COMPACT MONOTONICALLY METACOMPACT SPACES ARE METRIZABLE

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Abstract. Monotonically metacompact spaces were recently introduced as an extension of the concept of monotonically compact spaces. In this note we answer a question of Popvassilev, and Bennett, Hart, and Lutzer, by showing that every compact, Hausdorff, monotonically (countably) metacompact space is metrizable. We also show that certain countable spaces fail to be monotonically (countably) metacompact.

1. Introduction

A space X is *monotonically (countably) metacompact* if there is a function r that assigns to each (countable) open cover $\mathcal{U}$ of X a point-finite open refinement $r(\mathcal{U})$ covering X such that if $\mathcal{V}$ is a (countable) open cover of X and $\mathcal{V}$ refines $\mathcal{U}$, then $r(\mathcal{V})$ refines $r(\mathcal{U})$. The function r is called a *monotone (countable) metacompactness operator.*[1]

This property was first introduced by Popvasillev[7] in 2009, who showed that ω_1 and $\omega_1 + 1$ are not monotonically countably metacompact. Bennett, Hart, and Lutzer [1] examined the property, showed that any metric space as well as any metacompact Moore space was monotonically metacompact, and discovered several other results relating to monotone metcompactness in LOTS and GO-spaces; in particular, they proved that every monotonically metacompact compact LOTS is metrizable. More recently Liang-Xue Peng and Hui Li [6] have extended some results for monotonically compact and monotonically Lindelöf spaces to monotonically metacompact spaces; specifically, they showed that every monotonically normal, monotonically countably metacompact space is hereditarily paracompact, and that any compact, monotonically meta-Lindelöf T_2 space is first countable.

In this paper, we show that any compact, Hausdorff, monotonically countably metacompact space is metrizable. This answers an open question posed in both [7] and [1], and this result was inspired by the result that any monotonically compact Hausdorff space is metrizable, proved by G. Gruenhage in [4] . We also show that neither the sequential fan nor the single ultrafilter spaces are monotonically (countably) metacompact; this should be compared with the results of Levy and Matveev [8] that under CH, the sequential fan is monotonically Lindelöf and there is a single ultrafilter space which is monotonically Lindelöf.

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[1]It should be noted that there is another definition of monotone countable metacompactness, given by Good, Knight, and Stares [2]. That definition is entirely unrelated to the one studied here, and neither version implies the other.

For two collections $\mathcal{U}$ and $\mathcal{V}$ we write $\mathcal{U} \prec \mathcal{V}$ to mean $\mathcal{U}$ refines $\mathcal{V}$, or in other words, for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that $U \subset V$.

2. Main Results

In Lemma 2.1 below, following the same method as outlined in [4], we show that monotonically (countably) metacompact spaces have a certain property that can be described through neighborhood-pair assignments.

For a space X let $\mathcal{P}_X$ be the collection of all triples $p = (x^p, U_0^p, U_1^p)$ where U_0^p, U_1^p are open in X, and $x^p \in U_0^p \subset \overline{U_0^p} \subset U_1^p$.

Lemma 2.1. *Suppose X is monotonically (countably) metacompact. Then to each $p \in \mathcal{P}_X$ one can assign an open V^p satisfying:*

i. $x^p \in V^p \subset U_1^p$;

ii. *Whenever (countable) $\mathcal{Q} \subset \mathcal{P}_X$, then either $\bigcap_{q \in \mathcal{Q}} V^q = \emptyset$, or there exists a $\mathcal{Q}' \subset \mathcal{Q}$, with $\mathcal{Q}'$ finite, such that for any $q \in \mathcal{Q}$ there exists $q' \in \mathcal{Q}'$ such that either $V^q \subset U_1^{q'}$ or $V^q \cap U_0^{q'} = \emptyset$.*

Proof. Let X be monotonically (countably) metacompact, r the monotonically (countably) metacompact operator, and $\mathcal{P}_X$ defined as above. Notice that for every $p \in \mathcal{P}_X$, $\mathcal{U}^p = \{U_1^p,\ X \setminus \overline{U_0^p}\}$ is an open cover of X. Also note that $r(\mathcal{U}^p)$ is a point-finite refinement of $\mathcal{U}^p$.

