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Topological predomains and qcb spaces are not closed under sobrification

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In this note we show that *topological predomains* and *qcb spaces* as introduced in (Simpson 2003) are not closed under sobrification. As a consequence Σ-replete topological predomains need not be sober, i.e. in general Σ-repletion is not given by sobrification.

1. Background

In (Simpson 2003) A. Simpson introduced the category **PreDom** of *topological predomains* as a framework for denotational semantics containing also most classical spaces, namely all countably based T_0 spaces.

Countably based T_0 spaces are isomorphic to subspaces of $\mathcal{P}\omega$ where the latter is endowed with the Scott topology. A *qcb space* (as introduced in (Menni and Simpson 2002)) is a T_0 space which appears as quotient of a countably based T_0 space. In (Schröder 2003) qcb spaces have been characterized as those sequential spaces X for which there exists a *countable pseudobase*, i.e. a countable subset $\mathcal{B}$ of $\mathcal{P}(X)$ such that for every converging sequence $(x_n) \to x$ and open neighbourhood U of x there exists a $B \in \mathcal{B}$ with $B \subseteq U$ and (x_n) eventually in B. We write **QCB** for the category of qcb spaces and continuous maps. It has been stated in (Simpson 2003) and proved in (Battenfeld 2004) that **QCB** is equivalent to $\mathbf{ExPer}_\Sigma(\mathcal{P}\omega)$, the category of Σ-extensional per's over Scott's $\mathcal{P}\omega$ (where Σ is the Sierpinski space). A *topological predomain* is a qcb space X which is also a *monotone convergence space*, i.e. X is a directed complete partial order (dcpo) w.r.t. the specialization order $\sqsubseteq_X$ and every $U \in \mathcal{O}(X)$ is Scott open (w.r.t. $\sqsubseteq_X$).

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Then, obviously, the category **PreDom** of topological predomains and continuous maps is equivalent to $\mathbf{CE}_\Sigma(\mathcal{P}\omega)$, the category of *complete* Σ-*extensional pers* over $\mathcal{P}\omega$.[†]

In (Hyland 1990; Taylor 1991) M. Hyland and P. Taylor independently introduced the notion of Σ-repleteness which, intuitively, means "as complete as possible" and entails ω-completeness as discussed above. In realizability models Σ-replete objects can be characterized as the least small internally complete full subcategory (of the ambient universe) containing Σ.

In this note we consider Σ-replete objects in the realizability model over $\mathcal{P}\omega$, i.e. within **PreDom** since Σ-replete objects are contained within $\mathbf{CE}_\Sigma(\mathcal{P}\omega)$ (see (Phoa 1990)). Formally, the notion of Σ-repleteness looks very similar to the notion of sobriety. Recall that a space X is sober iff every continuous map $e : X \to Y$ is already a homeomorphism whenever $e^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a bijection. Analogously, a qcb space X is Σ-replete iff every map $e : X \to Y$ in **QCB** is a homeomorphism whenever $\Sigma^e : \Sigma^Y \to \Sigma^X$ is bijective.[‡] Thus a sober qcb space is also Σ-replete. An example of a topological predomain which is not Σ-replete is the non-sober dcpo (with its Scott topology) discussed in (Johnstone 1981) whose sobrification coincides with its Σ-repletion since sobrification adds a single new point which can be obtained as limit of a sequence of point filters. Motivated by these observations in (Simpson 2003) A. Simpson raised the question whether Σ-replete and sober coincide for qcb spaces, i.e. whether in **QCB** repletion amounts to sobrification. Since 2003 I. Battenfeld, M. Schröder, A. Simpson and the second named author tried to prove that the answer to this question is positive. The reason for this hope was a(n unpublished) result by M. Schröder saying that sobrifications of qcb spaces admit countable pseudobases. Thus, one "simply" had to show that sobrification of qcb spaces preserves the property of sequentiality. However, all attempts in this vein appeared as erronous. Finally, in summer 2005 the second named author—following a suggestion of K. Keimel—contacted the first named author who fairly quickly came up with a counterexample providing a negative answer to the question above. This counterexample is described in the next section.

