METRIZABLE SUBSPACES OF SPACES
HAVING A POINT-COUNTABLE BASE

GARY GRUENHAGE

Abstract. In [1], van Douwen, Lutzer, Pelant, and Reed asked if every regular space with a point-countable base can written as the union of \( \leq \mathfrak{c} \)-many metrizable subspaces. They also asked the same question for closed metrizable subspaces. In this note, we construct a counterexample to the second question; the first question remains open.

1. Introduction

In 1980, E.K. van Douwen, D.J. Lutzer, J. Pelant, and G.M. Reed [1] obtained the following:

Theorem 1. Any \( \sigma \)-space is the union of \( \leq \mathfrak{c} \)-many closed metrizable subspaces.

Theorem 2. Any \( T_1 \)-space with a \( \sigma \)-point-finite base, or more generally, any quasi-developable \( T_1 \)-space, is the union of \( \leq \mathfrak{c} \)-many metrizable subspaces.

They also asked the following question:

Question 1. Is every regular space with a point-countable base the union of \( \leq \mathfrak{c} \)-many (closed) metrizable subspaces?

In this note, we show that the answer to the part of Question 1 about closed metrizable subspaces is “no”. The question without “closed” is still open!

Our example does not have a \( \sigma \)-point-finite base. In [1] an example is given of a quasi-developable space which is not the union of \( \leq \mathfrak{c} \)-many closed metrizable subspaces. But it seems the following natural question is unsolved:

Question 2. Is there a \( T_1 \)-space with \( \sigma \)-point-finite base which is not the union of \( \leq \mathfrak{c} \)-many closed metrizable subspaces?

We show that no example similar to our point-countable base example could serve as a counterexample to Question 2.

We also mention the following question raised in [1] which as far as I know is unsolved.

Question 3. Does Theorem 1 remain true if “\( \sigma \)-space” is replaced by “semi-stratifiable space” or “semi-metric space”?

We wish to thank Steve Watson for reminding the author of the idea of splitting points, which is one of the keys to our example.

Key words and phrases. metrizable, point-countable base.

Research partially supported by NSF grant DMS 0072269.

1
2. The example

**Example.** For any infinite cardinal \( \kappa \), there is a regular space \( X \) with a point-countable base which is not a union of \( \leq \kappa \)-many closed metrizable subspaces.

**Proof.** Let \( \lambda = (2^\omega)^+ \), and let \( \mathcal{A} \) be a maximal almost-disjoint family of countably infinite subsets of \( \lambda \). That is, every pair of distinct elements of \( \mathcal{A} \) have finite intersection, and every infinite subset of \( \lambda \) meets some member of \( \mathcal{A} \) in an infinite set. Let

\[
X = \mathcal{A} \cup \{(\alpha, B) : (\alpha \in \lambda) \land (B \subseteq \mathcal{A}) \land (|B| \leq \omega)\}.
\]

Let the pairs \((\alpha, B)\) be isolated points. For each \( A \in \mathcal{A} \), choose an indexing \( \{a_0, a_1, \ldots\} \) of \( A \), and let

\[
B(A, n) = \{A\} \cup \{(\alpha, B) \in X : (\alpha \in \{a_i\}_{i \geq n} \land A \in B)\}
\]

be the \( n \)th member of a countable neighborhood base at \( A \).

Note that \( B(A, 0) \supset B(A, 1) \supset \ldots \) and \( \bigcap_{n \in \omega} B(A, n) = \{A\} \). Hence \( X \) is T1.

**Fact 1.** Each \( B(A, n) \) is clopen, hence \( X \) is completely regular. If \( A' \in \mathcal{A} \setminus \{A\} \), then there exists \( k \in \omega \) such that \( \{a_i\}_{i \geq k} \cap \{a_i\}_{i \geq n} = \emptyset \). Then \( B(A', k) \cap B(A, n) = \emptyset \).

Call a set \( H \) a \( G_\kappa \)-set if \( H \) is the intersection of \( \leq \kappa \)-many open sets.

**Fact 2.** \( X \) is the union of \( \leq \kappa \)-many closed metrizable subsets iff \( \mathcal{A} \) is a \( G_\kappa \)-set. The sufficiency is clear, so we prove the necessity. Suppose \( X = \bigcup_{\alpha < \kappa} M_{\alpha} \), where each \( M_{\alpha} \) is closed in \( X \) and metrizable. Then \( M_{\alpha} \setminus \mathcal{A} \) is open in the metrizable space \( M_{\alpha} \), so \( M_{\alpha} \setminus \mathcal{A} = \bigcup_{n \in \omega} F_{\alpha n} \), where each \( F_{\alpha n} \) is closed in \( M_{\alpha} \) and hence in \( X \). Then \( \mathcal{A} = \bigcap\{X \setminus F_{\alpha n} : \alpha < \kappa, n \in \omega\} \).