Set $V^p = \bigcap\{V \in r(\mathcal{U}^p) : x^p \in V\}$. Since $r(\mathcal{U}^p)$ is point-finite, V^p is open and $x^p \in V^p \subset U_1^p$.

Now let $\mathcal{Q} \subset \mathcal{P}_X$, ($|\mathcal{Q}| = \omega$), and assume that $\bigcap_{q \in \mathcal{Q}} V^q \neq \emptyset$.

Let $\mathcal{U} = \bigcup\{\mathcal{U}^q : q \in \mathcal{Q}\}$. $\mathcal{U}$ is a (countable) open cover of X, and for all $q \in \mathcal{Q}$, $\mathcal{U}^q \prec \mathcal{U}$. Thus $r(\mathcal{U}^q) \prec r(\mathcal{U})$ for all $q \in \mathcal{Q}$.

Let $t \in \bigcap_{q \in \mathcal{Q}} V^q$, and let $\mathcal{U}' = \{U \in r(\mathcal{U}) : t \in U\}$. Since $r(\mathcal{U})$ refines $\mathcal{U}$, for each $U \in \mathcal{U}'$ there is some $q(U) \in \mathcal{Q}$ such that $\{U\} \prec \mathcal{U}^{q(U)}$. Let $\mathcal{Q}' = \{q(U) : U \in \mathcal{U}'\}$.

We show that this $\mathcal{Q}'$ satisfies the conclusion of condition (ii). Suppose $q \in \mathcal{Q}$. We have $\{V^q\} \prec r(\mathcal{U}^q) \prec r(\mathcal{U})$, and $t \in V^q$, so there is some $U \in \mathcal{U}'$ such that $V^q \subset U$. Then from $\{V^q\} \prec \{U\} \prec \mathcal{U}^{q(U)}$, we have $V^q \subset U_1^{q(U)}$ or $V^q \cap U_0^{q(U)} = \emptyset$, as desired. □

A weaker version of the above property implied by monotone countable metacompactness is given below - the proof of it follows in the same manner as Theorem 2.3 in [4]. This version is sometimes easier to work with, since one need not worry about dealing with $\mathcal{P}_X$ and the triples.

Lemma 2.2. *Let X be a monotonically countably metacompact T_3-space, and $Y \subset X$. If for each $y \in Y$, U_y is some open neighborhood containing y, then there exists an open neighborhood V_y of y with $V_y \subset U_y$ such that if $Y' \subset Y$, and $\bigcap_{y \in Y'} V_y \neq \emptyset$, then there is a finite $Y'' \subset Y'$ such that $Y' \subset \bigcup_{y \in Y''} U_y$.*

Proof. Let U'_y be an open neighborhood of y such that $\overline{U'_y} \subset U_y$. Then $p(y) = (y, U'_y, U_y) \in \mathcal{P}_X$. Set $V_y = V^{p(y)} \cap U'_y$, where $V^{p(y)}$ is as in Lemma 2.1; also suppose $Y' \subset Y$, and $\bigcap_{y \in Y'} V_y \neq \emptyset$. Suppose that no finite $Y'' \subset Y'$ satisfying the conclusion of the lemma exists. Then we can find $y_0, y_1, ... \in Y'$ such that $y_n \notin \bigcup_{i<n} U_{y_i}$. Let $\mathcal{Q} = \{(y_n, U'_{y_n}, U_{y_n}) : n \in \omega\}$. Then there must be a finite subset $\mathcal{Q}'$ of $\mathcal{Q}$ satisfying the conclusion of Lemma 2.1. Let $n \in \omega$ be such that $(y_i, U'_{y_i}, U_{y_i}) \in \mathcal{Q}'$ implies that $i < n$. By Lemma 2.1, there must be $i < n$ such that either $V^{p(y_n)} \subset U_{y_i}$ or $V^{p(y_n)} \cap U'_{y_i} = \emptyset$. Now, $y_n \in V^{p(y_n)} \setminus U_{y_i}$, so $V^{p(y_n)} \not\subset U_{y_i}$. But $V^{p(y_n)} \cap U'_{y_i} \supset V_{y_n} \cap V_{y_i}$ and $V_{y_n} \cap V_{y_i} \neq \emptyset$. So $V^{p(y_n)} \cap U'_{y_i} \neq \emptyset$, a contradiction, and hence the lemma holds. □

The following lemma will be useful in proving our main result and is a good example of the use of Lemma 2.2. Recall that a space X has *caliber* ω_1 if for any uncountable collection of nonempty open sets $\mathcal{U} = \{U_\alpha\}_{\alpha \in \omega_1}$ there exists an uncountable $A \subset \omega_1$ such that $\bigcap_{\alpha \in A} U_\alpha \neq \emptyset$.