2. A Σ-replete qcb space which is not sober

We construct a comparatively simple topological predomain X whose sobrificaton $\mathsf{Sob}(X)$ is not sequential and, accordingly, not even in **QCB**.

The underlying set of X is $\mathbb{N} \times \mathbb{N}$. For $p = (n, m) \in X$ and $f : \{i \in \mathbb{N} \mid i > n\} \to \mathbb{N}$ let $U(p, f) = \{p\} \cup \{(i, j) \in \mathbb{N}^2 \mid i > n \text{ and } j \geq f(i)\}$. Obviously $p \in U(p, f)$. A subset U of X is called **open** iff for every $p \in U$ there is an f with $U(p, f) \subseteq U$. Obviously, we have $U(p, \max(f, g)) \subseteq U(p, f) \cap U(p, g)$. Moreover, if $q \in U(p, f)$ then $U(q, g) \subseteq U(p, f)$ for some g. Thus, sets of the form $U(p, f)$ are open themselves and for

[†] It is well-known (Phoa 1990) that $\mathbf{ExPer}_\Sigma(\mathcal{P}\omega)$ and $\mathbf{CE}_\Sigma(\mathcal{P}\omega)$ are both small internally complete subcategories of $\mathsf{RT}(\mathcal{P}\omega)$, the realizability topos over $\mathcal{P}\omega$, and thus give both rise to models of polymorphic λ-calculus.

[‡] For qcb spaces Z the exponential Σ^Z is $\mathcal{O}(Z)$ endowed with the Scott topology (see (Simpson 2003; Battenfeld 2004).

every $q \in U(p_1, f_1) \cap U(p_2, f_2)$ there is a g with $U(q, g) \subseteq U(p_1, f_1) \cap U(p_2, f_2)$. Thus, open sets are closed under finite intersections. As open sets are closed under arbitrary unions the open subsets of X form a topology on X. The set of the form $U(p, f)$ a basis for this topology.

It is easy to see that X is a T_1 space. Thus, the specialization order on X is discrete. Accordingly, X is a topological predomain provided it is sequential and has a countable pseudobase.

Lemma 2.1. For every $A \subseteq X$ and $p \in X \setminus A$ we have $p \in \overline{A}$ iff $A \cap (\{i\} \times \mathbb{N})$ is infinite for some $i > \pi_0(p)$.

Proof. Let $A \subseteq X$ and $p \in X \setminus A$.

For the forward direction suppose that $p \in \overline{A}$ and $A \cap (\{i\} \cap \mathbb{N})$ is finite for all $i > \pi_0(p)$. Then there exists an f with $A \cap U(p, f) = \emptyset$ contradicting $p \in \overline{A}$ since $U(p, f)$ is an open neighbourhood of p.

For the reverse direction suppose $A \cap (\{i\} \times \mathbb{N})$ is infinite for some $i > \pi_0(p)$. For showing that $p \in \overline{A}$ suppose U is an open neighbourhood of p. Then there exists f with $U(p, f) \subseteq U$. Then, since $A \cap (\{i\} \times \mathbb{N})$ is infinite, there exists a $j \geq f(i)$ with $(i, j) \in A$. Thus $(i, j) \in A \cap U(p, f) \subseteq A \cap U$ as desired. □

Lemma 2.2. X is a Fréchet space, i.e. for every $p \in \overline{A}$ there is a sequence (a_n) in A converging to p.

Proof. Suppose $p \in \overline{A}$. W.l.o.g. assume that $p \notin A$. Then by Lemma 2.1 there exists an $i > \pi_0(p)$ such that $A \cap (\{i\} \times \mathbb{N})$ is infinite. Let $\phi : \mathbb{N} \to \mathbb{N}$ be strictly increasing with $\{(i, \phi(n)) \mid n \in \mathbb{N}\} = A \cap (\{i\} \times \mathbb{N})$. Let $a_n := (i, \phi(n)) \in A$. We show that $(a_n) \to p$.

