MONOTONE COVERING PROPERTIES AND PROPERTIES THEY IMPLY

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Dedicated to Alan Dow, a fantastic mathematician who has made the second author smile uncountably many times.

Abstract. We study properties of spaces that were proven in an earlier paper ["Monotonically metacompact compact Hausdorff spaces are metrizable", Topology and its Applications **160** (2013), no. 1, 45 – 49] to follow from monotonic metacompactness. We show that all of the results of that earlier paper that follow from the monotonic covering property follow just from these weaker properties. The results we obtain are either strengthenings of earlier results or are new even for the monotonic covering property. In particular, some corollaries are that monotonically countably metacompact spaces are hereditarily metacompact, and separable monotonically countably metacompact spaces are metrizable. It follows that the well-known examples of stratifiable spaces given by McAuley and Ceder are not monotonically countably metacompact; we show that they are not monotonically meta-Lindelöf either. Finally, we answer a question of Gartside and Moody by exhibiting a stratifiable space which is monotonically paracompact in the locally finite sense, but not monotonically paracompact in the sense of Gartside and Moody.

1. Introduction

A space X is *monotonically (countably) metacompact [meta-Lindelöf]* if there is a function r that assigns to each (countable) open cover $\mathcal{U}$ of X a point-finite [point-countable] open refinement $r(\mathcal{U})$ covering X such that if $\mathcal{V}$ is an open cover of X and $\mathcal{V}$ refines $\mathcal{U}$, then $r(\mathcal{V})$ refines $r(\mathcal{U})$. The function r is called a *monotone (countable) metacompactness [meta-Lindelöfness] operator.*[1]

In [6] we introduced a couple of neighborhood assignment properties possessed by monotonically (countably) metacompact spaces. For a space X let $T(X)$ be the collection of all triples $p = (x^p, U_0^p, U_1^p)$ where U_0^p, U_1^p are open in X, and $x^p \in U_0^p \subset \overline{U_0^p} \subset U_1^p$.

Lemma 1.1. [6] *Let X be monotonically (countably) metacompact. Then to each $p \in T(X)$ one can assign an open V^p satisfying:*

i $x^p \in V^p \subset U_1^p$;

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[1]Of course, every space is trivially "monotonically countably meta-Lindelöf" with $r(\mathcal{U}) = \mathcal{U}$.

ii *Whenever $\mathcal{Q} \subset T(X)$, ($\mathcal{Q}$ countable), then either $\bigcap_{q \in \mathcal{Q}} V^q = \emptyset$, or there exists a $\mathcal{Q}' \subset \mathcal{Q}$, with $\mathcal{Q}'$ finite, such that for any $q \in \mathcal{Q}$ there exists $q' \in \mathcal{Q}'$ such that either $V^q \subset U_1^{q'}$ or $V^q \cap U_0^{q'} = \emptyset$.*

Lemma 1.2. [6] *Let X be a monotonically (countably) metacompact T_3-space, and $Y \subset X$. For each $y \in Y$, if U_y is some open neighborhood containing y, then there exists an open neighborhood V_y of y with $V_y \subset U_y$ such that if $Y' \subset Y$, (Y' countable), and $\bigcap_{y \in Y'} V_y \neq \emptyset$, then there is a finite $Y'' \subset Y'$ such that $Y' \subset \bigcup_{y \in Y''} U_y$.*

These two lemmas were the main tools in proving the results of [6], and a similar lemma was the main tool of [9]. Thus it is natural to investigate just how strong the properties indicated by the conclusion of these lemmas are. Let (A) denote the property of the conclusion of Lemma 1.1, and (B) the conclusion of Lemma 1.2.[2] We thank an anonymous referee for pointing out simpler versions of (A) and (B) which will be formulated later. These simpler formulations are close in spirit to one of Borges' characterizations of monotone normality in terms of neighborhoods assigned to pairs (x, U) where U is an open neighborhood of x.

In this paper, we show that virtually all results of [6] following from a hypothesis of monotone (countable) metacompactness actually follow from (A), and some follow from (B). In particular, a compact Hausdorff space satisfying (A) must be metrizable. Furthermore, we prove that a space satisfying (B) is hereditarily metacompact, a fact which was not previously known about monotone metacompact spaces; the analogue for meta-Lindelöf also holds. We also prove that every separable space satisfying (A), hence every separable monotonically countably metacompact space, must be metrizable. The corollary that countable monotonically countably metacompact spaces are metrizable answers a question in [6].

Bennett, Hart, and Lutzer [4, Question 4.13] asked which stratifiable spaces are monotonically (countably) metacompact, and asked specifically about the well-known examples of stratifiable spaces due to Ceder and McAuley. We show that Ceder and McAuley's examples do not satisfy (A) or the meta-Lindelöf version of (A), hence are neither monotonically countably metacompact nor monotonically meta-Lindelöf. Finally, we answer a question of Gartside and Moody [8] (repeated by Stares [16]) by showing that a certain stratifiable space is monotonically paracompact in the locally finite sense, but not in the sense of Gartside and Moody. We end the paper with a question that indicates that there may be something fundamental yet to be proven about the structure of these classes of spaces.