Lemma 2.3. *If the T_3-space X is monotonically countably metacompact, and has caliber ω_1, then X is hereditarily Lindelöf.*[2]

Proof. If X is not hereditarily Lindelöf, it contains a right-separated subspace $\{x_\alpha : \alpha < \omega_1\}$. For each $\alpha \in \omega_1$, let U_{x_α} be an open neighborhood of x_α such that $U_{x_\alpha} \cap \{x_\beta : \beta > \alpha\} = \emptyset$. Now let V_{x_α} be as in Lemma 2.2. Since X has caliber ω_1, there must be an uncountable $A \subset \omega_1$ such that $\bigcap\{V_{x_\alpha} : \alpha \in A\} \neq \emptyset$. From Lemma 2.2, there must be a finite $A' \subset A$ such that $\{x_\alpha : \alpha \in A\} \subset \bigcup_{\beta \in A'} U_{x_\beta}$, which is a contradiction. Hence X is hereditarily Lindelöf. □

We are now ready to prove the primary result of this paper.

Theorem 2.4. *Let X be compact T_2 and monotonically countably metacompact. Then X is metrizable.*

This theorem is immediate from the next two lemmas.

Lemma 2.5. *Let X be compact T_2 and monotonically countably metacompact. Then X has caliber ω_1.*

Proof. Assume the hypotheses and suppose X does not have caliber ω_1. Then there exists a collection of nonempty open sets $\hat{\mathcal{U}} = \{U_\alpha : \alpha \in \omega_1\}$ such that any uncountable subcollection of $\hat{\mathcal{U}}$ has empty intersection. In other words, $\hat{\mathcal{U}}$ is a point-countable collection.

Pick $x_0 \in U_0$, and let $\alpha_0 = 0$. Since $\hat{\mathcal{U}}$ is point countable, there exists an $\alpha_1 \in \omega_1$ such that $x_0 \notin U_{\alpha_1}$. Suppose that x_{α_γ} and U_{α_γ} have been defined for each $\gamma < \delta$, where $\delta < \omega_1$, such that:

(i) $x_{\alpha_\gamma} \in U_{\alpha_\gamma}$
(ii) $\gamma < \gamma' < \delta \Rightarrow x_{\alpha_\gamma} \notin U_{\alpha_{\gamma'}}$

[2]The proof of this lemma is essentially identical to the proof of Theorem 2.4 of [4].

By point-countability, there exists an α_δ such that $U_{\alpha_\delta} \cap \{x_{\alpha_\gamma} : \gamma < \delta\} = \emptyset$. Now choose $x_{\alpha_\delta} \in U_{\alpha_\delta}$. In this manner, we get an uncountable collection $A \subset \omega_1$ and $x_\alpha \in U_\alpha$ for every $\alpha \in A$, but $x\alpha \notin U_\beta$ for any $\beta > \alpha$, where $\beta \in A$.

By regularity, for each $\alpha \in A$ we can find a U'_α such that $x_\alpha \in U'_\alpha \subset \overline{U'_\alpha} \subset U_\alpha$. Let $\mathcal{U}_\alpha = \{U_\alpha,\ X \setminus \overline{U'_\alpha}\}$, for each $\alpha \in A$. Then $\mathcal{U}_\alpha$ is an open cover of X for each α, and $r(\mathcal{U}_\alpha) \prec \mathcal{U}_\alpha$.

Since X is compact, for each $\alpha \in A$ there exists a finite $\mathcal{V}_\alpha \subseteq \{V \in r(\mathcal{U}_\alpha) : V \subset X \setminus \overline{U'_\alpha}\}$ such that $X \setminus U_\alpha \subset \bigcup \mathcal{V}_\alpha$. Also notice that $x_\alpha \notin \bigcup \mathcal{V}_\alpha$.

