MONOTONICALLY COMPACT AND MONOTONICALLY LINDELÖF SPACES

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Abstract. We answer questions of Bennett, Lutzer, and Matveev by showing that any monotonically compact LOTS is metrizable, and any first-countable Lindelöf GO-space is monotonically Lindelöf. We also show that any compact monotonically Lindelöf space is first-countable, and is metrizable if scattered, and that separable monotonically compact spaces are metrizable.

1. Introduction

A space $X$ is monotonically compact (mC) (resp., monotonically Lindelöf (mL)) if one can assign to each open cover $U$ a finite (resp., countable) open refinement $r(U)$ such that $r(V)$ refines $r(U)$ whenever $V$ refines $U$. It is easily seen that separable metric spaces are mL and compact metric spaces are mC.

What is not yet clear is just what nonmetric spaces are mC or mL. H. Bennett, D.J. Lutzer, and M. Matveev [2] showed that separable GO-spaces, the lexicographic square, and some Suslin lines are mL, while $\omega_1 + 1$ is not mL. R. Levy and Matveev [8],[9] showed that $\beta\omega$, the one-point compactification of an uncountable discrete space, and some countable spaces are not mL, while under CH, there is a countable nonmetrizable space which is mL.

In this paper, we show that any monotonically compact linearly ordered space (LOTS) is metrizable, and any first-countable Lindelöf GO-space is monotonically Lindelöf. These results answer several questions in [2],[1], and [10]. Matveev asked in [7] whether every monotonically compact Hausdorff space is metrizable. This remains unsolved; we obtain the partial results that any compact monotonically Lindelöf space is first-countable and is metrizable if scattered, and that separable monotonically compact spaces are metrizable.

We refer the reader to [10] for an excellent survey of these properties which includes a long list of open questions.

All spaces are assumed to be regular and $T_1$.

2. Neighborhood-pair assignments

In Theorem 2.1 below, we show that mC and mL spaces have a certain property which can be phrased in terms of a kind of neighborhood-pair assignment. It may be interesting to note that every result on mC and mL spaces in this paper, with the exception of Theorem 5.1, can be derived from this neighborhood-pair assignment property.

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1A GO-space is a space which is homeomorphic to a subspace of a linearly ordered space.
Given a space $X$, let $\mathcal{P}_X$ denote the collection of all triples $p = (x^p, U_0^p, U_1^p)$ such that $U_0^p$ and $U_1^p$ are open and $x^p \in U_0^p \subset \overline{U_0^p} \subset U_1^p$.

**Theorem 2.1.** Suppose $X$ is $mC$ ($mL$). Then to each $p \in \mathcal{P}_X$, one can assign an open $V^p$ satisfying:

(i) $x^p \in V^p \subset U_1^p$;

(ii) Whenever $Q \subset \mathcal{P}_X$, there is a finite (resp., countable) $Q' \subset Q$ such that, for any $q \in Q$, there is some $q' \in Q'$ with $V^q \subset U_1^q$ or $V^q \cap U_0^q = \emptyset$.

**Proof.** For each $p \in \mathcal{P}_X$, let $U_p = \{U_1^p, X \setminus \overline{U_0^p}\}$. Let $V^p$ be any member of $r(U_p)$, where $r$ is the operator which witnesses $mC$ (resp., $mL$), which contains $x^p$. Clearly $V^p \subset U_1^p$.

Suppose $Q \subset \mathcal{P}_X$. Let $U_Q = \bigcup_{p \in Q} U_p$. Since $r(U_Q)$ refines $U_Q$, for each $O \in r(U_Q)$, there is some $q_0 \in Q$ such that $O \subset U_1^{q_0}$ or $O \cap U_0^{q_0} = \emptyset$. Let $Q' = \{q_0 : O \in r(U_Q)\}$.

Now consider $q \in Q$. Note that $U_q$ refines $U_Q$. Thus $V^q$ is a subset of some member $O$ of $r(U_Q)$. Then condition (ii) holds by taking $q' = q_0$. □

We illustrate the use of Theorem 2.1 by giving the following alternate proof of a result of Levy and Matveev.