All spaces are assumed to be regular and T_1.

2. Properties (A),(B),(C), and (D)

We first note simpler equivalent versions of the properties indicated by the conclusion of the lemmas stated in the introduction.

Definition 2.0. Let $P(X)$ be the collection of all pairs (x, U) where $x \in X$ and U is an open neighborhood of X. A space X has *property (A)* (resp., *property (B)*)if to each $(x, U) \in P(X)$, one can assign an open set $V(x, U)$ such that $x \in$

[2] It would seem that there should be countable versions of (A) and (B), corresponding to monotonically countably metacompact, but as we shall see, the countable versions are equivalent to the unrestricted versions.

$V(x,U) \subset U$ and such that for any collection $\{(x_\alpha, U_\alpha) : \alpha \in A\} \subset P(X)$, either $\bigcap_{\alpha\in A} V(x_\alpha, U_\alpha) = \emptyset$, or there exists a finite $A' \subset A$ such that for any $\alpha \in A$ there exists $\beta \in A'$ with $V(x_\alpha, U_\alpha) \subset U_\beta$ (resp., $\{x_\alpha : \alpha \in A\} \subset \bigcup_{\beta\in A'} U_\beta$).

It should be clear that (A) implies (B). It is useful to note that given an operator $V(x,U)$ satisfying (A) or (B), any other operator V' such that $V'(x,U) \subset V(x,U)$ satisfies the same property. So in proofs we may if we wish assume that $V(x,U)$ is a certain kind of basic open set.

The following proposition shows that property (A) as formulated in Definition 2.0 is equivalent to the property of the conclusion of Lemma 1.1.

Proposition 2.1. *The following are equivalent:*

(a) *Property (A);*
(b) *To each $p \in T(X)$ one can assign an open V^p satisfying:*
 (i) $x^p \in V^p \subset U_1^p$;
 (ii) *Whenever $\mathcal{Q} \subset T(X)$, then either $\bigcap_{q\in\mathcal{Q}} V^q = \emptyset$, or there exists a $\mathcal{Q}' \subset \mathcal{Q}$, with $\mathcal{Q}'$ finite, such that for any $q \in \mathcal{Q}$ there exists $q' \in \mathcal{Q}'$ such that either $V^q \subset U_1^{q'}$ or $V^q \cap U_0^{q'} = \emptyset$.*

Proof. (a)⇒(b). Assume $V(x,U)$ witnesses X has (A). Notice that for any $p = (x^p, U_0^p, U_1^p) \in T(X)$, the open set $V(x, U_0)$ will suffice for the V^p in (b).

(b)⇒(a). Let V^p witness X satisfies (b). Given (x,U), apply regularity to find a U_0 such that $p = (x, U_0, U) \in T(X)$. Then let $V(x,U) = V^p \cap U_0$. Note that if $q = (x, U_0, U)$ and $q' = (x', U_0', U')$, then $V^q \cap U_0^{q'} = \emptyset$ implies that $V(x,U) \cap V(x',U') = \emptyset$. It follows that the $V(x,U)$'s satisfy property (A). □

A similarly easy argument, which we omit, shows that (B) is equivalent to the property of the conclusion of Lemma 1.2.[3]

There are really two properties in the conclusion of Lemma 1.1 (also 1.2), one corresponding to the parenthetical "countable" restriction. Let us temporarily define properties "countable (A)" and "countable (B)" to be the same as (A) and (B), respectively, but with the index set A restricted to countable sets. The same argument as in Proposition 2.1 shows that countable (A) is equivalent to the conclusion of the countable version of Lemma 1.1, and similarly for countable (B) and Lemma 1.2. However, it is our good fortune that these countable versions are equivalent to the originals.

Proposition 2.2. *"countable (A)" is equivalent to (A) and "countable (B)" is equivalent to (B).*

Proof. Obviously (A) and (B) imply their countable versions. For the other direction, assume "countable (A)", witnessed by operator $V(x,U)$, and suppose (A) fails. Then there is a collection $\{(x_\alpha, U_\alpha) : \alpha \in A\} \subset P(X)$ such that $\bigcap_{\alpha\in A} V(x_\alpha, U_\alpha) \neq \emptyset$, and for any finite $A' \subset A$ there exists $\alpha \in A$ such that for any $\beta \in A'$, $V(x_\alpha, U_\alpha) \not\subseteq U_\beta$.