Since we have uncountably many finite collections $\mathcal{V}_\alpha$, there exists $n \in \omega$ and an $A' \subset A$, with $|A'| = \omega_1$, such that $|\mathcal{V}_\alpha| = n$ for all $\alpha \in A'$. Denote $\mathcal{V}_\alpha = \{V_{\alpha,1},\ V_{\alpha,2},\ \ldots\ V_{\alpha,n}\}$ for all $\alpha \in A'$.

Let $A'' \subset A'$ have order type ω. Note that for $s < t$ in A'', we have $x_s \notin U_t$, hence $x_s \in \bigcup \mathcal{V}_t$. Put $\{s,t\}$ in Pot i if $x_s \in V_{t,i}$. From Ramsey's Theorem, there is an infinite $B \subset A''$ and an $m \leq n$ such that $s < t \in B$ implies that $\{s,t\} \in$ Pot m, and thus $x_s \in V_{t,m}$. If $s_0 = \min B$ then $x_{s_0} \in V_{t,m}$ for all $t \in B \setminus \{s_0\}$.

Let $\mathcal{U} = \bigcup\{\mathcal{U}_t : t \in B \setminus \{s_0\}\}$. If $x_{s_0} \in V' \in r(\mathcal{U})$, then there is some $t \in B \setminus \{s_0\}$ such that $V' \subset U_t$ or $V' \subset (X \setminus \overline{U'_t})$. Since $x_{s_0} \notin U_t$, the latter must hold and so $x_t \notin V'$. Thus, for every $V' \in \{V \in r(\mathcal{U}) : x_{s_0} \in V\}$ there exists a $k(V') \in B \setminus \{s_0\}$ such that $x_{k(V')} \notin V'$.

Let $d \in B$ such that $d > k(V')$, for every $V' \in \{V \in r(\mathcal{U}) : x_{s_0} \in V\}$. We have $x_{s_0} \in V_{d,m} \in r(\mathcal{U}_d) \prec r(\mathcal{U})$, so there exists a $V' \in r(\mathcal{U})$ such that $V_{d,m} \subset V'$. But $k(V') < d$ implies $x_{k(V')} \in V_{d,m} \subset V'$, which is a contradiction. □

Lemma 2.6. *Let X be compact T_2 and monotonically countably metacompact. If X has caliber ω_1, then X is metrizable.*

Proof. The proof of this lemma is essentially identical to the proof that monotonically compact Hausdorff spaces having property (K) are metrizable, shown in [4]. For the benefit of the reader, we repeat the argument here.

Assume that X is compact, T_2, monotonically countably metacompact with operator r, and has caliber ω_1.

By Lemma 2.3, X is perfectly normal. Suppose X is not metrizable. Choose $x_0, y_0 \in X$ such that $x_0 \neq y_0$, and let U_0 be an open neighborhood of x_0 with $y_0 \notin \overline{U}_0$. Suppose $\alpha < \omega_1$ and $x_\beta,\ y_\beta,$ and U_β have been chosen for each $\beta < \alpha$. There cannot exist a countable collection of open sets in X that separates points in the T_0 sense, for otherwise, by perfect normality, we would also then have a collection that separates points in the T_1 sense, making X metrizable (see, e.g., Theorem 7.6 of [3]). Therefore there are points x_α and y_α, $x_\alpha \neq y_\alpha$, such that if $\beta < \alpha$, then $U_\beta \cap \{x_\alpha, y_\alpha\} = \emptyset$, or $\{x_\alpha, y_\alpha\} \subset U_\beta$. Now let U_α be an open neighborhood of x_α with $y_\alpha \notin \overline{U}_\alpha$. Thus we have defined $x_\alpha,\ y_\alpha$, and U_α for each $\alpha \in \omega_1$.

For each $\alpha \in \omega_1$, let U'_α be open such that $x_\alpha \in U'_\alpha \subset \overline{U'_\alpha} \subset U_\alpha$. Let $p_\alpha = (x_\alpha, U'_\alpha, U_\alpha)$, and let V^{p_α} be as in Lemma 2.1.

