$ is a closed subset of $\mathbb{R}^2\times\mathbb{R}^2 = R^4$, so ϕ^* is defined.

Lemma 3.3. *In V, if $\phi : \mathbb{R}^2\to\mathbb{R}^2$ is a homeomorphism of $\mathbb{R}^2$ into $\mathbb{R}^2$, then so is ϕ^*. Also, $\phi^*(U^*) = (\phi(U))^*$ for any open set U in $\mathbb{R}^n$.*

Proof. In V, let K be any compact subset of $\mathbb{R}^2$. Then $\phi\restriction K$ is uniformly continuous, and $(\phi\restriction K)^* = \overline{(\phi\restriction K)}$. It is an elementary excercise to show that if ϕ is a uniformly continuous map from a subset $E\subset R^2$ into R^2, then $\overline{\phi}$ is a function with domain $\overline{E}$, and $\overline{\phi}$ is continuous on $\overline{E}$. So $\phi^* = \overline{\phi} = \bigcup_{n<\omega}\overline{\phi\restriction[-n,n]^2}$ is a continuous function from $(\mathbb{R}^2)^{V[G]}$ into $(\mathbb{R}^2)^{V[G]}$. Since $(\phi^*)^{-1} = (\phi^{-1})^*$ is by the same argument a continuous function, ϕ^* is a homeomorphism.

To see the last statement consider again a compact K in $\mathbb{R}^n$. Then $(\phi(K))^* = \overline{\phi(K)} = \overline{\phi}(\overline{K}) = \phi^*(K^*)$. If $U\subset\mathbb{R}^n$ is open, let $U = \bigcup_{n\in\omega} K_n$, where K_n is compact. Then $(\phi(U))^* = (\phi(\bigcup_{n\in\omega} K_n))^* = (\bigcup_{n\in\omega}\phi(K_n))^* = \bigcup_{n\in\omega}(\phi(K_n))^* = \bigcup_{n\in\omega}\phi^*(K_n^*) = \phi^*(\bigcup_{n\in\omega} K_n^*) = \phi^*((\bigcup_{n\in\omega} K_n)^*) = \phi^*(U^*)$. $\square$

Now, given an inverse semi-limit $M = M(\{\phi_{\beta\alpha}\})$ in V with standard base $\mathcal{B}$, we can consider the manifold $M(\{\phi_{\beta\alpha}^*\})$, which we denote by M^*, in a generic extension $V[G]$. Let ϕ_α^* be defined from the $\phi_{\beta\alpha}^*$'s in $V[G]$ in the same way that ϕ_α was defined in V. If $B = \phi^{-1}(E)\in\mathcal{B}$, let $B^* = (\phi_\alpha^*)^{-1}(E^*)$. Then $\{B^* : B\in\mathcal{B}\}$ is the standard base for M^*.

Lemma 3.4. *Results of countable Boolean operations in V on elements of $\mathcal{B}$ are in a given containment relationship if and only if the results of the same Boolean operations on the corresponding elements of $\mathcal{B}^*$ are in the same containment relationship.*

Proof. Let $\{B_n\}_{n\in\omega}$ be an enumeration (in V) of all members of appearing in the Boolean operations. Let $B_n = \phi_{\beta_n}^{-1}(E_n)$, where $E_n \in \mathcal{E}$. Let $\alpha > \sup_{n\in\omega}\beta_n$. Now it is enough to show that the results of the afore-mentioned Boolean operations on the sets $\phi_\alpha(B_n) = \phi_{\beta_n\alpha}(E_n)$ are in a given containment relationship if and only if it is true for $\phi^*_\alpha(B^*_n) = \phi^*_{\beta_n\alpha}(E^*_n) = (\phi_\alpha(E_n))^*$, $n < \omega$. But this is true by Lemma 3.1. □

Lemma 3.4 implies the transfer of some statements about uncountable subfamilies of $\mathcal{B}$ to the corresponding subfamilies of $\mathcal{B}^*$. For example, if $\mathcal{A} \subset \mathcal{B}$ is a cover of M, then $\mathcal{A}^* = \{A^* : A \in \mathcal{A}\}$ is a cover of M^* . Indeed, it is enough to show that each M^*_α gets covered by $\mathcal{A}^*$. To see this, let $\mathcal{A}_0 \subset \mathcal{A}$ be a countable cover of M_α in V. By Lemma 3.4, $\{A^* : A \in \mathcal{A}_0\}$ covers M^*_α.