Pick $\alpha_0 \in A$. Then there is $\alpha_1 \in A$ such that $V(x_{\alpha_1}, U_{\alpha_1}) \not\subseteq U_{\alpha_0}$. If α_i is defined for $i < n$, choose $\alpha_n \in A$ such that for each $i < n$, $V(x_{\alpha_n}, U_{\alpha_n}) \not\subseteq U_{\alpha_i}$. Then $\{(x_{\alpha_n}, U_{\alpha_n}) : n \in \omega\}$ witnesses the failure of countable (A), contradiction. Thus (A) holds. A similar argument shows countable (B) implies (B). □

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[3] We thank the referee for the formulations of (A) and (B) as given above and suggesting that they are equivalent to the conclusions of Lemma 1.1 and 1.2, respectively.

Corollary 2.3. *Monotonically countably metacompact spaces satisfy property (A).*

Countably metacompact is a much weaker property than metacompact, but Proposition 2.2 suggests we ask:

Question 2.4. *Is there a monotonically countably metacompact space that is not monotonically metacompact?*

If you attempt to find a counterexample by finding a monotonically countably metacompact space which is not metacompact, you will fail, as we shall soon see.

Any space with only one non-isolated point satisfies (B): let $V(x, U) = \{x\}$ if x is isolated, and $V(x, U) = U$ otherwise. The one-point compactification of an uncountable discrete space satisfies (B), but by Theorem 3.1 does not satisfy (A) and hence is not monotonically countably metacompact. But we do not know any example of a space satisfying (A) which is not monotonically metacompact.

It's also not known if monotonically (countably) metacompact is a hereditary property or not (it is closed hereditary), even in linearly ordered spaces ([4, Question 4.9]), but these properties are.

Proposition 2.5. *The properties (A) and (B) are hereditary.*

Proof. We prove (A) is hereditary and leave similarly easy verification of (B) to the reader. Suppose the operator $V(x, U)$ witnesses X has property (A). Let $Y \subset X$. For each relatively open subset U of Y, choose an open set U^* in X such that $U^* \cap Y = U$. Now define $V(y, U) = V(y, U^*) \cap Y$. Suppose $\{(y_\alpha, U_\alpha) : \alpha \in A\} \subset P(Y)$, and that $\bigcap_{\alpha\in A} V(y_\alpha, U_\alpha) \neq \emptyset$. Then $\bigcap_{\alpha\in A} V(y_\alpha, U^*_\alpha) \neq \emptyset$, so there exists a finite $A' \subset A$ such that for any $\alpha \in A$ there exists $\beta \in A'$ with $V(y_\alpha, U^*_\alpha) \subset U^*_\beta$. But $V(y_\alpha, U^*_\alpha) \subset U^*_\beta$ implies $V(y_\alpha, U) \subset U_\beta$, so the same finite set A' witnesses that property (A) holds in Y. □

The next result shows that (B) has some surprising strength.

Lemma 2.6. *Every space X satisfying (B) is hereditarily metacompact.*

Proof. We have already shown that property (B) is hereditary, so all that is needed is to show (B) implies metacompact.

Let X have property (B) witnessed by the operator $V(x, U)$, and let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be an open cover of X for some ordinal κ. For all $x \in X$, let $\alpha(x)$ be the first α such that $x \in U_\alpha$. Let $V_\alpha = \bigcup\limits_{\alpha(x)=\alpha} V(x, U_\alpha)$, and let $\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$. Then $\mathcal{V}$ is clearly an open refinement of $\mathcal{U}$.

We claim that $\mathcal{V}$ is point-finite. Suppose otherwise. Then there exists some $x \in X$ and some infinite $A \subset \kappa$ such that $x \in V_\alpha$ for all $\alpha \in A$. We can assume that A has order type ω. For each $\alpha \in A$, choose $x_\alpha \in X$ such that $x \in V(x_\alpha, U_\alpha) \subset V_\alpha$. Since $x \in \bigcap\{V(x_\alpha, U_\alpha) : \alpha \in A\}$, we must have a finite $A' \subset A$ such that $\{x_\alpha : \alpha \in A\} \subset \bigcup_{\alpha\in A'} U_\alpha$. Let $\beta \in A$ such that $\beta > \max(A')$. Then β is least such that $x_\beta \in U_\beta$, so x_β is not covered by $\{U_\alpha : \alpha \in A'\}$, a contradiction. Thus $\mathcal{V}$ is point-finite, and X is metacompact, hence hereditarily metacompact. □

Corollary 2.7. *Monotonically countably metacompact spaces are hereditarily metacompact.*

There are versions of Lemmas 1.1 and 1.2 for monotonically meta-Lindelöf spaces, whose conclusions replace the word "finite" in 1.1 and 1.2 with "countable", and whose proofs are otherwise essentially identical to the proofs of 1.1 and 1.2. Indeed, one may follow the proof of 1.1 as given by Lemma 2.1 in [6], replacing the line "Set $V^p = \bigcap\{V \in r(\mathcal{U}) : x^p \in V\}$" with "Choose any $V \in r(\mathcal{U})$ with $x^p \in V\}$ and set $V^p = V$". The rest of the proof works nearly word for word as in [6] (see also [5], where this is written out in full).

