METRIZABLE SUBSPACES OF SPACES HAVING A POINT-COUNTABLE BASE

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Abstract. In [1], van Douwen, Lutzer, Pelant, and Reed asked if every regular space with a point-countable base can written as the union of $\leq \mathfrak{c}$-many metrizable subspaces. They also asked the same question for closed metrizable subspaces. In this note, we construct a counterexample to the second question; the first question remains open.

1. Introduction

In 1980, E.K. van Douwen, D.J. Lutzer, J. Pelant, and G.M. Reed [1] obtained the following:

Theorem 1. *Any σ-space is the union of $\leq \mathfrak{c}$-many closed metrizable subspaces.*

Theorem 2. *Any T_1-space with a σ-point-finite base, or more generally, any quasi-developable T_1-space, is the union of $\leq \mathfrak{c}$-many metrizable subspaces.*

They also asked the following question:

Question 1. *Is every regular space with a point-countable base the union of $\leq \mathfrak{c}$-many (closed) metrizable subspaces?*

In this note, we show that the answer to the part of Question 1 about closed metrizable subspaces is "no". The question without "closed" is still open!

Our example does not have a σ-point-finite base. In [1] an example is given of a quasi-developable space which is not the union of $\leq \mathfrak{c}$-many closed metrizable subspaces. But it seems the following natural question is unsolved:

Question 2. *Is there a T_1-space with σ-point-finite base which is not the union of $\leq \mathfrak{c}$-many closed metrizable subspaces?*

We show that no example similar to our point-countable base example could serve as a counterexample to Question 2.

We also mention the following question raised in [1] which as far as I know is unsolved.

Question 3. *Does Theorem 1 remain true if "σ-space" is replaced by "semi-stratifiable space" or "semi-metric space"?*

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2. The example

Example. *For any infinite cardinal κ, there is a regular space X with a point-countable base which is not a union of $\leq \kappa$-many closed metrizable subspaces.*

Proof. Let $\lambda = (2^{2^\kappa})^+$, and let $\mathcal{A}$ be a maximal almost-disjoint family of countably infinite subsets of λ. That is, every pair of distinct elements of $\mathcal{A}$ have finite intersection, and every infinite subset of λ meets some member of $\mathcal{A}$ in an infinite set. Let

$$X = \mathcal{A} \cup \{(\alpha, \mathcal{B}) : (\alpha \in \lambda) \wedge (\mathcal{B} \subset \mathcal{A}) \wedge (|\mathcal{B}| \leq \omega)\}.$$

Let the pairs $(\alpha, \mathcal{B})$ be isolated points. For each $A \in \mathcal{A}$, choose an indexing $\{a_0, a_1, \dots\}$ of A, and let

$$B(A, n) = \{A\} \cup \{(\alpha, \mathcal{B}) \in X : (\alpha \in \{a_i\}_{i \geq n}) \wedge A \in \mathcal{B}\}$$

be the n^{th} member of a countable neighborhood base at A.

Note that $B(A, 0) \supset B(A, 1) \supset \dots$ and $\bigcap_{n \in \omega} B(A, n) = \{A\}$. Hence X is T_1.

Fact 1. Each $B(A, n)$ is clopen, hence X is completely regular. If $A' \in \mathcal{A} \setminus \{A\}$, then there exists $k \in \omega$ such that $\{a'_i\}_{i \geq k} \cap \{a_i\}_{i \geq n} = \emptyset$. Then $B(A', k) \cap B(A, n) = \emptyset$.

Call a set H a G_κ*-set* if H is the intersection of $\leq \kappa$-many open sets.

Fact 2. X is the union of $\leq \kappa$-many closed metrizable subsets iff $\mathcal{A}$ is a G_κ-set. The sufficiency is clear, so we prove the necessity. Suppose $X = \bigcup_{\alpha < \kappa} M_\alpha$, where each M_α is closed in X and metrizable. Then $M_\alpha \setminus \mathcal{A}$ is open in the metrizable space M_α, so $M_\alpha \setminus \mathcal{A} = \bigcup_{n \in \omega} F_{\alpha n}$, where each $F_{\alpha n}$ is closed in M_α and hence in X. Then $\mathcal{A} = \bigcap \{X \setminus F_{\alpha n} : \alpha < \kappa, n \in \omega\}$.