Since X has caliber ω_1, there is an uncountable $A \subset \omega_1$ such that $\bigcap\{V^{p_\alpha} \cap U'_\alpha : \alpha \in A\} \neq \emptyset$. For $\beta < \alpha \in A$, we will put $\{\alpha, \beta\}$ in Pot I if $V^{p_\alpha} \not\subset U_\beta$; otherwise put $\{\alpha, \beta\}$ in Pot II.

We claim that there can be no infinite subset A' of A which is homogeneous for Pot I. If there were, we may assume A' has order type ω. Then from Lemma 2.1, there must be a finite $A'' \subset A'$ such that for any $\alpha \in A'$ there is some $\beta \in A''$ such

that $V^{p_\alpha} \subset U_\beta$ (since $\bigcap\{V^{p_\alpha} \cap U'_\alpha : \alpha \in A\} \neq \emptyset$, the alternative $V^{p_\alpha} \cap U'_\beta = \emptyset$ never holds). But then choosing $\alpha \in A'$ with $\alpha > \beta$ for every $\beta \in A''$ yields a contradiction.

Hence by Erdos' theorem $\omega_1 \to (\omega, \omega_1)^2$, there is an uncountable $A'' \subset A'$ that is homogenous for Pot II. In other words, $\beta < \alpha \in A''$ implies that $V^{p_\alpha} \subset U_\beta$.

Applying hereditarily Lindelöf, we must have a $\gamma < \omega_1$ so that for every $\mu,\ \nu \in \omega_1 \setminus \gamma$,

$$\overline{\bigcup_{\alpha \in A'' \setminus \mu} \{x_\alpha, y_\alpha\}} = \overline{\bigcup_{\alpha \in A'' \setminus \nu} \{x_\alpha,\ y_\alpha\}}$$

If β is the least element of $A'' \setminus \gamma$, then $y_\beta \in \overline{\bigcup_{\alpha \in A'' \setminus \beta} \{x_\alpha, y_\alpha\}} = \overline{\bigcup_{\alpha \in A'' \setminus (\beta+1)} \{x_\alpha,\ y_\alpha\}}$. For each $\alpha \in A'' \setminus (\beta + 1)$, we have that $U_\beta \supset V^{p_\alpha} \ni x_\alpha$ and thus $\{x_\alpha, y_\alpha\} \subset U_\alpha$. Thus $U_\beta \supset \bigcup_{\alpha \in A'' \setminus (\beta+1)} \{x_\alpha, y_\alpha\}$ and so $\overline{U}_\beta \supset \overline{\bigcup_{\alpha \in A'' \setminus (\beta+1)} \{x_\alpha, y_\alpha\}}$, and consequently $y_\beta \in \overline{U}_\beta$, a contradiction. □

3. Some countable spaces.

Recall that a space X is *monotonically Lindelöf* if there is a function r assigning to each open cover $\mathcal{U}$ a countable open refinement $r(\mathcal{U})$ covering X such that $\mathcal{V} \prec \mathcal{U}$ implies $r(\mathcal{V}) \prec r(\mathcal{U})$. Levy and Matveev [8] showed that, assuming the Continuum Hypothesis, the sequential fan is monotonically Lindelöf and there is a free ultrafilter $\mathcal{F}$ on ω such that the single ultrafilter space $\omega \cup \{\mathcal{F}\}$ is monotonically Lindelöf. It is apparently unknown if the sequential fan is monotonically Lindelöf in ZFC, or if there is any single ultrafilter space which is not monotonically Lindelöf. In this section we will show in ZFC that none of these spaces are monotonically (countably) metacompact.

We start with a lemma that will be useful for both examples.

Lemma 3.1. *Let X be a regular space, let $p \in X$, and let $\mathcal{B}_p$ be an open neighborhood base at p. Suppose that for every $\theta : \mathcal{B}_p \mapsto \mathcal{B}_p$, there is a sequence $U_0, U_1, U_2, \ldots$ in $\mathcal{B}_p$ such that, for each $n \in \omega$, $\theta(U_n) \not\subseteq U_i$ holds for all $i < n$. Then X is not monotonically countably metacompact.*

Proof. Suppose X were monotonically countably metacompact. For each $U \in \mathcal{B}_p$, choose U' open such that $p \in U' \subset \overline{U'} \subset U$. Then $\{(p, U', U) : U \in \mathcal{B}_p\} \subset \mathcal{P}_X$, so it follows from Lemma 2.1 that for each $U \in \mathcal{B}_p$, there is an open $V(U)$ with $p \in V(U) \subset U$ such that, whenever $\mathcal{Q} \subset \mathcal{B}_p$ is countable, there is a finite $\mathcal{Q}' \subset \mathcal{Q}$ such that, for every $U \in \mathcal{Q}$, there is some $Q \in \mathcal{Q}'$ with $V(U) \subset Q$. (Since p is in every $V(U)$, U', and U, the other alternatives in Lemma 2.1 do not occur.)