**Theorem 2.2.** [8] The one-point compactification of an uncountable discrete space is not $mC$.

**Proof.** Let $X = \omega_1 \cup \{\infty\}$ be compact Hausdorff, where $\omega_1$ has the discrete topology. For each $\alpha < \omega_1$, the triple $p_\alpha = (\infty, X \setminus \{\alpha\}, X \setminus \{\alpha\})$ is in $\mathcal{P}_X$. Let $V^{p_\alpha}$ be as in Theorem 2.1. Then $V^{p_\alpha} = X \setminus F_\alpha$ for some finite $F_\alpha \subset \omega_1$. There is an uncountable $A \subset \omega_1$ such that $\{F_\alpha : \alpha \in A\}$ is free, i.e., $\alpha \neq \beta \in A$ implies $\alpha \neq F_\beta$ (see, e.g., Theorem A3.5 of [5]).

Let $Q = \{p_\alpha : \alpha \in A\}$, and suppose $Q'$ is the countable subset of $Q$ guaranteed by 2.1. Pick $\alpha \in A \setminus \{\beta : p_\beta \in Q'\}$. By 2.1, there must be some $p_\beta \in Q'$ such that $V^{p_\alpha} \subset X \setminus \{\beta\}$ or $V^{p_\beta} \cap X \setminus \{\beta\} = \emptyset$. The latter clearly fails, so the former must hold. However, $\beta \in V^{p_\alpha}$ (since $\beta \notin F_\alpha$) and $\beta \notin X \setminus \{\beta\}$, contradiction. □

**Remark.** Levy and Matveev also showed that the one-point Lindelöfization of a discrete space of cardinality greater than $\omega_1$ is not $mL$ [8]. This can also be obtained by an argument similar to the above.

The following is a somewhat weaker form of Theorem 2.1 that is often useful. Recall that a collection $\mathcal{A}$ of sets is **linked** if for every $A_1, A_2 \in \mathcal{A}$, $A_1 \cap A_2 \neq \emptyset$.

**Theorem 2.3.** Suppose $X$ is $mC$ ($mL$), let $Y \subset X$, and for each $y \in Y$, let $N(y)$ be an open neighborhood of $y$. Then there is an open neighborhood $M(y)$ of $y$ contained in $N(y)$ such that the following holds:

(*) If $Y' \subset Y$ and $\{M(y) : y \in Y'\}$ is linked, then there is a finite (resp., countable) $Y'' \subset Y'$ such that $Y' \subset \bigcup_{y \in Y''} N(y)$.

**Proof.** We only show the proof for $mC$, as $mL$ is similar. Let $N'(y)$ be an open neighborhood of $y$ such that $\overline{N'(y)} \subset N(y)$. Then $p(y) = (y, N'(y), N(y)) \in \mathcal{P}_X$. Let $M(y) = V^{p(y)} \cap N'(y)$, where $V^{p(y)}$ is as in Theorem 2.1. Suppose $Y' \subset Y$ and $\{M(y) : y \in Y'\}$ is linked. Suppose there is no finite $Y'' \subset Y'$ satisfying the conclusion of the theorem. Then one can find $y_0, y_1, \ldots$ in $Y'$ such that $y_n \notin \bigcup_{i<n} N(y_i)$. Let $Q = \{(y_n, N'(y_n), N(y_n)) : n \in \omega\}$. Then there must be a
If \( x \in \) compact monotonically Lindelöf spaces are first-countable. \( \alpha \in \) separable mL spaces, and under MA, every CCC space satisfies property \( K \) (see, e.g., Theorem 4.2 of [11]).

We illustrate the use of 2.3 with the following result.\(^2\)

**Theorem 2.4.** If \( X \) is mL and has property \( K \), then \( X \) is hereditarily Lindelöf.

**Proof.** If \( X \) is not hereditarily Lindelöf, it contains a right-separated subspace \( \{x_\alpha : \alpha < \omega_1\} \). For each \( \alpha < \omega_1 \), let \( N(x_\alpha) \) be an open neighborhood of \( x_\alpha \) such that \( N(x_\alpha) \cap \{x_\beta : \beta > \alpha\} = \emptyset \). Let \( M(x_\alpha) \) be as in 2.3.

