BASE MULTIPLICITY IN COMPACT AND GENERALIZED COMPACT SPACES

Z. Balogh* and G. Gruenhage**

Abstract. We show that a compact Hausdorff space is metrizable if it has a base $\mathcal{B}$ such that every countably infinite subset of X is contained in at most countably many members of $\mathcal{B}$. We show that the same statement for countably compact spaces is consistent with and independent of ZFC. These results answer questions stated in [AJRS]. We prove some strenthenings of these theorems. We also consider generalizations of our results to higher cardinalities as well as to wider classes of spaces.

Introduction

"Space" in this paper will stand for "regular T_1-space". If κ and λ are cardinals, then let us say that a family $\mathcal{B}$ of sets is *κ-in-$\leq \lambda$* if for every set A of cardinality κ, $A \subset B$ holds for no more than λ members $B \in \mathcal{B}$. κ-in-$< \lambda$ is similarly defined. We will also say "point-countable" in place of "one-in-$\leq \omega$", "ω-in-countable" in place of "ω-in-$\leq \omega$" and "ω-in-finite" in place of "ω-in-$< \omega$". Our paper started by considering the following two problems from [AJRS]:

Problem 1. *If X is a compact space with an ω-in-countable base, then is X metrizable?*

Problem 2. *If X is a countably compact space with an ω-in-countable base, then is X compact metrizable?*

We are going to show that the answer is "yes" to Problem 1 (Corollary 2.5) and "it depends on your set theory" to Problem 2 (Section 4). The subject matter of this paper is to report about these and some stronger and related results.

In Section 1 we show that if $\lambda > \omega$ is a regular cardinal and X is a compact space of weight $\geq \lambda$, then every base for X contains λ members whose intersection contains either an open set or a perfect preimage of 2^κ (Theorem 1.3). The key step is, of course, Shapirovski's famous mapping theorem. Section 2 then uses Section 1 to prove, among other things, that a compact space with a κ-in-$\leq \kappa$ base has weight $\leq \kappa$ (Theorem 2.4). In Section 3 we extend the metrization theorems to locally compact spaces and paracompact p-spaces, including an improvement on a result of Peregudov (Theorem 3.4). Section 4 answers Problem 2 above.

In Section 5 we show that an initially $\mathfrak{c}$-compact space with a $\mathfrak{c}$-in-$\leq \mathfrak{c}$ base has weight $\leq \mathfrak{c}$ (Theorem 5.3). We also show that for any fixed infinite cardinal κ,

---

Key words and phrases. .

*Partially supported by NSF grant DMS-9623391.

**Partially supported by NSF grant DMS-9704849.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

it is consistent to have the analogous statement fail for κ (Corollary 5.5). (Note, however, that in the resulting model, $\mathfrak{c} = \kappa^+$.)

Finally, in many of the theorems mentioned above, "base" can be weakened to "T_1-separating open cover".

The following notation and terminology is used throughout. If $\mathcal{U}$ is a collection of subsets of a set X, and $x \in X$, we let $\mathcal{U}_x$ denote $\{U \in \mathcal{U} : x \in U\}$. For any set X and cardinal κ, $[X]^\kappa$ denotes the set of all subsets of X having cardinality κ; $[X]^{\leq\kappa}$ and $[X]^{<\kappa}$ are similarly defined. We let $Fn(\kappa, 2)$ denote the set of all functions from a finite subset of κ into $\{0, 1\} = 2$, and for $\sigma \in Fn(\kappa, 2)$ we let $[\sigma] = \{x \in 2^\kappa : \sigma \subset x\}$.

1. INTERSECTIONS OF BASIC OPEN SETS IN COMPACT SPACES

We will need the following lemma which is well known for $\lambda = \omega_1$. Since a similar proof works for any regular $\lambda > \omega$, we only sketch the proof.

Lemma 1.1. *Suppose that $\lambda > \omega$ is a cardinal, and X is an initially $< \lambda$–compact space with a point–$< \lambda$ T_1-separating open cover $\mathcal{B}$. Then X is a compact space of weight $< \lambda$.*

Proof. Let us first show that every open cover $\mathcal{U} \subset \mathcal{B}$ of X has a finite subcover. Suppose $\mathcal{U}$ is a counterexample. Since X is initially $<\lambda$-compact, $\mathcal{U}$ has no subcover of size $< \lambda$. Since $\mathcal{U}$ is point-$< \lambda$, by induction we can pick $\langle x_n \rangle_{n\in\omega}$ such that $n < m$ in ω implies $x_m \notin st(x_n, \mathcal{U})$. Then $\langle x_n \rangle_{n\in\omega}$ is a closed discrete collection of points in X, contradicting the countable compactness of X.

Next we prove that $|\mathcal{B}| < \lambda$. Suppose indirectly that $|\mathcal{B}| \geq \lambda$. By our first step above and since $\mathcal{B}$ is T_1-separating in X, we conclude that every $B \in \mathcal{B}$ can be included in a finite cover $\mathcal{V} \subset \mathcal{B}$. Since $|\mathcal{B}| \geq \lambda$, there are at least λ many finite covers $\mathcal{V} \subset \mathcal{B}$. Hence there is a Δ-system $\langle \mathcal{V}_\alpha \rangle_{\alpha<\lambda}$ of such finite covers with root $\mathcal{V}^*$. Since $\bigcup \mathcal{V}^* \neq X$, it follows that $\mathcal{B}$ is **not** point–$< \lambda$ at any point $x \in X \setminus \bigcup \mathcal{V}^*$, contradiction.