Call a collection $\mathcal{U}$ of sets *point-uncountable* if every $x \in \cup\mathcal{U}$ is a member of uncountably many members of $\mathcal{U}$. Recall that Rudin's manifold, indeed any separable perfectly normal manifold, is hereditarily separable [RZ]. We need the following combinatorial property of hereditarily separable spaces.

Lemma 3.5. *Let X be hereditarily separable. Then every uncountable collection $\mathcal{U}$ of open subsets of X contains a point-uncountable subcollection.*

Proof. Assume not. Then we can inductively define $p_\alpha \in X$ and $U_\alpha \in \mathcal{U}$ with $p_\alpha \in U_\alpha$ such that $U_\alpha \cap \{p_\beta : \beta < \alpha\} = \emptyset$. (Choose a point p_α which is in at least one but only countably many members of $\mathcal{U} \setminus \{U \in \mathcal{U} : p_\beta \in U$ for some $\beta < \alpha\}$.) Choose $\alpha < \omega_1$ such that $\overline{\{p_\beta\}}_{\beta<\alpha} \supset \{p_\beta : \beta < \omega_1\}$. Then $p_\beta \in U_\alpha$ for some $\beta < \alpha$, a contradiction. □

From now on, let P be the poset for adding κ-many Cohen reals to V, i.e., P is the set of all finite functions from κ to 2 ordered by extension. Let G be a P-generic filter over V.

Lemma 3.6. *In V, let $\mathcal{B} = \{B_\alpha\}_{\alpha<\omega_1}$ be the standard basis for the inverse semi-limit $M(\{\phi_{\beta\alpha}\})$, and suppose that every uncountable collection of members of $\mathcal{B}$ contains a point-uncountable subcollection. Then in $V[G]$, for every uncountable $A \subset \omega_1$, there is an uncountable $H \subset \omega_1$ in V such that $\bigcup_{\alpha\in H} B^*_\alpha \subset \bigcup_{\alpha\in A} B^*_\alpha$.*

Proof. Let $\dot{A}$ be a P-name such that $\dot{A}_G = A$ and $1 \Vdash \dot{A} \in [\omega_1]^{\omega_1}$. Let $\mathcal{A}$ be all subsets of ω_1 in V, and suppose indirectly that there is no H as stated in Lemma 3.6. Then there is a $p \in G$ such that

$$p \Vdash \forall H \in \check{\mathcal{A}}(\bigcup_{\alpha\in H} \dot{B}^*_\alpha \not\subset \bigcup_{\alpha\in\dot{A}} \dot{B}^*_\alpha).$$

Since $p \Vdash$ "$\dot{A}$ is uncountable", there is in V an $H \in [\omega_1]^{\omega_1}$ and a system $\langle p_\alpha\rangle_{a\in H}$ of elements of P extending p such that

(a) $\{\mathrm{dom}\ p_\alpha\}_{\alpha\in H}$ forms a Δ-system with root Δ;
(b) There is $r \in P$ with $p_\alpha \restriction \Delta = r$ for all $\alpha \in H$; and
(c) $p_\alpha \Vdash \check{\alpha} \in \dot{A}$.

By hypothesis, we may assume that $\{B_\alpha : \alpha \in H\}$ is point-uncountable. We will prove

$$r \Vdash \bigcup_{\alpha\in H} \dot{B}^*_\alpha \subset \bigcup_{\alpha\in\dot{A}} \dot{B}^*_\alpha \tag{1}$$

which will contradict $r \leq p$ (which holds since each p_α extends p) . Suppose indirectly that there are $r' \leq r$ and $\beta \in H$ such that

$$r' \Vdash \dot{B}^*_\beta \not\subset \bigcup_{\alpha\in\dot{A}} \dot{B}^*_\alpha. \tag{2}$$

In V, by the point-uncountability of $\{B_\alpha\}_{\alpha\in H}$, and since B_β is first-countable, each $x \in B_\beta$ has a neighborhood contained in uncountably many members of $\{B_\alpha\}_{\alpha\in H}$. Hence there is a countable sub-collection $\{C_n\}_{n\in\omega}$ of $\mathcal{B}$ such that $B_\beta = \bigcup_{n\in\omega} C_n$ and each C_n is contained in uncountably many members of $\{B_\alpha\}_{\alpha\in H}$. Since $1 \Vdash \bigcup_{n\in\omega} \dot{C}^*_n = \dot{B}^*_\beta$, there is $r'' \leq r'$ and $k < \omega$ such that

$$r'' \Vdash \dot{C}^*_k \not\subset \bigcup_{\alpha\in\dot{A}} \dot{B}^*_\alpha. \tag{3}$$