If in (A) and (B) we replace "finite" with "countable", we obtain properties we shall call *property (C)* and *property (D)*, respectively, which are equivalent to the conclusions of the meta-Lindelöf versions of 1.1 and 1.2. Clearly (A)⇒(C)⇒(D) and (B)⇒(D) (so (A), and hence monotonically countably metacompact, implies every one of (A),(B),(C), and (D)).

In the same way as in Lemma 2.6, we have:

Lemma 2.8. *Every space X satisfying (D), and hence every monotonically meta-Lindelöf space, is hereditarily meta-Lindelöf.*

It is known that every hereditarily metacompact, monotonically normal space is hereditarily paracompact. This comes from the combination of the known fact that no stationary subset of a regular uncountable cardinal is metacompact (in fact, it isn't even meta-Lindelöf), combined with the result of Balogh and Rudin that a monotonically normal space is hereditarily paracompact if and only if it does not contain a copy of such a stationary subset [2]. Since a monotonically meta-Lindelöf space is hereditarily meta-Lindelöf, and a monotonically countably metacompact space is hereditarily metacompact, then such a space cannot contain a copy of a stationary subset of a regular uncountable cardinal. Thus we get the following corollary, which answers a question of Bennett, Hart, and Lutzer [4, Question 4.14]:

Corollary 2.9. *Any monotonically normal, monotonically meta-Lindelöf or monotonically countably metacompact space is hereditarily paracompact.*

Finally, we mention another consequence of (B) and (D). Recall that a space has *caliber* ω_1 (*caliber* (ω_1, ω)) if every uncountable collection $\mathcal{U}$ of non-empty open sets contains an uncountable (infinite) subcollection $\mathcal{V}$ such that $\bigcap \mathcal{V} \neq \emptyset$; equivalently, every point-countable (point-finite) collection of non-empty open sets is countable. Note that separable spaces have caliber ω_1 (and hence (ω_1, ω) too).

Lemma 2.10. *If X satisfies (B) and has caliber (ω_1, ω), or (D) and has caliber ω_1, then X is hereditarily Lindelöf.*[4]

Proof. Suppose that X has caliber ω_1 ((ω_1, ω)) and that $V(x, U)$ witnesses X satisfying (D) ((B)). If X is not hereditarily Lindelöf, it contains a right-separated subspace $\{x_\alpha : \alpha < \omega_1\}$. For each $\alpha \in \omega_1$, let U_α be an open neighborhood of x_α such that $U_\alpha \cap \{x_\beta : \beta > \alpha\} = \emptyset$. Since X has caliber ω_1 ((ω_1, ω)), there must be an uncountable (infinite of order type ω) $A \subset \omega_1$ such that $\bigcap\{V(x_\alpha, U_\alpha) : \alpha \in A\} \neq \emptyset$. By (D) ((B)), there must be a countable (finite) $A' \subset A$ such that $\{x_\alpha : \alpha \in A\} \subset \bigcup_{\beta \in A'} U_\beta$, but this is impossible. Hence X is hereditarily Lindelöf. □

The following corollary is immediate:

Corollary 2.11. *Suppose X is a separable space. If X is monotonically meta-Lindelöf or monotonically countably metacompact, then X is hereditarily Lindelöf.*

The part of the above corollary concerning monotonically meta-Lindelöf spaces is due to Li and Peng [14].

[4] The proof of this lemma is essentially identical to the proof of Lemma 2.3 of [6].

3. Metrization

In this section we prove two metrization theorems. The main result of [6] was that compact monotonically countably metacompact spaces are metrizable, answering question of Popvassilev [15] which was repeated in [4]. Our first metrization theorem here shows that (A) suffices to do the job.

Theorem 3.1. *Let X be compact and satisfy (A). Then X is metrizable.*

Proof. Lemma 2.6 in [6] states that a compact Hausdorff monotonically countably metacompact space is metrizable if it has caliber ω_1. This lemma only used Lemma 1.1, i.e., (A), in its proof. So X cannot have caliber ω_1, i.e, there exists an uncountable, point-countable collection $\mathcal{U} = \{U_\alpha\}_{\alpha \in \omega_1}$ of open sets in X. By thinning out this collection if necessary, we can assume that there exists $p_\alpha \in U_\alpha$ such that for all $\beta > \alpha$, $p_\alpha \notin U_\beta$.

Since X is regular, we can find open U_α^0 such that $p_\alpha \in U_\alpha^0 \subset \overline{U_\alpha^0} \subset U_\alpha$. Notice that the collection $\{U_\alpha^0\}_{\alpha \in \omega_1}$ is also point-countable. Let $W_\alpha = X \setminus \overline{U_\alpha^0}$.