Thus we have proved that $|\mathcal{B}| < \lambda$. Then the family of all finite intersections of the sets $X \setminus B, B \in \mathcal{B}$, forms a network. Hence X is an initially $< \lambda$–compact space with $nw(X) < \lambda$, so X is compact of weight $< \lambda$. □

Lemma 1.2. *Suppose that $\lambda > \omega$ is a regular cardinal, X is a compact space, and $\mathcal{B}$ is a T_1-separating open cover of X. Then*

$$X_0 = \{x \in X : |\mathcal{B}_x| < \lambda\}$$

is a compact subspace of weight $< \lambda$.

Proof. Note that if $A \subset X_0$ and $|A| < \lambda$, then $\overline{A} \subset X_0$. (Indeed, $x \in \overline{A}$ and $|\mathcal{B}_x| \geq \lambda$ would imply $|\mathcal{B}_y| \geq \lambda$ for some $y \in A$, in contradiction with $A \subset X_0$). Hence X_0 is initially $< \lambda$-compact. By Lemma 1.1, X_0 is a compact subspace of weight $< \lambda$. □

Theorem 1.3. *Suppose that $\lambda > \omega$ is a regular cardinal, X is a compact T_2-space of weight $\geq \lambda$ and $\mathcal{B}$ is a T_1-separating open cover for X. Then there is a $\mathcal{C} \in [\mathcal{B}]^\lambda$*

such that $\bigcap \mathcal{C}$ *contains either a nonempty open subset of* X *or a perfect preimage of* 2^λ.

Proof. Let $X_0 = \{x \in X : |\mathcal{B}_x| < \lambda\}$ be as in the conclusion of Lemma 1.2. Since $\omega(X) \geq \lambda$, it follows that $X \setminus X_0 \neq \emptyset$. Let W be an open subset of X such that $W \subset \overline{W} \subset X \setminus X_0$, and $Z = \overline{W}$ is compact. We will consider two cases.

Case 1. Suppose that there is a $z \in Z$ such that $\pi\chi(z, Z) < \lambda$. Then by $|\mathcal{B}_z| \geq \lambda$ we conclude that there is a relatively open subset V of Z such that $\mathcal{C} = \{B \in \mathcal{B}_z : B \supset V\}$ has cardinality $\geq \lambda$. Then $U = V \cap W$ is a nonempty open subset of X which is contained in λ many members of $\mathcal{B}$.

Case 2. Suppose now that $\pi\chi(z, Z) \geq \lambda$ for every $z \in Z$. Then by Shapirovski's famous theorem ([Sha]; see also [Ju$_1$]) there is a compact subset Z_1 of Z and a continuous, irreducible, onto map $f : Z_1 \longrightarrow 2^\lambda$. Fix a point $z \in Z_1$. Since f is an irreducible perfect map, every neighborhood of z in Z_1 contains the full preimage of a basic open subset $[\phi]$ of 2^λ. Hence there are distinct $\langle B_\alpha \rangle_{\alpha<\lambda}$ in $\mathcal{B}_z$ and a sequence $\langle \phi_\alpha \rangle_{\alpha<\lambda}$ in $Fn(\lambda, 2)$ such that

(a) $f^{\leftarrow}([\phi_\alpha]) \subset B_\alpha$ for $\alpha < \lambda$;
(b) $\langle \mathrm{dom}(\phi_\alpha) \rangle_{\alpha<\lambda}$ forms a Δ-system with root r;
(c) $\phi_\alpha \restriction r$ is the same for every $\alpha < \lambda$;
(d) with the notation $A = \bigcup_{\alpha<\lambda} \mathrm{dom}(\phi_\alpha)$, we have $|\lambda \setminus A| = \lambda$.

Then $g = \bigcup_{\alpha<\lambda} \phi_\alpha \in 2^A$ and $T = \{s \in 2^\lambda : s \restriction A = g\}$ is homeomorphic to 2^λ. Finally,

$$f^{\leftarrow}(T) \subset \bigcap_{\alpha<\lambda} f^{\leftarrow}([\phi_\alpha]) \subset \bigcap_{\alpha<\lambda} B_\alpha.$$

□

The next simple example shows that λ in Theorem 1.3 needs to be regular.

Example 1.4. *For every singular cardinality* λ*, there is a compact space* X *of weight* λ *and a base* $\mathcal{B}$ *of* X *such that for every* $\mathcal{C} \in [\mathcal{B}]^\lambda, \cap\mathcal{C} = \emptyset$.

Proof. Indeed, let $\mu = cf(\lambda)$ and let $\langle \lambda_\nu \rangle_{\nu<\mu}$ be an increasing sequence of cardinals cofinal in λ. Then the one-point compactification of the topological sum $\oplus_{\nu<\mu} 2^{\lambda_\nu}$, with its natural base $\mathcal{B}$, is as required. □

2. Base multiplicity in compact spaces

Let $\theta \geq \omega$ be a cardinal. We'll say that a space is $< \theta$*-scattered* if every closed subspace contains a nonempty open set of cardinality $< \theta$.