The next fact is immediate from the fact that \( \mathcal{A} \) is a maximal almost-disjoint family.

**Fact 3.** For each \( F : \mathcal{A} \to \omega \), the set \( \lambda \setminus \bigcup\{\{a_i\}_{i \geq F(A)} : A \in \mathcal{A}\} \) is finite.

The remainder of the proof is devoted to proving:

**Fact 4.** \( \mathcal{A} \) is not a \( G_\kappa \)-set.

Suppose \( \mathcal{F} \) is a collection of \( \kappa \)-many functions \( F : \mathcal{A} \to \omega \). For \( F \in \mathcal{F} \), let \( U(F) = \bigcup_{A \in \mathcal{A}} B(A, F(A)) \). We need to show that \( \mathcal{A} \neq \bigcap_{F \in \mathcal{F}} U(F) \).

For each \( A \in \mathcal{A} \), define \( \Theta_A \in \omega^\mathcal{F} \) by \( \Theta_A(F) = F(A) \), and for each \( \Theta \in \omega^\mathcal{F} \), let \( \mathcal{A}_\Theta = \{A \in \mathcal{A} : \Theta_A = \Theta\} \). Then \( \mathcal{A}_\Theta : \Theta \in \omega^\mathcal{F} \) is a partition of \( \mathcal{A} \).

**Claim 1.** There exists \( \Theta \in \omega^\mathcal{F} \) and \( \alpha < \lambda \) such that, for each \( n \in \omega \), there is some \( A \in \mathcal{A} \) with \( \Theta_A = \Theta \) and \( \alpha \in \{a_i\}_{i \geq n} \).

Suppose the claim fails. Fix \( \alpha < \lambda \). Then for every \( \Theta \in \omega^\mathcal{F} \), there is \( n(\alpha, \Theta) \in \omega \) such that

\[
\alpha \not\in \bigcup\{\{a_i\}_{i \geq n(\alpha, \Theta)} : \Theta_A = \Theta\}
\]

and hence

\[
\alpha \not\in \bigcup\{\{a_i\}_{i \geq n(\alpha, \Theta_A)} : A \in \mathcal{A}\}.
\]
Proposition 1. It follows that, for each \( \alpha < \lambda \), there exists \( G : \mathcal{A} \to \omega \) such that

1. \( \alpha \not\in \bigcup \{ \{ a_i \}_{i \geq G(A)} : A \in \mathcal{A} \} \);
2. \( G \upharpoonright \mathcal{A}_\emptyset \) is constant for each \( \Theta \in \omega^\mathcal{F} \).

There are not more than \( |\omega^\mathcal{F}| = 2^\omega \)-many such \( G \)'s, and \( \lambda = (2^\omega)^+ \), so there must exist such a \( G \) with \( |\lambda \setminus \bigcup_{A \in \mathcal{A}} \{ a_i \}_{i \geq G(A)}| = \lambda \), contradicting Fact 3. This proves Claim 1.

Now, let \( \tilde{\Theta} \in \omega^\mathcal{F} \) and \( \tilde{\alpha} < \lambda \) be as in Claim 1. Then, for each \( n \in \omega \), there exists \( A(n) = \{ a_{ni} \}_{i \in \omega} \in \mathcal{A} \) with \( \Theta_{A(n)} = \tilde{\Theta} \) and \( \tilde{\alpha} \in \{ a_{ni} \}_{i \geq n} \). Let \( \mathcal{B} = \{ A(n) \}_{n \in \omega} \).

The next claim completes the proof of Fact 4 and the example.

Claim 2. \( (\tilde{\alpha}, \mathcal{B}) \in U(F) \) for every \( F \in \mathcal{F} \).

Fix \( F \in \mathcal{F} \), and let \( k = \tilde{\Theta}(F) \). Then for each \( n \in \omega \), \( F(A(n)) = \Theta_{A(n)}(F) = \tilde{\Theta}(F) = k \). Also \( \tilde{\alpha} \in \{ a_{ni} \}_{i \geq n} \) and \( A(n) \in \mathcal{B} \), so if \( n \geq k \), we have

\[
(\tilde{\alpha}, \mathcal{B}) \in B(A(n), n) \subset B(A(n), k) = B(A(n), F(A(n))) \subset U(F).
\]

The version of Question 1 in which the metrizable subspaces are not required to be closed may well be the more interesting one, as clearly any \( X \) of the form above, i.e., \( D \cup I \), where \( D \) is closed discrete and \( I \) a set of isolated points, is useless for that question. It is also easy to show that such a space cannot provide an answer to Question 2.