Consider the collection $\{V(x, W_\alpha) : x \in W_\alpha\}$. Since X is compact, there must be a finite subcollection $\mathcal{V}_\alpha = \{V(x_{\alpha,i} W_\alpha) | i = 1, ..., k_\alpha\}$ covering $X \setminus U_\alpha$. There exists $W \subset \omega_1$ with $|W| = \omega_1$, and $k \in \omega$, such that $k_\alpha = k$ for all $\alpha \in W$. We will induct on the value of k to get a contradiction and finish the proof. However, we will first prove the following claim that will allow us to more efficiently complete the induction:

Claim. *Let F be a proper subset of $\{1, 2, \dots, k\}$. Suppose F has the following property:*

($*$) *There is an uncountable subset W' of W such that for any $\beta < \alpha \in W'$, we have $p_\beta \notin \bigcup_{i \in F} V(x_{\alpha,i}, W_\alpha)$.*

Then there is some F' with $F \subsetneq F' \subseteq \{1, 2, \dots, k\}$ such that F' satisfies ($*$).

Proof of Claim. Assume $F \subset \{1, 2, \dots, k\}$ and that F satisfies ($*$) above. Let W' be as in ($*$).

If $\beta_0 = \min W'$, then $p_{\beta_0} \in X \setminus U_\alpha$ for all $\alpha \in W' \setminus \{\beta_0\}$, and from ($*$), $p_{\beta_0} \notin V(x_{\alpha,i}, W_\alpha)$ for any $i \in F$. So for all $\alpha \in W' \setminus \{\beta_0\}$ there exists $j_\alpha \in \{1, 2, ..., k\} \setminus F$ such that $p_{\beta_0} \in V(x_{\alpha,j_\alpha}, W_\alpha)$.

Since $|W' \setminus \{\beta_0\}| = \omega_1$, there is $j \in \omega$ such that $j_\alpha = j$ for uncountably many α. Let $\hat{W} = \{\alpha \in W' \setminus \{\beta_0\} : p_{\beta_0} \in V(x_{\alpha,j}, W_\alpha)\}$. Notice that $\bigcap_{\alpha \in \hat{W}} V(x_{\alpha,j}, W_\alpha) \neq \emptyset$.

For $\beta < \alpha \in \hat{W}$, put $\{\alpha, \beta\}$ in Pot I if $p_\beta \in V(x_{\alpha,j}, W_\alpha)$ and in Pot II otherwise. By Erdös' theorem $\omega_1 \to (\omega, \omega_1)^2_2$ from partition calculus, either there exists an infinite $B \subset \hat{W}$ such that B is homogeneous for Pot I, or there exists an uncountable $B \subset \hat{W}$ such that B is homogenous for Pot II.

Suppose B is infinite homogeneous for Pot I; we may assume B has order type ω. Then $\beta < \alpha \in B$ implies that $p_\beta \in V(x_{\alpha,j}, W_\alpha)$. By countable (A), there is a finite $B' \subset B$ such that for each $\alpha \in B$ there exists $\alpha' \in B'$ with $V(x_{\alpha,j}, W_\alpha) \subset W_{\alpha'} = X \setminus \overline{U_{\alpha'}^0}$. Choose $\alpha \in B$ with $\alpha > \max(B')$. We must have $V(x_{\alpha,j}, W\alpha) \subset (X \setminus \overline{U_{\alpha'}^0})$. But this is a contradiction since $p_{\alpha'} \in V(x_\alpha, j, W_\alpha) \cap U_{\alpha'}^0$.

Thus there exists an uncountable $B \subset \hat{W}$ such that B is homogenous for Pot II. Then if we let $F' = F \cup \{j\}$, then B is an uncountable subset of W such that for

all $\beta < \alpha \in B$ we have $p_\beta \notin \bigcup_{i \in F'} V(x_{\alpha,j}, W_\alpha)$. And thus F' satisfies $(*)$.
This proves the Claim.

We now complete the proof of the theorem. If $F_0 = \emptyset$, then F trivially satisfies $(*)$. From the claim there exists $F_1 \supsetneq F_0$ such that F_1 satisfies $(*)$. If $F_1 \neq \{1, 2, ..., k\}$, then apply the claim again to F_1 to get a superset F_2. We can continue in this way until we get $F_n = \{1, 2, \ldots, k\}$ satisfying $(*)$. But if $\{1, 2, \ldots, k\}$ satisfies $(*)$, we get our desired contradiction: for then we have an uncountable $W' \subset W$ such that $\forall \beta < \alpha \in W'$, we have $p_\beta \notin \bigcup_{i=1}^{k} V(x_{\alpha,i}, W_\alpha)$. But this then implies that $p_\beta \in U_\alpha$, since $\bigcup_{i=1}^{k} V(x_{\alpha,i}, W_\alpha)$ covers $X \setminus U_\alpha$. This is a contradiction to our initial assumption on $p_\beta \notin U_\alpha$ for all $\alpha > \beta$. Hence X is metrizable. $\square$

The one-point compactification of an uncountable discrete space, which satisfies (B) (see the paragraph prior to Proposition 2.5), shows that (B) does not suffice in the above theorem.