Since \( X \) is has property \( K \), there must be an uncountable \( A \subset \omega_1 \) such that \( \{M(x_\alpha) : \alpha \in A\} \) is linked. Theorem 2.3 implies there is a countable \( A' \subset A \) such that \( \{x_\alpha : \alpha \in A\} \subset \bigcup_{\beta \in A} N(x_\beta) \), which is clearly impossible. \( \square \)

**Corollary 2.5.** Separable mL spaces, and under MA, CCC mL spaces, are hereditarily Lindelöf.

**Corollary 2.6.** [8] \( \beta \omega \) and \( \beta \omega \setminus \omega \) are not mL.

**Proof.** \( \beta \omega \) is separable and not hereditarily Lindelöf, and \( \beta \omega \setminus \omega \) contains a copy of \( \beta \omega \). \( \square \)

The next result improves an unpublished result of J. Vaughan, who showed that monotonically compact spaces are Fréchet.

**Theorem 2.7.** Compact monotonically Lindelöf spaces are first-countable.

**Proof.** Suppose \( X \) is compact and mL but not first-countable at \( p \). We will define a decreasing sequence \( H_\alpha, \alpha < \omega_1 \), of closed \( G_\delta \)-sets containing \( p \), and \( x_\alpha \in H_\alpha \), satisfying:

(i) \( x_\alpha \in H_\alpha \setminus H_{\alpha+1} \);  
(ii) \( \alpha \) a limit \( \Rightarrow H_\alpha = \bigcap_{\beta<\alpha} H_\beta \);  
(iii) If \( \alpha \) is a limit and there is \( x \in H_\alpha, x \neq p \), such that \( x \in \{x_\beta : \beta < \alpha\} \), then \( x_\alpha \in \{x_\beta : \beta < \alpha\} \).

To start, let \( H_0 = X \) and let \( x_0 \in H_0 \setminus \{p\} \). Suppose \( H_\beta \) and \( x_\beta \) have been defined for all \( \beta < \alpha \). If \( \alpha = \gamma + 1 \), let \( K \) be any closed \( G_\delta \)-set with \( p \in K \subset X \setminus \{x_\gamma\} \), let \( H_\alpha = H_\gamma \cap K \), and pick \( x_\alpha \in H_\alpha \setminus \{p\} \). If \( \alpha \) is a limit, let \( H_\alpha = \bigcap_{\beta<\alpha} H_\beta \). If there is \( x \in H_\alpha, x \neq p \), such that \( x \in \{x_\beta : \beta < \alpha\} \), then choose \( x_\alpha \in H_\alpha \cap \{x_\beta : \beta < \alpha\} \). Otherwise, let \( x_\alpha \) be any point of \( H_\alpha \setminus \{p\} \).

Now that we have \( x_\alpha \) and \( H_\alpha \) defined for all \( \alpha < \omega_1 \), let \( N(x_\alpha) = X \setminus H_{\alpha+1} \), and let \( M(x_\alpha) \) be as in Theorem 2.3. Let \( S = \{\alpha < \omega_1 : x_\alpha \in \{x_\beta : \beta < \alpha\}\} \).

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\(^2\)The author thanks the referee for comments which led to generalizing Theorems 2.4 and 3.1 from separable spaces to spaces having property \( K \).
Case 1. \( S \) is stationary. Then by the Pressing Down Lemma, there is \( \gamma < \omega_1 \) and an uncountable \( W \subset S \) such that \( x_\gamma \in M(x_\alpha) \) for all \( \alpha \in W \). By Theorem 2.3, there is a countable \( W' \subset W \) such that \( \{x_\alpha\}_{\alpha \in W} \subset \bigcup_{\beta \in W'} N(x_\beta) \). But if \( \alpha > \sup W' \), then \( x_\alpha \in H_{\beta+1} \) for all \( \beta \in W' \), hence \( x_\alpha \not\in N(x_\beta) \) for any \( \beta \in W' \), contradiction.

Case 2. \( S \) is nonstationary. Let \( C \) be a closed unbounded set missing \( S \). We claim that for any neighborhood \( U \) of \( p \), we have \( \{x_\alpha : \alpha \in C \} \cup \{p\} \) is the one-point compactification of an uncountable discrete space, which is not \( mL \).