Theorem 2.1. *Suppose that* $\lambda > \omega$ *is a regular cardinal,* $\omega \leq \theta \leq 2^\lambda$*, and* X *is a compact space with a* θ*-in-*$<\lambda$ T_1*-separating open cover* $\mathcal{B}$*. Then* $X_0 = \{x \in X : |\mathcal{B}_x| < \lambda\}$ *is a compact subspace of weight* $< \lambda$*, and* $X \setminus X_0$ *is* $< \theta$*-scattered and is the union of* $< \lambda$ *compact subspaces.*

Proof. Set $X_0 = \{x \in X : |\mathcal{B}_x| < \lambda\}$. By Lemma 1.2, X is compact and $w(X_0) < \lambda$. Since $\mathcal{B}$ is point-$< \lambda$ on X_0 and $w(X_0) < \lambda$, it follows that only $< \lambda$ many members of $\mathcal{B}$ intersect X_0. Thus X_0 is the intersection of $< \lambda$ many open sets each of which is the union of a finite cover of X_0 by members of $\mathcal{B}$.

To see that $X \setminus X_0$ is $< \theta$-scattered, let Z be a relatively closed subspace of $X \setminus X_0$. Fix any $z \in Z$, and an open subset W of X with $z \in W \subset \overline{W} \subset X \setminus X_0$. We are going to prove that $K = \overline{W \cap Z}$ has a nonempty relatively open subset of size $< \theta$ (and thus, Z has a nonempty open subset of size $< \theta$).

Suppose indirectly that every nonempty relatively open set in K has cardinality $\geq \theta$. We shall distinguish between two cases, and arrive at a contradiction in both.

Case 1. Suppose that $w(K) < \lambda$. Then by $|\mathcal{B}_z| \geq \lambda$, there is a relatively open neighborhood V of Z in K such that $\{B \in \mathcal{B}_z : B \supset V\} \geq \lambda$. Since $|V| \geq \theta$, this contradicts our assumption that $\mathcal{B}$ is a θ-in-$< \lambda$ family.

Case 2. Suppose that $w(K) \geq \lambda$. By applying Theorem 1.3 to K and the trace of $\mathcal{B}$ on K, it follows that there is a $\mathcal{C} \in [\mathcal{B}]^{\lambda}$ such that $\bigcap \mathcal{C}$ contains either a relatively open subset V of K or a perfect preimage P of 2^{λ}. Both V and P have cardinality $\geq \theta$, again in contradiction with our assumption that $\mathcal{B}$ is a θ-in-$< \lambda$ family. $\square$

Corollary 2.2. *Suppose that X is a compact space with a $\mathfrak{c}$-in-countable T_1-separating open cover. Then X is the disjoint union of a separable metrizable subspace X_0 and a scattered, locally compact, σ-compact space $X \setminus X_0$.*

Proof. Apply Theorem 2.1 with $\theta = \mathfrak{c}$ and $\lambda = \omega_1$. $\square$

Remark. $\omega_1 + 1$ with its order topology has a $\mathfrak{c}$-in-countable (even ω_1-in-countable) base but it is not metrizable.

Lemma 2.3. *Suppose that $\kappa \geq \omega$ is a cardinal and $K = Z \cup (K \setminus Z)$ is a space such that $|K| \geq \kappa^+$ and*

(a) *Z is closed subspace of weight $\leq \kappa$;*
(b) *if F is closed in K and $F \subset K \setminus Z$, the $|F| < \kappa$.*

Let $\mathcal{B}$ be a family of sets such that $B \in \mathcal{B}$ implies that $B \cap K$ is open in K and $x \in K$ implies $|\mathcal{B}_x| \geq \kappa^+$. Then there is an $S \in [K]^{\kappa}$ such that $S \subset B$ for at least κ^+ many $B \in \mathcal{B}$.

Proof. By (b) and $w(Z) \leq \kappa$ there is a point $x \in Z$ such that every neighborhood of x in X has cardinality $\geq \kappa^+$. Since $\chi(x, Z) \leq w(Z) \leq \kappa$ and $|\mathcal{B}_x| \geq \kappa^+$, there is an open subset $V \ni x$ in X such that the family $\mathcal{C} = \{B \in \mathcal{B}_x : B \supset \overline{V} \cap Z\}$ has cardinality $\geq \kappa^+$.

Fix a collection $\langle A_\alpha \rangle_{\alpha < \kappa}$ of pairwise disjoint subsets of $\overline{V}$. Note that by (b) we have $|\overline{V} \setminus B| < \kappa$ for every $B \in \mathcal{C}$. Hence every $B \in \mathcal{C}$ contains at least one (in fact, all but $< \kappa$ many) of the sets A_α. By $|\mathcal{C}| \geq \kappa^+$ it follows that there is an A_α which is in $\geq \kappa^+$ many member of $\mathcal{C}$. $\square$

Theorem 2.4. *Suppose $\kappa \geq \omega$ is an infinite cardinal and X is a compact space with a κ-in-$\leq \kappa$ T_1-separating open cover $\mathcal{B}$. Then $w(X) \leq \kappa$.*

Proof. Suppose indirectly that $w(X) \geq \kappa^+$. Apply Theorem 2.1 with $\lambda = \kappa^+$ and $\theta = \kappa$ to obtain a decomposition $X = X_0 \cup (X \setminus X_0)$ such that X_0 is compact of weight $\leq \kappa$ and $X \setminus X_0$ is $< \kappa$-scattered, is the union of $\leq \kappa$ compact subspaces and is such that $x \in X \setminus X_0$ implies $|\mathcal{B}_x| \geq \kappa^+$.

Let $U = \cup\{V \subset X \setminus X_0 : V \text{ is open in } X \text{ and } |V| < \kappa\}$. Since $X \setminus X_0$ has Lindelöf degree $\leq \kappa$ and weight $\geq \kappa^+$, it follows that $U \neq X \setminus X_0$. Since $X \setminus X_0$ is

$< \kappa$-scattered, there is a point $x \in (X \setminus X_0) \setminus U$ and a compact neighborhood K of x in $X \setminus X_0$ such that $|K \setminus U| < \kappa$. Note that by $x \notin U$, every neighborhood of x in K has cardinality $\geq \kappa$. We will consider two cases.