Suppose U is an open neighbourhood of p. Then $U(p, f) \subseteq U$ for some f. Let $n_0 \in \mathbb{N}$ with $\phi(n_0) \geq f(i)$. Then for all $n \geq n_0$ we have $\phi(n) \geq f(i)$ and thus $a_n = (i, \phi(n)) \in U(p, f) \subseteq U$. □

Let $\mathcal{B}_0$ be the collection of all $V_{i,n} := \{(i, j) \mid j \geq n\}$ with $i, n \in \mathbb{N}$. Let $\mathcal{B}$ be the set of all finite nonempty unions of elements of $\mathcal{B}_0 \cup \{\{x\} \mid x \in X\}$.

Lemma 2.3. $\mathcal{B}$ is a countable pseudobase for X.

Proof. Obviously $\mathcal{B}$ is countable since $\mathcal{B}_0$ and X are both countable.

For showing that $\mathcal{B}$ is a pseudobase for X suppose that $(p_n) \to p$ and U an open neighbourhood of p. Then $U(p, f) \subseteq U$ for some f. For $i > \pi_0(p)$ let $I_i = \{n \in \mathbb{N} \mid p_n \in V_{i,f(i)}\}$.

Next we show that almost all I_i are empty. For sake of contradiction suppose this were not the case. Then there exists a subsequence $q_n = p_{\phi(n)}$ of (p_n) with $\pi_0(q_n) > \pi_0(p) + 1$ for all $n \in \mathbb{N}$. Then $p \notin S := \{q_n \mid n \in \mathbb{N}\}$ but $p \in \overline{S}$ since $(q_n) \to p$. Then by Lemma 2.1 there is an $i > \pi_0(p)$ with $S \cap (\{i\} \times \mathbb{N})$ infinite. But, actually, by assumption on (q_n) we have $i > \pi_0(p) + 1$. Thus, again by Lemma 2.1 we have $(\pi_0(p) + 1, 0) \in \overline{S}$ since $(\pi_0(p) + 1, 0) \notin S$. Thus, there exists a subsequence $r_n = q_{\psi(n)}$ of (q_n) converging to $(\pi_0(p)+1, 0)$. But since $(r_n) \to p$ and limits of sequences are unique in T_1 spaces we have $p = (\pi_0(p) + 1, 0)$ which clearly is impossible.

Let $i_0 \in \mathbb{N}$ with $I_i = \emptyset$ for $i \geq i_0$. Then $B = \{p\} \cup \bigcup_{\pi_0(p)<j<i_0} V_{j,f(j)} \in \mathcal{B}$ and it holds that $B \subseteq U(p,f) \subseteq U$ and (a_n) is eventually in B as desired. □

Thus, since X is sequential and has a countable pseudobase the space X is in **QCB**.

Lemma 2.4. All irreducible closed subsets of X are either singletons or X.

Proof. Suppose C is an irreducible closed subset of X different from X.

By Lemma 2.1 if $C \cap (\{i\} \times \mathbb{N})$ is infinite then for all $j < i$ and $n \in \mathbb{N}$ we have $(j,n) \in \overline{C \cap (\{i\} \times \mathbb{N})} \subseteq \overline{C} = C$. Thus $C \cap (\{i\} \times \mathbb{N})$ is infinite for only finitely many i since otherwise $X = C$.

Thus precisely one of the following two conditions holds

(1) $C \cap (\{i\} \times \mathbb{N})$ is finite for all $i \in \mathbb{N}$
(2) there is a greatest $i \in \mathbb{N}$ with $C \cap (\{i\} \times \mathbb{N})$ infinite.

In case (1) every point of C is isolated in the subspace C. Thus C cannot be irreducible closed unless C is a singleton.
In case (2) every point of the infinite set $C \cap (\{i\} \times \mathbb{N})$ is isolated in the subspace C. But as irreducible closed sets contain at most one isolated point this is impossible. □

Thus $\mathsf{Sob}(X) = X \dot\cup \{\infty\}$ (where ∞ stands for the irreducible closed set X). The nonempty open sets of $\mathsf{Sob}(X)$ are those of the form $U \dot\cup \{\infty\}$ where $U \in \mathcal{O}(X) \setminus \{\emptyset\}$.

Lemma 2.5. As a subset of $\mathsf{Sob}(X)$ the set X is sequentially closed but not closed w.r.t. the sober topology. Thus the space $\mathsf{Sob}(X)$ is not sequential.