Thus, by Erdos’ theorem \( \omega_1 \rightarrow (\omega, \omega_1)^2 [3] \), there is an uncountable \( W' \subset W \) that is homogeneous for \( \omega_1 \). Applying hereditarily Lindelöf again, there must be \( \gamma < \omega_1 \) such that \( \forall \mu, \nu < \omega_1 \), \( x_\mu \not\in N(x_\nu) \).

Corollary 2.8. A compact scattered monotonically Lindelöf space is metrizable.

3. MC Spaces with Property \( K \) are Metrizable

Theorem 3.1. Every monotonically compact space having property \( K \) is metrizable.

Proof. Let \( X \) be monotonically compact and have property \( K \), and suppose \( X \) is not metrizable. Choose \( x_0 \neq y_0 \in X \), and let \( U_0 \) be an open neighborhood of \( x_0 \) with \( y_0 \not\in U_0 \).

Suppose \( \alpha < \omega_1 \) and \( x_\beta, y_\beta \) and \( U_\beta \) have been chosen for each \( \beta < \alpha \). No countable collection of open sets in \( X \) can separate points even in the \( T_0 \) sense, for otherwise, since \( X \) is perfectly normal by Theorem 2.4, one sees that there would be a countable collection which separates points in the \( T_1 \) sense, making \( X \) metrizable (see, e.g., Theorem 7.6 in [4]). Thus there are points \( x_\alpha \neq y_\alpha \) in \( X \) such that if \( \beta < \alpha \), then \( U_\beta \cap \{x_\alpha, y_\alpha\} = \emptyset \) or \( U_\beta \supset \{x_\alpha, y_\alpha\} \). Now let \( U_\alpha \) be an open neighborhood of \( x_\alpha \) with \( y_\alpha \not\in U_\alpha \). This defines \( x_\alpha, y_\alpha \) and \( U_\alpha \) for each \( \alpha < \omega_1 \).

Let \( U'_\alpha \) be open such that \( x_\alpha \in U'_\alpha \subset U'_\alpha \subset U_\alpha \). Then \( p_\alpha = (x_\alpha, U'_\alpha, U_\alpha) \in \mathcal{P}_X \).

Let \( V^{p_\alpha} = V_\alpha \) be as in Theorem 2.1.

By property \( K \), there is an uncountable \( W \subset \omega_1 \) such that \( \{V_\alpha \cap U'_\alpha : \alpha \in W \} \) is linked. For \( \beta < \alpha \in W \), put \( (\alpha, \beta) \) in Pot I if \( V_\alpha \not\subset U_\beta \); otherwise, put \( (\alpha, \beta) \) in Pot II.

We claim that there is no infinite subset \( A \) of \( W \) which is homogeneous for Pot I. Suppose there were. We may assume \( A \) has order type \( \omega \). By Theorem 2.1, there is a finite \( A' \subset A \) such that, for any \( \alpha \in A \), there is some \( \beta \in A' \) such that \( V_\alpha \subset U_\beta \) or \( V_\alpha \cap U'_\beta = \emptyset \). Since the latter never holds for \( \alpha, \beta \in W \), \( V_\alpha \subset U_\beta \) must hold. But then choosing \( \alpha \in A \) with \( \alpha > \beta \) for every \( \beta \in A' \) yields a contradiction.

Thus, by Erdos’ theorem \( \omega_1 \rightarrow (\omega, \omega_1)^2 [3] \), there is an uncountable \( W' \subset W \) that is homogeneous for Pot II; that is, \( \beta < \alpha \in W' \) implies \( U_\beta \supset V_\alpha \). Applying hereditarily Lindelöf again, there must be \( \gamma < \omega_1 \) such that \( \forall \mu, \nu < \omega_1 \), \( x_\mu \not\in N(x_\nu) \).
Let $\beta$ be the least element of $W' \setminus \gamma$. Then

$$y_\beta \in \bigcup \{x_\alpha, y_\alpha\}_{\alpha \in W \setminus \beta} = \bigcup \{x_\alpha, y_\alpha\}_{\alpha \in W \setminus (\beta + 1)}.$$  