Case 1. Suppose that $|K| = \kappa$. Then $w(K) \leq \kappa$. Since $|\mathcal{B}_x| \geq \kappa^+$, there is a neighborhood of x in K contained in $\geq \kappa^+$ members of $\mathcal{B}$, in contradiction with our assumption that $\mathcal{B}$ is κ in $\leq \kappa$.

Case 2. Suppose $|K| \geq \kappa^+$. Then Lemma 2.3 applied to $Z = K \setminus U$ gives us a contradiction to "$\mathcal{B}$ is κ-in-$\leq \kappa$". □

Corollary 2.5. *A compact space is metrizable if and only if it has an ω-in-countable base.*

Remark. Corollary 2.5 answers the compact part of Question 18 in [AJRS].

3. Metrizability in locally compact spaces and paracompact p-spaces

Theorem 3.1. *If X is a locally compact space with a $\mathfrak{c}$-in-countable base $\mathcal{B}$, then X has a closed metrizable subspace X_1 such that $X \setminus X_1$ is scattered.*

Proof. Let X_1 be the Cantor-Bendixon remainder of X (i.e., X_1 has no isolated points and $X \setminus X_1$ is scattered). By applying Corollary 2.2 to compact meighborhoods in X_1 it follows that X_1 is locally metrizable. Since X_1 is first countable and locally $\geq \mathfrak{c}$, it follows that $\mathcal{B}$ is point-countable on X_1, and hence, X_1 is metrizable (see [G], Theorem 7.2). □

Corollary 3.2. *A locally compact space X with no isolated points is metrizable if and only if it has a $\mathfrak{c}$-in-countable base.*

Theorem 3.3. *If X is a locally compact space with an ω-in-countable base $\mathcal{B}$, then X' is metrizable.*

Proof. By Corollary 2.5, X is locally second countable, and thus, first countable. Thus $\mathcal{B}$ is point-countable on X'. □

Examples.

1. The Prufer manifold [N] is locally compact, has no isolated points, and has a countable T_1-separating open cover. Thus in none of Theorem 3.1, Corollary 3.2 and Theorem 3.3 can we weaken "base" to "T_1-separating open cover".
2. Mrowka's Ψ has an ω-in-countable (even ω-in-≤ 1) base, so in Theorem 3.3, the conclusion cannot be strengthened to "X is metrizable".

Theorem 3.4. *Let $n \geq 1$ be a natural number. A locally compact space is metrizable if and only if it has an n-in-countable base.*

Proof. Only the "if" part needs proof. So let $\mathcal{B}$ be an n-in-countable base for X. By Corollary 2.5, X is locally second countable. By Theorem 3.3, $X' = \oplus_{\alpha<\lambda} C_\alpha$ with each C_α compact second countable. Let $I = X \setminus X'$ and fix, for every $\alpha < \lambda$, an open set $U_\alpha \supset C_\alpha$ such that $\overline{U}_\alpha$ is compact second countable and $\overline{U}_\alpha \cap (\cup_{\beta \neq \alpha} C_\beta) = \emptyset$.

Our proof is by induction on λ. Without loss of generality we can assume $w(X) = \lambda$. If $\lambda = \omega$, we are done. Suppose now that $\lambda > \omega$ and for every $\lambda' < \lambda$ we have already proved our theorem. We are going to distinguish between two cases.

Case 1. Suppose that λ is regular.

Then let

$$S = \{\alpha \in \lambda : \cup_{\beta<\alpha} U_\beta \neq \overline{\cup_{\beta<\alpha} U_\beta}\}.$$

If S is nonstationary in λ, then X is the free sum of open subspaces of weight $< \lambda$. Hence we are done by our inductive hypothesis.

If S is stationary in λ, then for every $\alpha \in S$, we can fix a point $x_\alpha \in \overline{\cup_{\beta<\alpha} U_\beta} \setminus \cup_{\beta<\alpha} U_\beta$. Note that $x_\alpha \in C_{\nu(\alpha)}$ for a unique $\nu(\alpha) \geq \alpha$. Let us fix, for each $\alpha \in S$, a $B_\alpha \in \mathcal{B}$ such that $x \in B_\alpha \subset U_{\nu(\alpha)}$. By the Pressing Down Lemma and by passing to a stationary subset of S, if necessary, we can assume that the $\nu(\alpha)$'s are all distinct and there is a fixed $\gamma < \lambda$ such that

$$|B_\alpha \cap (\cup_{\beta<\gamma} U_\beta) \cap I| \geq n \text{ for every } \alpha \in S.$$

Let $W_\gamma = (\cup_{\beta<\gamma} U_\beta) \cap I$. Since each U_β is second countable, it follows that $|W_\gamma| < \lambda$. Hence there is an $L \in [W_\gamma]^n$ such that $L \subset B_\alpha$ for uncountably many (in fact, $\geq \lambda$ many) $\alpha < \lambda$, in contradiction with our assumption that $\mathcal{B}$ is an n-in-countable base.

Case 2. Suppose that λ is singular.