Proof. Obviously X is not closed in $\mathsf{Sob}(X)$ since $\infty \in \overline{X} \setminus X$. Nevertheless X is a sequentially closed subset of $\mathsf{Sob}(X)$ which can be seen as follows.

For the sake of contradiction suppose (x_n) is a sequence in X converging to ∞ in $\mathsf{Sob}(X)$. As ∞ is in the closure of $S = \{x_n \mid n \in \mathbb{N}\}$ it is impossible that $S \cap (\{i\} \times \mathbb{N})$ is finite for all $i \in \mathbb{N}$. Thus, there exists an $i \in \mathbb{N}$ with $S \cap (\{i\} \times \mathbb{N})$ infinite. But then $U = \{(i,0)\} \cup \{(j,k) \in \mathbb{N}^2 \mid i < j\} \cup \{\infty\}$ is an open neighbourhood of ∞ in $\mathsf{Sob}(X)$ such that infinitely many elements of S (namely those of $S \cap (\{i\} \times \mathbb{N})$) are not in U. Thus (x_n) does not converge to ∞ in contradiction to our assumption containing all point filters. □

Theorem 2.6. The space X is a Σ-replete qcb space but not sober.

Proof. ¿From the above lemmas it is immediate that X is a topological predomain.

By an equalizer construction one can carve out from Σ^{Σ^X} the subobject $H(\Sigma^X, \Sigma)$ of all continuous $h : \Sigma^X \to \Sigma$ preserving finite meets and joins (w.r.t. the specialization order. Due to the closure properties of Σ-replete objects (namely closure under equalizers, see (Hyland 1990)) the object $H(\Sigma^X, \Sigma)$ is also Σ-replete. Obviously, the underlying set of $H(\Sigma^X, \Sigma)$ consists of the complete prime filters in $\mathcal{O}(X) = \Sigma^X$ and thus contains all point filters. From the characterization of regular monos (as given in (Simpson 2003; Battenfeld 2004) it follows that $H(\Sigma^X, \Sigma)$ is the sequentialization of $\mathsf{Sob}(X)$. As the greatest element of $H(\Sigma^X, \Sigma)$ is isolated it follows that $R(X) \cong X$. Thus, as claimed X is a Σ-replete object in **PreDom** which is not sober. □

A. Simpson has pointed out to us that our counterexample shows also that the category **CCG** of *core compactly generated spaces* is not closed under sobrifiation. The category **CCG** as introduced in (Escardó et.al. 2004) contains **QCB** as a full subcategory, namely as those core compactly generated spaces which admit a countable pseudobase. Obviously X is in **CCG**. If $\mathsf{Sob}(X)$ were in **CCG** as well then $\mathsf{Sob}(X)$ would be Σ-replete and thus a subspace of $\Sigma^{\Sigma^{\mathsf{Sob}(X)}} \cong \Sigma^{\Sigma^X}$ from which it follows that $\mathsf{Sob}(X)$ would have a countable pseudobase and thus would be in **QCB** contradicting Theorem 2.6.

3. Failure of P. Taylor's "Tentative Repletion Construction" for QCB

In the first half of the 1990's P. Taylor suggested that Σ-repletion of X can be obtained as the equalizer $E(X)$ of the maps $\eta_{\Sigma^2(X)}, \Sigma^2(\eta_X) : \Sigma^2(X) \to \Sigma^4(X)$.[§] It can be shown that $E(X)$ consists of all $F \in \Sigma^2(X)$ that commute with all operations $f : \Sigma^Y \to \Sigma$ from which it follows that $|E(X)|$ is a subset of $|\mathsf{Sob}(X)|$. Alternatively, one may characterize $E(X)$ as the regular subobject of $\Sigma^2(X)$ consisting of all $F \in \Sigma^2(X)$ with $\Phi(F) = F(\Phi \circ \eta_X)$ for all $\Phi \in \Sigma^3(X)$.

We will show that for the qcb space X introduced in section 2 the regular subobject $E(X)$ of $\Sigma^2(X)$ contains $\exists : \Sigma^X \to \Sigma : U \mapsto \bigsqcup\{\top \mid U \neq \emptyset\}$ from which it follows that $E(X)$ is not isomorphic to X. For this purpose we need the following lemma.