Now for each $\alpha \in W \setminus (\beta + 1)$, we have $U_\beta \supseteq V_\alpha \ni x_\alpha$, hence $\{x_\alpha, y_\alpha\} \subset U_\beta$. Thus $U_\beta \supset \bigcup \{x_\alpha, y_\alpha\}_{\alpha \in W \setminus (\beta + 1)}$, and so $\overline{U_\beta} \supset \bigcup \{x_\alpha, y_\alpha\}_{\alpha \in W \setminus (\beta + 1)}$, which puts $y_\beta \in \overline{U_\beta}$, contradiction. \qed

**Corollary 3.2.** Separable $mC$ spaces, and under $MA_{\omega_1}$, CCC $mC$ spaces, are metrizable.

4. Monotonically compact LOTS

The following result answers Question 10 in [1].

**Theorem 4.1.** A monotonically compact LOTS is metrizable.

**Proof.** Suppose $X$ is a LOTS which is monotonically compact. Let $T$ be the collection of closed intervals of $X$. Inductively define a map $C : 2^{<\omega_1} \to T$ as follows: Let $C(\emptyset) = X$. Suppose $C(\sigma)$ has been defined for all $\sigma \in 2^{<\alpha}$, where $\alpha < \omega_1$. If $\alpha$ is a limit ordinal and $\tau \in 2^\alpha$, let $C(\tau) = \bigcap_{\beta < \alpha} C(\tau \upharpoonright \beta)$. If $\alpha = \beta + 1$, let $I = C(\tau \upharpoonright \beta)$. If $I$ is a singleton or doubleton, let $C(\tau) = I$. Otherwise, let $a < b$ be the endpoints of $I$, and choose a point $c$ between $a$ and $b$. Then let $C(\tau) = [a, c]$ if $\tau(\beta) = 0$ and $C(\tau) = [c, b]$ if $\tau(\beta) = 1$.

Let $T$ be all elements $t$ of the range of $C$ that have more than 2 points, or exactly two points but every predecessor has more than two points. Then $T$ is a tree under reverse inclusion, and the $\alpha^\text{th}$ level of $T$ is all members of $C(2^\alpha)$ in $T$. Note that any two members of $T$ which are incomparable in the tree order are either disjoint or meet at an endpoint.

**Claim 1.** Every branch of $T$ is countable. If $T$ had an uncountable branch, then $X$ would contain a well-ordered increasing or decreasing sequence in type $\omega_1$, hence a closed copy of $\omega_1 + 1$. But $\omega_1 + 1$ is not monotonically compact $[2]$.

**Claim 2.** Every level of $T$ is countable. Suppose not, and let $\alpha$ be least such that the $\alpha^\text{th}$ level $L_\alpha$ of $T$ is uncountable. Note that $\alpha$ must be a limit ordinal. For each $I = [a_I, b_I] \in L_\alpha$, let $N(a_I) = (-\infty, b_I)$ and $N(b_I) = (a_I, \infty)$. Let $M(a_I)$ and $M(b_I)$ be as guaranteed by Theorem 2.3.

Let $\sigma_I \in 2^{\alpha}$ be such that $I = C(\sigma_I)$. Since $C(\sigma_I) = \bigcap_{\beta < \alpha} C(\sigma_I \upharpoonright \beta)$, there is $\beta_I < \alpha$ such that $C(\sigma_I \upharpoonright \beta_I) \subset M(a_I) \cup I \cup M(b_I)$. Note that $C(\sigma_I \upharpoonright \beta_I)$ is a member of $T$ below level $\alpha$. Since by assumption there are only countably many such, there are $t \in T$ and an infinite subset $J$ of $L_\alpha$ such that $C(\sigma_I \upharpoonright \beta_I) = t$ for every $I \in J$. For $J, J' \in J$, define $J < J'$ iff $a_J < a_{J'}$. There is a sequence $J_0, J_1, \ldots$ in $J$ which is either increasing or decreasing. W.l.o.g., suppose it is increasing. Then $\bigcap_{i \in \omega} M(a_{J_i})$ contains the left endpoint of $t$, but no finite subcollection of $\{N(a_{J_i}) : i \in \omega\}$ covers $\{a_{J_i} : i \in \omega\}$. This contradicts 2.3.