In [6], the authors proved that the sequential fan and the so-called "single ultrafilter space" are not monotonically countably metacompact. Levy and Matveev [11] had shown that, assuming the Continuum Hypothesis, the sequential fan is monotonically Lindelöf and there is a free ultrafilter such that the single ultrafilter space is monotonically Lindelöf. It is apparently unknown if the sequential fan is monotonically Lindelöf in ZFC, or if there is any single ultrafilter space which is not monotonically Lindelöf. Levy and Matveev asked if it is consistent that every countable monotonically Lindelöf space is metrizable, and we asked if it might be true in ZFC that every countable monotonically countably metacompact space is metrizable. Actually, it would have been reasonable to ask for more to be true, i.e., that separable monotonically countably metacompact spaces are metrizable, as the only classes of monotonically (countably) metacompact spaces given in [4] (metacompact Moore spaces and non-Archimedean spaces) are metrizable if separable. Here we prove that this more optimistic conjecture is indeed so.

Theorem 3.2. *Let X be a separable space. If X satisfies (A), then X is metrizable.*

Proof. Let $V(x, U)$ be an operator witnessing that the separable space X satisfies (A). Suppose X is not metrizable; then it does not have a countable base.

Let $(x_0, U_0) \in P(X)$. If (x_β, U_β) has been defined for all $\beta < \alpha$, let $(x_\alpha, U_\alpha) \in P(X)$ witness that $\{V(x_\beta, U_\beta) : \beta < \alpha\}$ is not a base for X, i.e., $x_\alpha \in V_\beta$ implies $V_\beta \not\subset U_\alpha$. By using regularity to shrink U_α if necessary, we may assume that U_α was chosen so that $x_\alpha \in V_\beta$ implies $V_\beta \not\subset \overline{U}_\alpha$.

By Lemma 2.11, X is hereditarily Lindelöf. Let

$$S_0 = \{\beta < \omega_1 : \exists \alpha > \beta (x_\alpha \in V(x_\beta, U_\beta))\}.$$

That S_0 is uncountable is easily seen by applying the hereditarily Lindelöf property to tails of the sequence $V(x_\alpha, U_\alpha)$, $\alpha < \omega_1$. For each $\beta \in S_0$, let $\beta' = \min\{\alpha > \beta : x_\alpha \in V(x_\beta, U_\beta)\}$, and choose a point $x(0, \beta) \in (V(x_\beta, U_\beta) \setminus \overline{U}_{\beta'}) \cap D$, where D is a countable dense set. There are a point $d_0 \in D$ and an uncountable subset T_0 of S_0 such that $x(0, \beta) = d_0$ for each $\beta \in T_0$. Let α_0 be such that $d_0 \notin U_{\alpha_0}$ (e.g, $\alpha_0 = \beta'$ for some (any) $\beta \in T_0$).

Now let

$$S_1 = \{\beta \in T_0 : \exists \alpha \in T_0(\alpha > \beta \text{ and } x_\alpha \in V(x_\beta, U_\beta))\}.$$

Note S_1 is uncountable. For each $\beta \in S_1$, let $\beta' = \min\{\alpha \in T_0 : \alpha > \beta \text{ and } x_\alpha \in V(x_\beta, U_\beta)\}$, and choose a point $x(1, \beta) \in (V(x_\beta, U_\beta) \setminus \overline{U}_{\beta'}) \cap D$. There are a point $d_1 \in D$ and an uncountable subset T_1 of S_1 such that $x(1, \beta) = d_1$ for each $\beta \in T_1$. Pick $\beta \in T_1$ with $\beta > \alpha_0$ and let $\alpha_1 = \beta'$. Note that $d_1 \notin U_{\alpha_1}$ since $\beta \in T_1$, but $d_0 \in V(x_{\alpha_1}, U_{\alpha_1})$ since $\alpha_1 \in T_0$.

Suppose d_i, S_i, T_i, and α_i have been selected for each $i \leq n$. Let

$$S_{n+1} = \{\beta \in T_n : \exists \alpha \in T_n(\alpha > \beta \text{ and } x_\alpha \in V(x_\beta, U_\beta))\}.$$

Note S_{n+1} is uncountable. For each $\beta \in S_{n+1}$, let $\beta' = \min\{\alpha \in T_n : \alpha > \beta \text{ and } x_\alpha \in V(x_\beta, U_\beta)\}$, and choose a point $x(n+1, \beta) \in (V(x_\beta, U_\beta) \setminus \overline{U}_{\beta'}) \cap D$. There are a point $d_{n+1} \in D$ and an uncountable subset T_{n+1} of S_{n+1} such that $x(n+1, \beta) = d_{n+1}$ for each $\beta \in T_{n+1}$. Pick $\beta \in T_{n+1}$ with $\beta > \alpha_n$ and let $\alpha_{n+1} = \beta'$.