The next fact is immediate from the fact that $\mathcal{A}$ is a maximal almost-disjoint family.

Fact 3. For each $F : \mathcal{A} \to \omega$, the set $\lambda \setminus \cup\{\{a_i\}_{i \geq F(A)} : A \in \mathcal{A}\}$ is finite.

The remainder of the proof is devoted to proving:

Fact 4. $\mathcal{A}$ is not a G_κ-set.

Suppose $\mathcal{F}$ is a collection of κ-many functions $F : \mathcal{A} \to \omega$. For $F \in \mathcal{F}$, let $U(F) = \bigcup_{A \in \mathcal{A}} B(A, F(A))$. We need to show that $\mathcal{A} \neq \bigcap_{F \in \mathcal{F}} U(F)$.

For each $A \in \mathcal{A}$, define $\Theta_A \in \omega^{\mathcal{F}}$ by $\Theta_A(F) = F(A)$, and for each $\Theta \in \omega^{\mathcal{F}}$, let $\mathcal{A}_\Theta = \{A \in \mathcal{A} : \Theta_A = \Theta\}$. Then $\{\mathcal{A}_\Theta : \Theta \in \omega^{\mathcal{F}}\}$ is a partition of $\mathcal{A}$.

Claim 1. There exists $\Theta \in \omega^{\mathcal{F}}$ and $\alpha < \lambda$ such that, for each $n \in \omega$, there is some $A \in \mathcal{A}$ with $\Theta_A = \Theta$ and $\alpha \in \{a_i\}_{i \geq n}$.

Suppose the claim fails. Fix $\alpha < \lambda$. Then for every $\Theta \in \omega^{\mathcal{F}}$, there is $n(\alpha, \Theta) \in \omega$ such that

$$\alpha \notin \cup\{\{a_i\}_{i \geq n(\alpha, \Theta)} : \Theta_A = \Theta\}$$

and hence

$$\alpha \notin \cup\{\{a_i\}_{i \geq n(\alpha, \Theta_A)} : A \in \mathcal{A}\}.$$

It follows that, for each $\alpha < \lambda$, there exists $G : \mathcal{A} \to \omega$ such that

(1) $\alpha \notin \cup\{\{a_i\}_{i \geq G(A)} : A \in \mathcal{A}\}$;
(2) $G \upharpoonright \mathcal{A}_\Theta$ is constant for each $\Theta \in \omega^{\mathcal{F}}$.

There are not more than $|\omega^{\omega^{\mathcal{F}}}| = 2^{2^\kappa}$-many such G's, and $\lambda = (2^{2^\kappa})^+$, so there must exist such a G with $|\lambda \setminus \cup_{A \in \mathcal{A}}\{a_i\}_{i \geq G(A)}| = \lambda$, contradicting Fact 3. This proves Claim 1.

Now, let $\bar{\Theta} \in \omega^{\mathcal{F}}$ and $\bar{\alpha} < \lambda$ be as in Claim 1. Then, for each $n \in \omega$, there exists $A(n) = \{a_{ni}\}_{i \in \omega} \in \mathcal{A}$ with $\Theta_{A(n)} = \bar{\Theta}$ and $\bar{\alpha} \in \{a_{ni}\}_{i \geq n}$. Let $\mathcal{B} = \{A(n)\}_{n \in \omega}$.

The next claim completes the proof of Fact 4 and the example.

Claim 2. *$(\bar{\alpha}, \mathcal{B}) \in U(F)$ for every $F \in \mathcal{F}$.*

Fix $F \in \mathcal{F}$, and let $k = \bar{\Theta}(F)$. Then for each $n \in \omega$, $F(A(n)) = \Theta_{A(n)}(F) = \bar{\Theta}(F) = k$. Also $\bar{\alpha} \in \{a_{ni}\}_{i \geq n}$ and $A(n) \in \mathcal{B}$, so if $n \geq k$, we have

$$(\bar{\alpha}, \mathcal{B}) \in B(A(n), n) \subset B(A(n), k) = B(A(n), F(A(n))) \subset U(F).$$

□

The version of Question 1 in which the metrizable subspaces are not required to be closed may well be the more interesting one, as clearly any X of the form above, i.e., $D \cup I$, where D is closed discrete and I a set of isolated points, is useless for that question. It is also easy to show that such a space cannot provide an answer to Question 2.