**Claim 3.** $T$ is countable. Suppose $T$ is uncountable. By Claim 2, it has uncountable height. For each countable limit ordinal $\alpha$, choose $t_\alpha = [a_\alpha, b_\alpha]$ in the $\alpha^\text{th}$ level of $T$, let $N(a_\alpha) = (-\infty, b_\alpha)$ and $N(b_\alpha) = (a_\alpha, \infty)$, and let $M(a_\alpha), M(b_\alpha)$ be as guaranteed by Theorem 2.3.

As in the proof of Claim 2, there is a predecessor $s_\alpha$ of $t_\alpha$ in $T$ such that $s_\alpha \subset M(a_\alpha) \cup t_\alpha \cup M(b_\alpha)$. By the Pressing Down Lemma, there is an uncountable $W' \subset \omega_1$ such that the level of $s_\alpha$ is the same ordinal for every $\alpha \in W'$. Then
since each level is countable, there are \( s \in T \) and uncountable \( W \subset W' \) such that 
\( s_\alpha = s \) for each \( \alpha \in W \). For \( \alpha < \beta \in W \), put \( \{ \alpha, \beta \} \) in Pot II if \( t_\beta \subset t_\alpha \), otherwise put \( \{ \alpha, \beta \} \) in Pot I. Since there is no uncountable chain in \( T \), it follows from the partition relation \( \omega_1 \to (\omega_1, \omega_1)^2 \) that there is an infinite \( A \subset W \) homogeneous for Pot I; i.e., for distinct \( \alpha, \beta \in A \), \( t_\alpha \) and \( t_\beta \) are incomparable, and hence viewed as intervals are either disjoint or meet at an endpoint. As in the proof of Claim 2, there is a sequence \( t_{\alpha_0}, t_{\alpha_1}, \ldots \) that is either increasing or decreasing in the order \( s < t \) if \( s < \inf t \), and the rest of the argument is finished as in Claim 2.

Now we complete the proof of the theorem. Note that for each \( x \in X \), there is a branch of \( T \) consisting of a decreasing well-ordered sequence of intervals containing \( x \) which either intersect to \( x \), or there is a last member of the branch which has exactly two points, one of which is \( x \). Thus, if to \( T \) we add the points of the 2-element members of \( T \), we obtain a countable network for \( X \). But a compact space with the countable network is metrizable. \( \square \)

5. First countable Lindelöf GO-spaces

The following result improves the result in [2] that separable GO-spaces are mL, and answers several questions in [2], [1], and [10]. In particular, it shows that any Suslin line is mL.

**Theorem 5.1.** Every first countable Lindelöf GO-space is monotonically Lindelöf.

**Proof.** Let \( Y \) be a first-countable Lindelöf GO-space. Then \( Y \) is a dense subspace of a compact LOTS \( X[6] \). Note that \( X \) is first-countable. To see this, if not, then \( X \) would contain a copy of \( \omega_1 + 1 \). The point in \( X \) corresponding to the point \( \omega_1 \) could not be in \( Y \) by first countability of \( Y \). But on the other hand, this point must be in \( Y \) else \( Y \) would not be Lindelöf.

Construct a tree of open intervals of \( X \) as follows.

Let \( X = [a, b] \), \( D_0 = \{a, b\} \), and \( L_0 = \{(a, b)\} \). Let \( t = (a, b) \). Let \( \Delta_i(a) \) be \( \{a', \beta\} \), where \( a' \) is the immediate successor of \( a \) if such exists, else \( \Delta_i(a) \) is a sequence in \( t \) converging to \( a \) from the right. Define \( \Delta_i(b) \) similarly, making sure \( sup(\Delta_i(a)) < inf(\Delta_i(b)) \) if possible, which it is unless \( t \) contains only one point \( c \), in which case \( \Delta_i(a) = \Delta_i(b) = \{c\} \). Then let \( D_i = \Delta_i(a) \cup \Delta_i(b) \).

Let \( D_1 = D_t \) where \( t = (a, b) \) is the only member of \( L_0 \), and let \( L_1 \) be the (nonempty) convex components of \( X \setminus D_1 \cup \{a, b\} \).