Note that $d_{n+1} \notin U_{\alpha_{n+1}}$ since $\beta \in T_{n+1}$, but $\{d_0, d_1, \ldots, d_n\} \subset V(x_{\alpha_{n+1}}, U_{\alpha_{n+1}})$ since $\alpha_{n+1} \in T_i$ for for every $i \leq n$. Thus $V_{\alpha_{n+1}} \not\subset U_{\alpha_i}$ for each $i \leq n$. It follows that the sequence (x_n, U_n) with $U_n = U_{\alpha_n}$ and $x_n = x_{\alpha_n}$ for all n violates (A). This contradiction completes the proof. □

Corollary 3.3. *A separable monotonically countably metacompact space is metrizable.*

The most interesting related open question is the one of Levy and Matveev:

Question [11]**.** Is it consistent that every countable monotonically Lindelöf space is metrizable?

4. Stratifiable spaces

One of the more examined problems involving monotonic metacompactness has been the attempt to discover well-known properties which imply monotonically metacompact. as we have mentioned, Bennett and Lutzer [4] showed that every metacompact Moore space and every non-Archimedean space is monotonically metacompact. But which other properties will give the same result? The class of stratifiable spaces is a natural one to look at as they are both monotonically normal and hereditarily paracompact. Indeed, Bennett and Lutzer asked which stratifiable spaces were monotonically metacompact, and in particular, if well-known examples of stratifiable spaces given by Ceder and McAuley have this property. Cedar and McAuley's spaces are separable, so we now know by Theorem 3.2 that they do not satisfy (A) and hence are not monotonically countably metacompact. Here we show that Ceder and McAuley's spaces do not satisfy (C) either, hence are not monotonically meta-Lindelöf. Then we give an example of a stratifiable space which is not only monotonically metacompact but is monotonically paracompact "in the locally finite sense" but is not monotonically paracompact in the sense of Gartside and Moody [8]; this answers a question in [8].

Ceder and McAuley's spaces are quite similar; the following description of these spaces is taken from [1].

McAuley's "bow-tie" space is defined as $X = X_0 \cup X_1$ where $X_0 = \{(x, 0) \mid x \in \mathbb{R}\}$, and $X_1 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. A basic open neighborhood for a point $x = (x, 0) \in X_0$ is a "bowtie" of the form

$$M\left(x, \frac{1}{n}\right) = \{x\} \cup \left\{(x', y') \in X \ : \ y' < \frac{1}{n}|x' - x_0| < \frac{1}{n^2}\right\}$$

for $n \in \mathbb{N}$.The points in X_1 have the usual Euclidean open balls as basic neighborhoods.

The Ceder space uses the same underlying set for X, and as in the McAuley space, basic open neighborhoods of points above the x-axis are Euclidean open balls. A basic open neighborhood for a point $x = (x_0, 0) \in X_0$ is anything of the form $C\left(x, \frac{1}{n}\right) = \{x\} \cup \left\{(x', y') \in X \ : \ y' < n - \sqrt{n^2 - (x' - x)^2}, |x' - x| < \frac{1}{n}\right\}$, for $n \in \mathbb{N}$. So a basic neighborhood of a point on the x-axis in Ceder's space is a bow-tie with the upper boundary an arc of a circle tangent to the x-axis at the point $(x, 0)$.

Example 4.1. *McAuley's "bow-tie" space and Ceder's space do not satisfy (C), hence are not monotonically meta-Lindelöf.*

Proof. We prove this for McAuley's space X; the proof for Ceder's space is virtually identical. For each $x = (x, 0) \in X_0$, let $M(x, 1)$ be its basic open neighborhood as defined above. This is a "bow-tie" with center point x. Let $V(x, U)$ be an operator such that $x \in V(x, U) \subset U$; we will show that V cannot satisfy (C) or countable (A). We may assume $V(x, M(x, 1))$ is also a bow-tie at x, and further that $V(x, M(x, 1))) \cap X_0$ is an interval on the x-axis with rational endpoints. Then there is an uncountable subset A of X_0 such that $V(x, M(x, 1))) \cap X_0 = V(x', M(x', 1))) \cap X_0$ for each $x, x' \in A$. Note that for each $x, x' \in A$, $V(x, M(x, 1)) \not\subseteq M(x', 1)$ since each point of $V(x, M(x, 1))) \cap X_0$ other than x contains points in the upper half plane directly above it. It follows that the collection $\{(x, M(x, 1)) : x \in A\}$ witnesses the negation of (C). □

In 1993 Gartside and Moody [8] gave the first definition of monotonic paracompactness. In their definition, they used star-refinements instead of locally finite refinements. In other words, they defined X to be monotonically paracompact if to each open cover $\mathcal{U}$ one can assign an open star-refinement $r(\mathcal{U})$ such that $\mathcal{U}$ refines $\mathcal{V}$ implies $r(\mathcal{U})$ refines $r(\mathcal{V})$. They then went on to show that for T_1 spaces, monotonically paracompact (by their definition) was equivalent to a space being protometrizable. They also asked whether defining monotonically paracompact via locally finite refinements (i.e., replace "star-refinement" by 'locally finite refinement") would be equivalent.