Proposition 1. *Let X be a T_1-space of the form $D \cup I$, where D is closed discrete and I a set of isolated points. If X has a σ-point-finite base, then X is the union of $\leq \mathfrak{c}$-many closed metrizable subspaces.*

Proof. It is easy under the assumptions to find a local base $\{B(d, n) : n \in \omega\}$ at each $d \in D$ and disjoint subsets D_n of D, such that:

(1) $D = \bigcup_{n \in \omega} D_n$;
(2) For each $n \in \omega$, $\{B(d, 0) : d \in D_n\}$ is point-finite;
(3) For each $n \in \omega$, $B(d, n) \subseteq B(d, 0)$.

Then for each $\alpha \in I$ and each $n \in \omega$, there is $f : \omega \to \omega$ such that $\alpha \notin \bigcup_{n \in \omega}(\bigcup_{d \in D_n} B(d, f(n)))$. It follows that D is the intersection of $\leq \mathfrak{c}$-many open sets, and hence X is the union of $\leq \mathfrak{c}$-many closed metrizable (in fact, discrete) subspaces. □

One might wonder if the choice of $\lambda = (2^{2^\kappa})^+$ is the least possible cardinal that would work to get our example. We don't know. But we do know that taking $\lambda = 2^\kappa$ would often be too small to work.

Proposition 2. *Suppose $k^\omega = \kappa$ and $\nu = 2^\kappa$. If $|I| \leq \nu$ and $X = D \cup I$ is a T_1-space with a point- countable base, where I is a set of isolated points and D is closed discrete, then D is can be written as the intersection of $\leq \kappa$-many open sets.*

Proof. W.l.o.g., no point of D is isolated. Let $\mathcal{B}$ be a point-countable base for X. For each $d \in D$, choose $B_d \in \mathcal{B}$ with $B_d \cap D = \{d\}$. For each $i \in I$, let

$D(i) = \{d : i \in B_d\}$. Then each $D(i)$ is countable and $D = \bigcup_{i \in I} D(i)$. It follows that $|D| \leq |I|$.

A minor variation of the Hewitt-Marczewski-Pondiczery theorem on the density of product spaces implies that there is a set $\mathcal{F}$ of κ-many functions $f : D \to \omega$ such that any function from a countable subset of ν to ω is extended by some member of $\mathcal{F}$. To see this directly, identify D with $\nu = 2^\kappa$ as the power of the discrete space $\{0, 1\}$, but with the topology obtained by declaring all sets that are G_δ-sets in the Tychonoff product topology to be open. By $k^\omega = \kappa$, the weight of this space is κ. Let $\mathcal{C}$ be any basis for this space of cardinality κ. Then let $\mathcal{P}$ be the collection of all partitions $P = \{P_0, P_1, \dots\}$ of the space such that, for $i \geq 1$, $P_i = C_i \setminus \bigcup_{1 \leq j < n} C_j$, where each $C_i \in \mathcal{C}$, and $P_0 = 2^\kappa \setminus \bigcup_{i \geq 1} C_i$. Again by $\kappa^\omega = \kappa$, we have $|\mathcal{P}| = \kappa$. Now let $\mathcal{F}$ be all functions $f : 2^\kappa \to \omega$ such that, for some partition $P \in \mathcal{P}$, f is constant on each member of P. Then $\mathcal{F}$ is easily checked to be the desired collection.

Now let $B(d, n)$, $n < \omega$, be a countable decreasing neighborhood base at $d \in D$ such that $B(d, 0) \cap D = \{d\}$ and the collection $\{B(d, 0) : d \in D\}$ is point-countable. For each $f \in \mathcal{F}$, let $U(f) = \bigcup_{d \in D} B(d, f(n))$. Each $U(f)$ is of course an open superset of D. Let $a \in I$, and let $D_a = \{d \in D : a \in B(d, 0)\}$. There is a function $g : D_a \to \omega$ such that $a \notin \bigcup_{d \in D_a} B(d, g(d))$. Pick any $f \in \mathcal{F}$ that extends g. Then $a \notin U(f)$. Hence $D = \bigcap_{f \in \mathcal{F}} U(f)$. $\square$

References

[1] van Douwen, E. K., Lutzer, D. J., Pelant, J.,and G. M. Reed, *On unions of metrizable subspaces*, Canad. J. Math. **32** (1980), 76–85.

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