Now let $\gamma \in H$ be such that $q = r'' \cup p_\gamma \in P$ and $C_k \subset B_\gamma$. Then

$$q \Vdash \text{“}\dot{C}^*_k \not\subset \bigcup_{\alpha\in\dot{A}} \dot{B}^*_\alpha \text{ and } \check{\gamma} \in \dot{A}\text{”}. \tag{4}$$

But $1 \Vdash \dot{C}^*_k \subset \dot{B}^*_\gamma$, so we have a contradiction. □

Corollary 3.7. *In V, let $M = M(\{\phi_{\beta\alpha}\})$ be non-metrizable, and suppose that $M \setminus \cup\mathcal{A}$ compact for every uncountable $\mathcal{A} \subset \mathcal{B}$. In $V[G]$, let $M^* = M(\{\phi^*_{\beta\alpha}\})$. Then M^* is countably compact, perfectly normal, and hereditarily separable.*

Proof. Since M is non-metrizable, the standard basis $\mathcal{B}$ has cardinality ω_1, so we can let $\mathcal{B} = \{B_\alpha\}_{\alpha<\omega_1}$. Suppose that in $V[G]$, $A \subset \omega_1$ is uncountable; we shall show that $M^* \setminus \bigcup_{\alpha\in A} B^*_\alpha$ is compact. Indeed, by Lemma 3.6 there is an uncountable $H \subset \omega_1$ in V such that $C = M^* \setminus \bigcup_{\alpha\in A} B^*_\alpha \subset M^* \setminus \bigcup_{\alpha\in H} B^*_\alpha$. There is a finite $\mathcal{F} \subset \mathcal{B}$ such that, in V, $M \setminus \bigcup_{\alpha\in H} B_\alpha \subset \cup\mathcal{F}$, i.e., $\{B_\alpha\}_{\alpha\in H} \cup \mathcal{F}$ is a cover of M. So $\{B^*_\alpha\}_{\alpha\in H} \cup \{F^* : F \in \mathcal{F}\}$ is a cover of M^* , whence $\{F^* : F \in \mathcal{F}\}$ is a finite cover of C. As C is closed, C must be compact.

To see that M^* is countably compact, let us consider a countable cover $\mathcal{V}$ of M^*. Then there is a $V \in \mathcal{V}$ that contains uncountably many B^*_α's , making $M^* \setminus V$ compact.

M^* is perfectly normal since each of its closed sets is either compact or has second countable complement.

M^* is separable, for otherwise it would contain an uncountable collection of pairwise disjoint open sets. The complement of the union of this would be compact, making M^* metrizable.

Finally, M^* is hereditarily separable since it is separable and perfectly normal. □

Now we have the main result of this section:

Theorem 3.8. *Add any number of Cohen reals to a model of $ZFC + \diamondsuit$. Then in the resulting extension, there is a countably compact, perfectly normal, hereditarily separable, non-metrizable manifold.*

Proof. Let N be Rudin's manifold constructed from $\diamondsuit$. Then N is the strictly increasing union of open subspaces N_α, $\alpha < \omega_1$, such that each N_α is homeomorphic to $\mathbb{R}^2$ and such that any open set which is not contained in some N_α has compact complement. (The latter claim'follows from the claim on page 654 of [N_1]).

Let $\theta_\alpha : N_\alpha \to \mathbb{R}^2$ be a homeomorphism, and for $\beta < \alpha$, let $\phi_{\beta\alpha} = \theta_\alpha \circ \theta_\beta^{-1}$. For $p \in N_\alpha$, let $h(p)$ be the point $\vec{x} \in M = M(\{\phi_{\beta\alpha}\})$ such that $x_\alpha = \theta_\alpha(p)$. It is easy to check that $h : N \to M$ is a homeomorphism such that $h(N_\alpha) = M_\alpha$ for all $\alpha < \omega_1$. So M satisfies the conditions of Corollary 3.7, and the proof is complete. □

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