Then note that by our inductive assumption, every open subspace of X of weight $< \lambda$ is the free sum of second countable subspaces. Also, X can be covered by $\mu = cf(\lambda) < \lambda$ many such subspaces. Since a second countable subspace can intersect only countably many pairwise disjoint open sets, it follows from a standard chaining argument that X is the free sum of subspaces of weight $\leq \mu$. By applying the induction hypothesis again, we conclude that X is metrizable. □

Corollary 3.5 ([P],[BD]). *A locally compact space with a 2-in-finite base is metrizable.*

By a result of Shiraki [Shi] a paracompact p-space (=paracompact M-space) with a point-countable T_1-separating open cover is metrizable. In Theorem 3.8 below we will show that "point-countable" here can be weakened to "ω-in-countable".

We will derive Theorem 3.9 from some stronger results that are interesting in their own right. Let us say that X is weakly Frechet-Urysohn iff every $x \in X'$ is the limit of a convergent sequence from $X \setminus \{x\}$. Chaber [C] proved that a submetacompact β-space with a point-countable T_1-separating open cover has a G_σ-diagonal. In the class of weakly Frechet-Urysohn spaces, his proof extends to ω-in-countable.

Lemma 3.6. *Suppose that X is a weakly Frechet-Urysohn submetacompact β-space with an ω-in-countable T_1-separating open cover $\mathcal{B}$. Then X has a G_δ-diagonal.*

Proof. Since X is weakly Frechet-Urysohn, it follows from "$\mathcal{B}$ is ω-in-countable" that $\mathcal{B}_x$ is countable for every $x \in X'$. From this point on we can follow Chaber's proof (see [G], Theorem 7.9) with the following minor changes:

[(a)] enumerate $\mathcal{B}_x$ as $\langle B_n(x)\rangle_{n\in\omega}$ for $x \in X'$ only;
[(b)] define $G(x) = g(0, x) \cap B_0(x)$, if $x \in X'$, and $G(x) = \{x\}$, if x is isolated in X;
[(c)] at the end of the proof, point out that $|G_0(x)| \geq 2$ implies that $x \in X'$.

□

Theorem 3.7. *A strict p-space X with an ω-in-countable T_1-separating open cover $\mathcal{B}$ is developable.*

Proof. By Corollary 2.5 every compact subspace of X is second countable. It follows that X is first countable. By first countability, and since strict p-spaces are submetacompact [Ji] and β-spaces ([G]), it follows from Lemma 3.6 that X has a G_δ-diagonal. A strict p-space with a G_δ-diagonal is developable ([Ji]). □

Theorem 3.8. *A space X is metrizable if and only if it is a paracompact p-space with an ω-in-countable T_1-separating open cover.*

Proof. Only the "if" part needs proof. Suppose X is a paracompact p-space with an ω-in-countable T_1-separating open cover. By Theorem 3.7, X is a paracompact developable space, and therefore, X is metrizable. □

Problem 3.9. *Suppose that X is a submetacompact β-space with an ω-in-countable base. Is X developable?*

We find this question particularly interesting, because if the answer is "yes", then the proof also gives a proof of "every compact space with an ω-in-countable base is second countable" which is very different from the one we gave in Corollary 2.5 (in that it does not use Shapirovski's Theorem).

4. Countably compact spaces

In this section, we give an example under CH of a countably compact space with an ω-in-countable base which is not metrizable (hence not compact), and show on the other hand that there are no such examples under $MA + \neg CH$ (or more generally, $\mathfrak{p} > \omega_1$). These results answer Problem 2 in the introduction.

Definition 4.1 [HJ]**.** Let $\kappa > \omega$. A subset X of 2^κ is called an HFD (*hereditarily finally dense*) if for each infinite $Y \subset X$, there exists a countable subset Z_Y of κ such that $\{y \restriction (\kappa \setminus Z_Y) : y \in Y\}$ is dense in $2^{\kappa \setminus Z_Y}$.

Theorem 4.2. *If $X \subset 2^\kappa$ is an HFD, then X has an ω-in-$< \kappa$ base.*

Proof. Let $X \subset 2^\kappa$ be an HFD. For each $W \in [\kappa]^\kappa$, the set $\{x \in X : x \restriction W \text{ is constant}\}$ is finite (else the HFD property would be violated).

Now let $\mathcal{B} = \{[\sigma] \cap X : \sigma \in Fn(\kappa, 2)\}$, and let $\mathcal{B}' = \{[\sigma_\alpha] \cap X : \alpha < \kappa\}$ be a κ-sized subcollection of $\mathcal{B}$. We claim that $\bigcap \mathcal{B}'$ is finite. Without loss of generality, we may assume that for some $\rho \in Fn(\kappa, 2)$, $\sigma_\alpha \cap \sigma_\beta = \rho$ whenever $\alpha < \beta < \kappa$. For each α, choose $\gamma_\alpha \in dom(\sigma_\alpha \setminus \rho)$ and suppose $\sigma_\alpha(\gamma_\alpha) = e_\alpha$. There is $W \in [\kappa]^\kappa$ with $e_\alpha = e_\beta$ for all $\alpha, \beta \in W$. Then any member of $\bigcap \mathcal{B}'$ is constant on W, hence $\bigcap \mathcal{B}'$ is finite. □

Remark. Note that, if κ is regular, then it suffices in Theorem 4.2 for X to have the following property: for each infinite $Y \subset X$, there exists $Z_Y \in [\kappa]^{<\kappa}$ such that $\{y \restriction (\kappa \setminus Z_Y) : y \in Y\}$ is dense in $2^{\kappa \setminus Z_Y}$.