Define $\theta : \mathcal{B}_p \mapsto \mathcal{B}_p$ such that $\theta(U) \subset V(U)$. By assumption, there is a sequence $U_0, U_1, U_2, \ldots$ such that , for each n, $\theta(U_n) \not\subseteq U_i$ for $i < n$. Let $\mathcal{Q} = \{U_n : n \in \omega\}$, and suppose $\mathcal{Q}'$ is the finite subset of $\mathcal{Q}$ guaranteed to exist by Lemma 2.1. Let n be greater than i for any $U_i \in \mathcal{Q}'$. Then $\theta(U_n)$, and hence $V(U_n)$,is not contained in Q for any $Q = U_i \in \mathcal{Q}'$, contradiction. Thus X is not monotonically countably metacompact. □

Now let $X = (\omega \times \omega) \cup \{\infty\}$ be the sequential fan, i.e., all points in $\omega \times \omega$ are isolated, while an open neighborhood of ∞ is of the form $B(\infty, f) = \{(m,n) \in (\omega \times \omega) \mid f(m) \leq n\}$, where $f \in \omega^\omega$.

For functions $f, g : \omega \to \omega$, we say $f \leq^* g$ iff the set $\{n \in \omega : f(n) > g(n)\}$ is finite. The following lemma is surely known; for the benefit of the reader, we include its easy proof.

Lemma 3.2. *If $\mathcal{F} \subset \omega^\omega$ is $\leq^*$-unbounded, and $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$ then there exists $n_0 \in \omega$ such that $\mathcal{F}_{n_0}$ is $\leq^*$-unbounded.*

Proof. Assume not. Then for each $n \in \omega$, there exists a $g_n \in \omega^\omega$ such that for every $f \in \mathcal{F}_n$ we have $f(i) < g_n(i)$ for all sufficiently large $i \in \omega$.

Define a function $g^* \in \omega^\omega$ by $g^*(k) = \sum_{i=0}^{k} g_i(k)$

Then $g_n \leq^* g^*$ for all $n \in \omega$, and it follows that $g \leq^* g^*$ for all $g \in \mathcal{F}$, which is a contradiction. □

Proposition 3.3. *The sequential fan is not monotonically countably metacompact.*

Proof. Suppose θ maps $\{B(\infty, f) : f \in \omega^\omega\}$ to itself. Let $g_f \in \omega^\omega$ be such that $B(\infty, g_f) = \theta(B(\infty, f))$.

For each $f \in \omega^\omega$, put $f \in \mathcal{F}_n$ if $g_f(0) = n$. By Lemma 3.2, there exists an $n_0 \in \omega$ such that $\mathcal{F}_{n_0}$ is $\leq^*$-unbounded in ω^ω. For each $f \in \mathcal{F}_{n_0}$ put $f \in \mathcal{F}_{n_0,n}$ if $g_f(1) = n$. There exists a $n_1 \in \omega$ such that $\mathcal{F}_{n_0,n_1}$ is $\leq^*$-unbounded. Continue on in this way: if $\mathcal{F}_\sigma$ has been defined, where $\sigma \in \omega^{<\omega}$, let $k = |\sigma|$ and partition $\mathcal{F}_\sigma$ by placing $f \in \mathcal{F}_\sigma$ into $\mathcal{F}_{\sigma^\frown n}$ if $g_f(k) = n$, where $^\frown$ here is the concatenation operator. Then there exists a $n_k \in \omega$ such that $\mathcal{F}_{\sigma^\frown n_k}$ is $\leq^*$-unbounded. Thus we have defined $g^* = (n_0, n_1, n_2, \dots) \in \omega^\omega$ such that for each $n \in \omega$ and each $f \in \mathcal{F}_{g^*\restriction n}$, we have $g_f \restriction_n = g^* \restriction_n$ and $\mathcal{F}_{g^*\restriction n}$ is $\leq^*$-unbounded in ω^ω.