We show here an example of a space that is monotonically paracompact under the locally finite definition (hence monotonically metacompact), and stratifiable, but not protometrizable and hence not monotonically paracompact by Gartside and Moody's definition.[5]

We remind the reader that a space X is *stratifiable* if for every open set U, one can assign a countable collection of open sets $\{U_n\}_{n \in \omega}$, such that:

[5]Popvassilev and Porter independently obtained a similar example, although their example is not stratifiable [15]. They also prove that monotonically paracompact spaces in the sense of Gartside and Moody are monotonically paracompact in the locally finite sense.

i. $\overline{U}_n \subset U$;
ii. $\bigcup_{n\in\omega} U_n = U$;
iii. $U_n \subset V_n$ whenever $U \subset V$.

Example 4.2. *Let $X = (\omega_1 \times \omega) \cup \{\infty\}$, where points of $\omega_1 \times \omega$ are isolated, and a basic open neighborhood of ∞ is of the form $B(\alpha, n) = \{(\beta, m) : \alpha \leq \beta, n \leq m\} \cup \{\infty\}$. Then X is stratifiable, and monotonically paracompact in the locally finite sense, but is not monotonically paracompact in the sense of Gartside and Moody.*

Proof. That X is stratifiable follows from the well-known and easy-to-prove fact that a space with a single non-isolated point is stratifiable if and only if the non-isolated point is a G_δ-point.

To see that X is monotonically paracompact in the locally finite sense, first notice that the local base at ∞ is order-isomorphic to $\omega_1 \times \omega$, which is well-founded. So we may define $r(\mathcal{U})$ as follows: let $m(\mathcal{U})$ be the maximal members of the local base at ∞ which are contained in some member of $\mathcal{U}$; then let $r(\mathcal{U}) = m(\mathcal{U}) \cup \{\{x\} : x \in X \setminus \cup m(\mathcal{U})\}$. Since incomparable subsets of $\omega_1 \times \omega$ are finite, $m(\mathcal{U})$ is finite, and it follows that $r(\mathcal{U})$ is locally finite. Finally, it is easy to check that if $\mathcal{U}$ refines $\mathcal{V}$, then $r(\mathcal{U})$ refines $r(\mathcal{V})$. Thus X is monotonically paracompact in the locally finite sense.

To see that X is not protometrizable, assume that it is. Then by definition (see [10, Def. 1.7]), X is paracompact and has an orthobase[6] $\mathcal{B}$. We can find $B_n \in \mathcal{B}$ containing ∞ such that $B_n \cap (\omega_1 \times n) = \emptyset$. Then $\bigcap_{n\in\omega} B_n = \{\infty\}$, but it is easy to see that $\{B_n\}_{n\in\omega}$ is not a local base at ∞. So $\mathcal{B}$ is not an orthobase, and X is not protometrizable. □

5. Question

The second author thanks Paul Gartside for reminding him of the class of spaces having a base $\mathcal{B}$ which is *Noetherian of subinfinite rank (NSR)*, i.e.,

(i) Every strictly increasing sequence (by set inclusion) of members of $\mathcal{B}$ is finite;
(ii) Every subcollection of $\mathcal{B}$ with nonempty intersection contains comparable elements.

If $\mathcal{B}$ is an NSR base, then one can define a monotone metacompactness operator by declaring $r(\mathcal{U})$ to be the maximal members of $\mathcal{B}$ that are contained in some member of $\mathcal{U}$; (i) guarantees that $r(\mathcal{U})$ covers and (ii) that it is point-finite. Metrizable spaces and nonarchimedean spaces, indeed every monotonically metacompact space that we know of, has an NSR base.

So we end with a question, parts of which were mentioned earlier in this paper. Let mM abbreviate monotonically metacompact and let mCM abbreviate monotonically countably metacompact. We have the following line of implications:

$$\text{NSR base} \Rightarrow \text{mM} \Rightarrow \text{mCM} \Rightarrow \text{property (A)}.$$

Question. Do any, or even all, of the above implications reverse?

[6]An *orthobase* for a space X is a base $\mathcal{B}$ for the topology on X such that for any collection $\mathcal{F} \subset \mathcal{B}$, either $\bigcap \mathcal{F}$ is open, or $\bigcap \mathcal{F} = \{x\}$ and $\mathcal{F}$ is a local base at x.

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