Corollary 4.3 (CH). *There is a countably compact non-metrizable space X with an ω-in-countable base.*

Proof. Hajnal and Juhasz have shown that under CH there is a countably compact HFD X in 2^{ω_1} (see [Ju$_2$]). Clearly no HFD is compact, so X is not metrizable. By 4.2, X has an ω-in-countable base. □

The following result shows that some set-theoretic hypothesis is necessary in Corollary 4.3. Recall that $\mathfrak{p}$ is the minimum cardinality of a collection $\mathcal{F}$ of subsets of ω such that $\cap\mathcal{F}'$ is infinite for every finite $\mathcal{F}' \subset \mathcal{F}$, but there is no infinite $A \subset \omega$ with $A \setminus F$ finite for every $F \in \mathcal{F}$.

Theorem 4.4 ($\mathfrak{p} > \omega_1$). *If X is countably compact and has an ω-in-countable T_1-separating open cover $\mathcal{B}$, then X is a compact metrizable space.*

Proof. Let

$$X^* = \{x \in X : \exists D \in [X]^\omega (x \in \overline{D} \setminus D)\}.$$

Note that $cl_X(A) \subset X^*$ for all $A \in [X^*]^\omega$; thus X^* is countably compact.

Claim: $\mathcal{B}$ is point-countable on X^*. To see this, suppose $y \in X^*$ and $|\mathcal{B}_y| > \omega$. Let $D \in [X]^\omega$ with $y \in \overline{D} \setminus D$. Let $\mathcal{B}'_y \in [\mathcal{B}_y]^{\omega_1}$. By $\mathfrak{p} > \omega_1$, there is an infinite $A \subset D$ such that $A \setminus B$ is finite for every $B \in \mathcal{B}'_y$. But then there is an uncountable $\mathcal{B}''_y \subset \mathcal{B}'_y$ with $|\bigcap \mathcal{B}''_y| = \omega$, contradiction.

Since X^* is countably compact and $\mathcal{B}$ is point-countable on X^*, it follows that X^* is compact and metrizable, and

$$|\{B \in \mathcal{B} : B \cap X^* \neq \emptyset\}| \leq \omega.$$

Thus X^* is G_δ in X. Note that each point of $X \setminus X^*$ is isolated: otherwise there would be some point $x \in X^* \setminus X$ which is the limit of some countable set, putting $x \in X^*$. It follows that $X \setminus X^*$ is countable, so $nw(X) \leq \omega$. Thus X is compact and metrizable. □

Problem 4.5. *Is it consistent that if X is a countably compact space with no isolated points that has a $\mathfrak{c}$-in-countable base (or T_1-separating open cover), then X is metrizable?*

5. Initially κ-compact spaces

The natural extension of Section 4 is to consider the following question for every cardinal $\kappa \geq \omega$.

Question. *If X is an initially κ-compact space with a κ-in-$\leq \kappa$ base, then is X compact?*

We have seen in Section 4 that for $\kappa = \omega$ the answer depends on your set theory. In contrast, we will show in this section that the answer is "yes" in ZFC if $\kappa = \mathfrak{c}$ (Theorem 5.3), and it is "yes" for every κ if X is assumed to be scattered (Theorem 5.2). On the other hand, for each fixed κ there is a model of ZFC (with $\mathfrak{c} = \kappa^+$) in which there is a noncompact initially κ-compact space with a κ-in-$\leq \kappa$ base (Corollary 5.5).

Lemma 5.1. *Let κ be an infinite cardinal. Suppose that X is an initially κ-compact and locally $< \kappa$ space. Then X is a compact space of cardinality $< \kappa$.*

Proof. We are going to prove Lemma 5.1 by induction of the scattered height (or Cantor-Bendixon length) of X that we are going to denote by $sch(X)$.

If $sch(X) = 1$, then X is finite. Suppose now that $sch(X) = \lambda > 1$ and for spaces with scattered height $< \lambda$ we are already done. We will consider two cases.

Case 1. If λ is limit ordinal, then, since X is locally $< \kappa$, it follows that $\mu = cf(\lambda) \leq \kappa$. Thus there is a strictly increasing open cover of X of length $\mu \leq \kappa$, contradicting our assumption that X is initially κ-compact.

Case 2. Suppose now that $\lambda = \mu + 1$ is a successor ordinal, i.e. there is a closed discrete subspace D of X such that $sch(X \setminus D) = \mu$. Since X is countably compact, D is finite. Since X is locally $< \kappa$, there is an open subset $U \supset D$ with $|U| < \kappa$. Since $X \setminus U$ is initially κ-compact and locally $< \kappa$ of scattered height $\leq \mu < \lambda$, it follows from our inductive hypothesis that $|X \setminus U| < \kappa$. Hence $|X| = |U| + |X \setminus U| < \kappa$. By initially κ-compactness, X is also compact. □

Theorem 5.2. *Let κ be an infinite cardinal. If X is a scattered initially κ-compact space with a κ-in-$\leq \kappa$ T_1-separating open cover $\mathcal{B}$, then X is a compact space of weight $\leq \kappa$.*

Remark. Note that a compact scattered space of weight $\leq \kappa$ also has cardinality $\leq \kappa$.

Proof. Let us set $X^* = \{x \in X : |\mathcal{B}_x| \leq \kappa\}$. Since $A \in [X^*]^{\leq \kappa}$ implies $\overline{A} \subset X^*$, it follows that X^* is initially κ-compact. By applying Lemma 1.1 to $\lambda = \kappa^+$ we conclude that X^* is a compact subspace of weight $\leq \kappa$. Since $w(X^*) \leq \kappa$, and $|\mathcal{B}_x| \leq \kappa$ for every $x \in X$, it follows that only $\leq \kappa$ many members of $\mathcal{B}$ intersect X^*. Since $\mathcal{B}$ is T_1-separating, it follows that X^* is the intersection of $\leq \kappa$ many open sets of X (each of which is the union of a finite open cover of X^* by members of $\mathcal{B}$). Hence it remains to prove the following claim to show $nw(X) \leq \kappa$, and thus, $w(X) \leq \kappa$.