There exists a $f_0 \in \omega^\omega$ such that $f_0 \not\leq^* g^*$, or in other words, $f_0(i) > g^*(i)$ for infinitely many $i \in \omega$. Let $i_0 \in \omega$ be first such that $f_0(i_0) > g^*(i_0)$.

Now $\mathcal{F}_{g^*\restriction(i_0+1)}$ is $\leq^*$-unbounded in ω^ω, so there exists a $f_1 \in \mathcal{F}_{g^*\restriction(i_0+1)}$ such that $f_1(i) > g^*(i)$ for infinitely many $i \in \omega$. Note $g_{f_1}(i_0) = g^*(i_0) < f_0(i_0)$, whence $B(\infty, g_{f_1}) \not\subset B(\infty, f_0)$.

Let $i_1 \in \omega$ be greater than i_0 such that $f_1(i_1) > g^*(i_1)$. Choose $f_2 \in \mathcal{F}_{g^*\restriction(i_1+1)}$ such that $f_2(i) > g^*(i)$ for infinitely many $i \in \omega$. Note that $g_{f_2} \restriction (i_1 + 1) = g^* \restriction (i_1 + 1)$, so $g_{f_2}(i_0) = g^*(i_0) < f_0(i_0)$ and $g_{f_2}(i_1) = g^*(i_1) < f_1(i_1)$, whence $B(\infty, g_{f_2}) \not\subset B(\infty, f_i)$ for any $i < 2$.

Continuing in this way, we obtain a sequence $f_0, f_1, f_2, \dots$ of elements of ω^ω such that, for each $n \in \omega$, $\theta(B(\infty, f_n)) = B(\infty, g_{f_n}) \not\subset B(\infty, f_i)$ for any $i < n$. By Lemma 3.1, the sequential fan is not monotonically countably metacompact. □

Let $X = \omega \cup \{\mathcal{F}\}$ be the single ultrafilter space, where $\mathcal{F}$ is a free ultrafilter on ω. The points of ω are isolated, and a basic neighborhood of $\mathcal{F}$ is $F \cup \{\mathcal{F}\}$, where $F \in \mathcal{F}$.

The proof that the single ultrafilter space is not monotonically metacompact is quite similar to the sequential fan. First, we need an analogue of Lemma 3.2. Recall that $A \subset^* B$ means $A \setminus B$ is finite.

Lemma 3.4. *If $\mathcal{F}$ is a free ultrafilter on ω, and if $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$, then there exists $n \in \omega$ such that there does not exist an infinite $A \subset \omega$ with $A \subset^* F$ for all $F \in \mathcal{F}_n$ (i.e., $\mathcal{F}_n$ has no infinite pseudo-intersection).*

Proof. Assume not. Then for all $n \in \omega$ there exists an infinite $A_n \subset \omega$ such that $A_n \subset^* F$ for all $F \in \mathcal{F}_n$. By shrinking the A_n's if necessary, we may assume that they are pairwise disjoint. Now, for each $F \in \mathcal{F}_n$, $F \cap A_n \neq \emptyset$. Thus, if $A = \bigcup_{n\in\omega} A_n$, we have that for all $F \in \mathcal{F}$, $F \cap A \neq \emptyset$. Thus $A \in \mathcal{F}$.

For each $n \in \omega$, partition A_n into infinite sets A_n^0 and A_n^1. Consider $A^0 = \bigcup_{n\in\omega} A_n^0$ and $A^1 = \bigcup_{n\in\omega} A_n^1$. For all $F \in \mathcal{F}$, $F \cap A^0 \neq \emptyset$ and $F \cap A^1 \neq \emptyset$. So A^0 and A^1 are in $\mathcal{F}$, but $A^0 \cap A^1 = \emptyset$, contradiction. Hence the lemma holds. □

Lemma 3.5. *Suppose that $\mathcal{F}$ is a collection of sets satisfying:*

(∗) *Whenever $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$, then there exists $n \in \omega$ such that there does not exist $A \subset \omega$, $|A| = \omega$, with $A \subset^* F$ for all $F \in \mathcal{F}_n$.*

Then whenever $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$, there exists $n \in \omega$ such that $\mathcal{F}_n$ satisfies (∗).