Lemma 3.1. The closure of $\{\eta_X(x) \mid x \in X\}$ in $\Sigma^2(X)$ contains $\exists$.

Proof. ¿From M. Schröder's work in (Schröder 2003) it follows that if $\mathcal{B}$ is a countable pseudobase for X then a countable pseudobase for Σ^X is given by the set of all finite intersections of sets of the form $[B] = \{U \in \Sigma^X \mid B \subseteq U\}$ with $B \in \mathcal{B}$.

Let $\mathcal{B}' = \mathcal{B} \cup \{\emptyset\}$ where $\mathcal{B}$ is defined as in section 2. Obviously, besides being a countable pseudobase for X the set $\mathcal{B}'$ is closed under finite unions and intersections. As $[B_1] \cap \cdots \cap [B_n] = [B_1 \cup \cdots \cup B_n]$ and $\mathcal{B}'$ is closed under finite unions the set $\mathcal{B}_1 = \{[B] \mid B \in \mathcal{B}'\}$ is a countable pseudobase for Σ^X and is closed under finite intersections. Thus, a countable pseudobase for $\Sigma^2(X)$ is given by $\mathcal{B}_2 = \{[[B]] \mid B \in \mathcal{B}'\}$ where $F \in [[B]]$ iff $F(U) = \top$ for all $U \in \Sigma(X)$ with $B \subseteq U$.

To show that $\exists$ is in the closure of $\{\eta_X(x) \mid x \in X\}$ suppose $\mathcal{U}$ is an open neighbourhood of $\exists$. Then since $\mathcal{B}_2$ is a pseudobase for $\Sigma^2(X)$ there exists a $B \in \mathcal{B}'$ with $\exists \in [[B]] \subseteq \mathcal{U}$. As $\exists(\emptyset) = \bot$ and $\exists \in [[B]]$ the set B must be nonempty. Let $x \in B$. Then $\eta_X(x) \in [[B]]$ because for $U \in \Sigma(X)$ from $B \subseteq U$ it follows that $x \in U$ and thus $\eta_X(x)(U) = \top$. Thus, since $[[B]] \subseteq \mathcal{U}$ we have $\eta(x) \in \mathcal{U}$. □

Theorem 3.2. The set $E(X) = \{F \in \Sigma^2(X) \mid \forall \Phi \in \Sigma^3(X).\ \Phi(F) = F(F \circ \eta_X)\}$ contains $\exists$. Thus X is not isomorphic to $E(X)$, i.e. P. Taylor's tentative repletion fails for X.

Proof. For showing that $\exists \in E(X)$ we have to show that $\Phi(\exists) = \exists(\Phi \circ \eta_X)$ for all $\Phi \in \Sigma^3(X)$. Let $\Phi \in \Sigma^3(X)$. We write $\mathcal{U}$ for the open set $\Phi^{-1}(\top)$. Suppose $\exists(\Phi \circ \eta_X) = \top$.

§ We write $\Sigma^n(X)$ for $\Sigma^{(-)}$ applied n-times to X.

Then $\Phi(\eta_X(x)) = \top$ for some $x \in X$. Thus $\Phi(\exists) = \top$ since $\eta_X(x) \sqsubseteq \exists$ and Φ preserves the specialization order. For the reverse direction suppose $\Phi(\exists) = \top$, i.e. $\exists \in \mathcal{U}$. Thus, by Lemma 3.1 we have $\eta_X(x) \in \mathcal{U}$ for some $x \in X$. Thus $\Phi(\eta_X(x)) = \top$ from which it follows that $\exists(\Phi \circ \eta_X) = \top$ as desired. □

Thus none of the simplifications suggested for Σ-repletion works for **QCB** and **PreDom**. We leave it for future investigations whether our counterexamples for realizability over $\mathcal{P}\omega$ can be extended to number realizability. They certainly work for function realizability since $\mathbf{ExPer}_\Sigma(\mathbb{N}^\mathbb{N}) \cong \mathbf{ExPer}_\Sigma(\mathcal{P}\omega)$ as follows from (Schröder 2003).

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