Claim. If F is a closed subset of X such that $F \subset X \setminus X^*$, then $|F| \leq \kappa$.

To prove this Claim, let us set

$$W = \bigcup \{V : V \text{ is relatively open in } F \text{ and } |V| < \kappa\}.$$

If $F = W$, then we are done by Lemma 2.1. If $F \neq W$, then fix an isolated point x of $F \setminus W$, and a closed neighborhood K of x in F such that $K \subset \{x\} \cup W$. Note that $|K| \geq \kappa$ by $x \notin W$ and that $|\mathcal{B}_x| \geq \kappa$ by $x \notin X^*$.

Suppose that $|K| = \kappa$. Since K is initially κ-compact, it follows that K is a compact space of weight $\leq \kappa$. Since $|\mathcal{B}_x| \geq \kappa^+$, there is a neighborhood of U of x in F such $x \in U \subset K$ and $U \subset B$ holds for $\geq \kappa^+$ many B in $\mathcal{B}$. Since $\mathcal{B}$ is κ-in-$\leq \kappa$, it follows that $|U| < \kappa$ in contradiction with $x \ni W$.

Finally, suppose that $|K| > \kappa$. Note that for each neighborhood U of x, $|K \setminus U| < \kappa$ by Lemma 5.1. Thus by Lemma 2.3, K does not have a κ-in-$\leq \kappa$ base, contradiction. □

Theorem 5.3. *Suppose that X is an initially $\mathfrak{c}$-compact space with a $\mathfrak{c}$-in-$\leq \mathfrak{c}$ T_1-separating open cover. Then X is a compact space of weight $\leq c$.*

Proof. Note first that every separable closed subspace of X is of weight $\leq \mathfrak{c}$ and is initially $\mathfrak{c}$-compact. Thus every separable subspace of X is compact.

Let

$$X^* = \{x \in X : |\mathcal{B}_x| \leq \mathfrak{c}\}.$$

As at the beginning of the proof of Theorem 5.2, it follows that X^* is a compact subspace of weight $\leq \mathfrak{c}$ and X^* is the intersection of $\leq \mathfrak{c}$ open subsets of X.

Claim 1. If $S \in [X \setminus X^*]^\omega$ and $\overline{S} \subset X \setminus X^*$, then $|\overline{S}| < \mathfrak{c}$.

Indeed, since $\overline{S}$ is compact it is enough to show that $\overline{S}$ is locally countable. Pick $x \in \overline{S}$. Since $x \notin X^*$, it follows that $|\mathcal{B}_x| \geq c^+$. Since $w(\overline{S}) \leq \mathfrak{c}$, x has a neighborhood V in $\overline{S}$ such that $V \subset B$ holds for $\geq c^+$ many $B \in \mathcal{B}_x$. Since $\mathcal{B}$ is $\mathfrak{c}$-in-$\leq \mathfrak{c}$, it follows that $|V| < \mathfrak{c}$.

Claim 2. $X \setminus X^*$ is scattered.

Suppose indirectly that Claim 2 is false, i.e. there is a relatively closed dense-in-itself subset F of $X \setminus X^*$. Fix an $x \in F$, and let V be an open set in X such that $x \in V \subset \overline{V} \subset X \setminus X^*$. Then $K = \overline{F \cap V}$ is a dense-in-itself initially $\mathfrak{c}$-compact subspace of X, so there is a continuous onto map $f : K \to 2^\omega$. Pick $S \in [K]^\omega$ in such a way that $\overline{f''S} = 2^\omega$. Then $f''\overline{S} = 2^\omega$, so $|\overline{S}| \geq \mathfrak{c}$. On the other hand $\overline{S} \subset K \subset X \setminus X^*$. By Claim 1, it follows that $|\overline{S}| < \mathfrak{c}$, contradiction.

Now, to finish the proof of Theorem 5.3, note that since X^* is the intersection of $\leq \mathfrak{c}$ open subsets of X, we conclude that $X \setminus X^*$ is the union of $\leq c$ initially $\mathfrak{c}$-compact subspaces. By Theorem 2.2, each of those initially $\mathfrak{c}$-compact subspaces has weight $\leq \mathfrak{c}$. Since $\omega(X^*) \leq \mathfrak{c}$ also holds, it follows that $nw(X) \leq \mathfrak{c}$. Since X is initially $\mathfrak{c}$-compact, we conclude that X is compact and thus $w(X) = nw(X) \leq \mathfrak{c}$. □

Theorem 5.4. *Suppose λ is an uncountable regular cardinal, and suppose $X \subset 2^\lambda$ has the following properties:*

(i) *For each infinite $Y \subset X$, there exists $Z_Y \in [\lambda]^{<\lambda}$ such that $\{y \restriction (\lambda \setminus Z_Y) : y \in Y\}$ is dense in $2^{\lambda \setminus Z_Y}$.*
(ii) *$X \cap [f] \neq \emptyset$ for every $f \in {}^{<\lambda}2$.*

Then X is a non-compact initially $< \lambda$-compact space with an ω-in-$< \lambda$ base.