Proof. Suppose that $\mathcal{F}$ satisfies (∗), and $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$, but that for all $n \in \omega$, $\mathcal{F}_n$ does not satisfy (∗). Then for each $n \in \omega$, $\mathcal{F}_n = \bigcup_{m\in\omega} \mathcal{F}_{n,m}$, and for each $m \in \omega$, there exists $A_m \subset \omega$, with $|A_m| = \omega$, such that $A_m \subset^* F$ for every $F \in \mathcal{F}_{n,m}$. So, $\mathcal{F} = \bigcup_{n\in\omega} \bigcup_{m\in\omega} \mathcal{F}_{n,m}$, and every $\mathcal{F}_{n,m}$ has infinite pseudointersection, which violates $\mathcal{F}$ satisfying (∗). □

Notice that a collection of sets $\mathcal{F}$ satisfying (∗) implies that $\mathcal{F}$ has no infinite pseudo-intersection.

Proposition 3.6. *The single ultrafilter space is not monotonically countably metacompact.*

Proof. Suppose θ maps $\{F \cup \{\mathcal{F}\} : F \in \mathcal{F}\}$ to itself. Let $G(F) \in \mathcal{F}$ be such that $G(F) \cup \{\mathcal{F}\} = \theta(F \cup \{\mathcal{F}\})$.

For each $n \in \omega$, let $\mathcal{F}_n = \{F \in \mathcal{F} \mid n \in G(F)\}$. Then $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$. By Lemma 3.4, $\mathcal{F}$ satisfies (∗) from Lemma 3.5, so there exists $n_0 \in \omega$ such that $\mathcal{F}_{n_0}$ satisfies (∗). Now, for each $n \in \omega \setminus \{n_0\}$, let $\mathcal{F}_{n_0,n} = \{F \in \mathcal{F}_{n_0} \mid n \in G(F)\}$. Applying Lemma 3.5 again, there is $n_1 \in \omega \setminus \{n_0\}$ such that $\mathcal{F}_{n_0,n_1}$ satisfies (∗). Having constructed $\sigma = (n_0, n_1, ..., n_k)$ such that $\mathcal{F}_\sigma$ satisfies (∗), in the same way we may find $n_{k+1} \in \omega \setminus ran(\sigma)$ such that $\mathcal{F}_{n_0,n_1,...,n_{k+1}}$ satisfies (∗).

Now let $s = (n_0, n_1, \dots)$ and let $S = ran(s)$. There exists $F_0 \in \mathcal{F}$ such that $S \not\subset^* F_0$, i.e., $S \setminus F_0$ is infinite. Let $i_0 \in \omega$ such that $n_{i_0} \in S \setminus F_0$. Then there is $F_1 \in \mathcal{F}_{n_0,\dots,n_{i_0}}$ such that $S \setminus (F_0 \cup F_1)$ is infinite (since no infinite set, in particular $S \setminus F_0$, can be a pseudo-intersection of $\mathcal{F}_{n_0,\dots,n_{i_0}}$) . Note that $G(F_1) \not\subset F_0$ since

$n_{i_0} \in G(F_1) \setminus F_0$. Let $n_{i_1} \in S \setminus (F_0 \cup F_1)$ with $i_1 > i_0$, and choose $F_2 \in \mathcal{F}_{n_0,\ldots,n_{i_1}}$ such that $S \setminus (F_0 \cup F_1 \cup F_2)$ is infinite. Then $G(F_2)$ is not a subset of F_0 or F_1.

Continuing in like manner, we get $F_0, F_1, \ldots$ in $\mathcal{F}$ such that $G(F_n) \not\subset F_i$ if $i < n$. Since $G(F_n) \cup \{\mathcal{F}\} = \theta(F_n \cup \{\mathcal{F}\})$, by Lemma 3.1 the single ultrafilter space is not monotonically countably metacompact. □

Levy and Matveev have asked whether it is consistent that every monotonically Lindelöf countable space is metrizable. Our results suggest the following related question, which could have a positive answer in ZFC.

Question 3.7. *If X is countable and monotonically (countably) metacompact, must X be metrizable?*

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