**Proposition 1.** Let \( X \) be a \( T_1 \)-space of the form \( D \cup I \), where \( D \) is closed discrete and \( I \) a set of isolated points. If \( X \) has a \( \sigma \)-point-finite base, then \( X \) is the union of \( \leq c \)-many closed metrizable subspaces.

**Proof.** It is easy under the assumptions to find a local base \( \{ B(d, n) : n \in \omega \} \) at each \( d \in D \) and disjoint subsets \( D_n \) of \( D \), such that:

1. \( D = \bigcup_{n \in \omega} D_n \);
2. For each \( n \in \omega \), \( \{ B(d, 0) : d \in D_n \} \) is point-finite;
3. For each \( n \in \omega \), \( B(d, n) \subseteq B(d, 0) \).

Then for each \( \alpha \in I \) and each \( n \in \omega \), there is \( f : \omega \to \omega \) such that \( \alpha \not\in \bigcup_{n \in \omega} (\bigcup_{d \in D_n} B(d, f(n))) \). It follows that \( D \) is the intersection of \( \leq c \)-many open sets, and hence \( X \) is the union of \( \leq c \)-many closed metrizable (in fact, discrete) subspaces.

One might wonder if the choice of \( \lambda = (2^\omega)^+ \) is the least possible cardinal that would work to get our example. We don’t know, but we do know that taking \( \lambda = 2^\omega \) would often be too small to work.

**Proposition 2.** Suppose \( k^\nu = \kappa \) and \( \nu = 2^\kappa \). If \( |I| \leq \nu \) and \( X = D \cup I \) is a \( T_1 \)-space with a point-countable base, where \( I \) is a set of isolated points and \( D \) is closed discrete, then \( D \) is can be written as the intersection of \( \leq \kappa \)-many open sets.

**Proof.** W.l.o.g., no point of \( D \) is isolated. Let \( \mathcal{B} \) be a point-countable base for \( X \). For each \( d \in D \), choose \( B_d \in \mathcal{B} \) with \( B_d \cap D = \{ d \} \). For each \( i \in I \), let
A minor variation of the Hewitt-Marczewski-Pondiczery theorem on the density of product spaces implies that there is a set \( F \) of \( \kappa \)-many functions \( f : D \to \omega \) such that any function from a countable subset of \( \nu \) to \( \omega \) is extended by some member of \( F \). To see this directly, identify \( D \) with \( 2^\kappa \) as the power of the discrete space \( \{0,1\} \), but with the topology obtained by declaring all sets that are \( G_\delta \)-sets in the Tychonoff product topology to be open. By \( k^\omega = \kappa \), the weight of this space is \( \kappa \).

Let \( \mathcal{C} \) be any basis for this space of cardinality \( \kappa \). Then let \( \mathcal{P} \) be the collection of all partitions \( P = \{P_0, P_1, \ldots\} \) of the space such that, for \( i \geq 1 \), \( P_i = C_i \setminus \bigcup_{1 \leq j < n} C_j \), where each \( C_i \in \mathcal{C} \), and \( P_0 = 2^\kappa \setminus \bigcup_{i \geq 1} C_i \). Again by \( k^\omega = \kappa \), we have \( |\mathcal{P}| = \kappa \). Now let \( \mathcal{F} \) be all functions \( f : 2^\kappa \to \omega \) such that, for some partition \( P \in \mathcal{P} \), \( f \) is constant on each member of \( P \). Then \( \mathcal{F} \) is easily checked to be the desired collection.

Now let \( B(d,n), n < \omega \), be a countable decreasing neighborhood base at \( d \in D \) such that \( B(d,0) \cap D = \{d\} \) and the collection \( \{B(d,0) : d \in D\} \) is point-countable. For each \( f \in \mathcal{F} \), let \( U(f) = \bigcup_{d \in D} B(d, f(n)) \). Each \( U(f) \) is of course an open superset of \( D \). Let \( a \in I \), and let \( D_a = \{d \in D : a \in B(d,0)\} \). There is a function \( g : D_a \to \omega \) such that \( a \notin \bigcup_{d \in D_a} B(d, g(d)) \). Pick any \( f \in \mathcal{F} \) that extends \( g \). Then \( a \notin U(f) \). Hence \( D = \bigcap_{f \in \mathcal{F}} U(f) \).

References


Department of Mathematics, Auburn University, AL 36849, U.S.A.
E-mail address: garyg@auburn.edu