Proof. Let X satisfy the hypotheses. By the remark following the proof of Theorem 4.2, X has an ω-in-$< \lambda$ base. Clearly X cannot be compact.

So it remains to show that X is initially $< \lambda$-compact. The argument is a mild generalization of the argument in case $\lambda = \omega_1$ (see [Ju$_2$]). We need to show that every infinite subset of X of regular cardinality $< \lambda$ has a complete accumulation point. To this end, fix a regular cardinal $\kappa < \lambda$, and $Y \in [X]^\kappa$. Using property (i) and regularity of λ, it is easy to see that for each $Z \in [X]^\kappa$ there is $\delta_Z < \lambda$ such that $\{z \restriction (\lambda \setminus \delta_Z) : z \in Z\}$ is κ-dense in $2^{\lambda \setminus \delta_Z}$ (consider splitting Z into κ-many infinite disjoint pieces).

Using again the regularity of λ, we can find $\gamma < \lambda$ satisfying:

(i) The projection $\pi_\gamma : Y \to 2^\gamma$ is one-to-one;
(ii) For each $\sigma \in Fn(\gamma, 2)$, if $|Y \cap [\sigma]| = \kappa$ then $\delta_{Y \cap [\sigma]} < \gamma$.

Now choose a complete accumulation point $f \in 2^\gamma$ of $\pi_\gamma(Y)$, and let $x \in X$ be an extension of f. We claim that x is a complete accumulation point of Y. Suppose $\sigma \in Fn(\lambda, 2)$ with $x \in [\sigma]$. Let $Z = Y \cap [\sigma \restriction \gamma]$; then $|Z| = \kappa$. By (ii), $\delta_Z < \gamma$, so $|Z \cap [\sigma \restriction (\lambda \setminus \gamma)]| = \kappa$. Thus $|Y \cap [\sigma]| = \kappa$, and the proof is complete. □

Remark. Note that the conditions on X in Theorem 5.4 imply that $2^{<\lambda} = \lambda = 2^\omega$.

Corollary 5.5. *For each infinite cardinal κ, it is consistent with ZFC that ($\mathfrak{c} = \kappa^+$ and) there is a non-compact initially κ-compact space X with an ω-in-$\leq \kappa$ (and* *hence κ-in-$\leq \kappa$) base.*

Proof. We will show that for any uncountable regular cardinal λ, it is consistent for there to be an X satisfying the conditions of Theorem 5.4. Corollary 5.5 then follows by taking $\lambda = \kappa^+$.

So, let λ be regular and uncountable, and let M be any model of set theory obtained by adding λ-many Cohen reals to a model of $2^{<\lambda} = \lambda$. Hajnal and Juhasz (see [Ju$_2$]) show that in M there is an HFD $Y = \{y_\alpha : \alpha < \lambda\} \subset 2^\lambda$ (the assumption that the ground model satifies $2^{<\lambda} = \lambda$ is irrelevant here). Note that M also satisfies $2^{<\lambda} = \lambda$ (see [K], Chapter VII, Lemma 5.13). Let ${}^{<\lambda}2 = \{f_\alpha : \alpha < \lambda\}$, and define $x_\alpha \in 2^\lambda$ by $x_\alpha(\beta) = f_\alpha(\beta)$ if $\beta \in dom(f_\alpha)$ and $x_\alpha(\beta) = y_\alpha(\beta)$ otherwise. Then X satisfies the conditions of Theorem 5.4. □

References

[AJRS] A. V. Arhangel'skii, W. Just, Reznichenko and P.J. Szeptyczki, *Sharp bases and weakly uniform bases versus point countable bases*, to appear.

[BDJSSz] Z. Balogh, S.W. Davis, W. Just, S.Shelah and P.J. Szeptyczki, *Strongly almost disjoint sets and weakly uniform bases*, preprint.

[BD] D. K. Burke and S. W. Davis, *On a Theorem of Peregudov*, Topology Proceedings **8** (1983), 237-240..

[C] J. Chaber, *On point-countable collections and monotonic properties*, Fund. Math. **94** (1977), 209-219.

[G] G. Gruenhage, *Generalized Metric Spaces*, Handbook of Set-theoretic Topology (K. Kunen and J.E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 423-501.

[HJ] A. Hajnal and I. Juhasz, *On hereditarily α-Lindelöf and α-separable spaces, II*, Fund. Math. **81** (1974), 148-158.

[K] K. Kunen, *Set Theory*, North-Holland, Amsterdam, 1980.

[Ji] S. Jiang, *Every strict p-space is θ-refinable*, Topology Proceedings **11** (1986), 309-316.

[Ju$_1$] I. Juhasz, *Cardinal Functions in Topology-Ten Years Later*, Mathematical Centre Tracts 123, Mathematisch Centrum, Amsterdam, 1983.

[Ju$_2$] I. Juhasz, *A survey of S and L spaces*, Colloq. Math. Soc. Janos Bolyai **23** (1978), 675-688.

[N] P.J. Nyikos, *The theory of non-metrizable manifolds*, Handbook of Set-theoretic Topology, North-Holland, Amsterdam, 1980, pp. 633-684.

[P] S. A. Peregudov, *On Metrizability in a class of topological spaces with a weakly uniform base*, Bull. Pol. Acad. Nauk **28** (1980), 609-612.

[Sha] B. Shapirovskii, *On the tightness, π-weight, and related notions*, Uc. Zap. Latv. Univ. **257** (1976), 88-89.

[Shi] T. Shiraki, *M-spaces, their generalization, and metrization theorems*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **11** (1971), 57-67.