Suppose a disjoint collection \( L_\beta \) of open intervals of \( X \), and a set \( D_\beta \) of points of \( X \), has been constructed for \( \beta < \alpha \), where \( \alpha < \omega_1 \), satisfying:

(i) If \( \beta < \gamma < \alpha \) and \( t \in L_\gamma \), there is \( s \in L_\beta \) with \( \bar{t} \subset s \);
(ii) If \( t = (a_t, b_t) \in L_\beta \), then \( D_{\beta+1} \cap t = D_\beta(a_t) \cup D_\beta(b_t) \), where \( D_\beta(a_t) \) and \( D_\beta(b_t) \) are countable relatively closed discrete subsets \( (a_t, b_t) \) defined similarly to the case \( t = (a, b) \in L_0 \) above. \( D_{\beta+1} = \bigcup_{t \in L_\beta} D_t \).
(iii) \( L_{\beta+1} = \{s : \exists t \in L_\beta(s \text{ is a convex component of } t \setminus D_t)\} \);
(iv) \( X \setminus \bigcup L_\beta = \bigcup_{\gamma \leq \beta} D_\gamma \cup \{x : |x \cap B_x| \leq 2\} \), where \( B_x \) is the unique branch of \( \bigcup_{\gamma < \beta} L_\gamma \) consisting of those intervals which contain \( x \).

Note that \( T_\alpha = \bigcup_{\beta < \alpha} L_\beta \) is a tree under reverse inclusion, with \( L_\beta \) the \( \beta^{th} \) level. Define \( L_\alpha \) and \( D_\alpha \) as follows. If \( \alpha = \beta + 1 \), then define \( D_\alpha \) and \( L_\alpha \) following (ii) above. If \( \alpha \) is a limit, then for every branch \( B \) in \( T_\alpha \) such that \( \cap B \) is an interval \([a, b]\) with \( (a, b) \neq \emptyset \), put \( (a, b) \) in \( L_\alpha \) and \( a, b \) in \( D_\alpha \). This completes the construction.
of the tree $T = \bigcup_{n<\omega_1} L_n$. Note that $T$ has no uncountable chains (else $X$ would contain a copy of $\omega_1 + 1$).

We will use $<_T$ for the tree order; so, if $s, t \in T$, then $s <_T t$ iff $s$ contains $t$ when viewed as subsets of $X$.

Now we describe a base at points in the $D_\alpha$’s, for $\alpha$ a successor. Let $x \in D_\alpha$, $\alpha = \beta + 1$. Then there is some $t = (a, b) \in L_\beta$ such that $x \in (a, b)$. Suppose $x \in D_t(a)$. If $x$ has an immediate predecessor $y$ (it could be, but is not necessarily, the case that $y = a$), let $P(x) = \{y\}$. Otherwise, there is a point $p$ in $D_t(a)$ just to the left of $x$, $s = (p, x) \in L_{\alpha + 1}$, and $D_s(x)$ is a sequence in $(p, x)$ converging to $x$. In this case, let $P(x) = D_\alpha(x)$. Define $R(x)$, a set of points to the right of $x$, similarly. Let $B(x)$ be all open intervals containing $x$ with endpoints in $P(x) \cup R(x)$. Then $B(x)$ is clearly a base at $x$. Define $B(x)$ for $x \in D_t(b)$ similarly. Note that it follows from this definition that if $x \neq y$, $x, y \in D_{\alpha + 1}$, $B_x \subset B(x)$, and $B_y \subset B(y)$, then $B_x \cap B_y = \emptyset$.

If $\mathcal{A}$ is a collection of subsets of $X$, and $Z \subset X$, then we let $\mathcal{A} \mid Z$ denote the collection $\{A \cap Z : A \in \mathcal{A}\}$. If $\mathcal{A}'$ is another collection of subsets of $X$, we write $\mathcal{A}' \prec \mathcal{A}$ to mean $\mathcal{A}'$ refines $\mathcal{A}$. Also, if $C \subset X$, we write $C \prec \mathcal{A}$ to mean $C \subset A$ for some $A \in \mathcal{A}$.

Now we proceed to define $r(U)$.

Given an open cover $U$ of $Y$, first define $r'(U)$ as follows.

(I) Put in $r'(U)$ all members of $\bigcup\{B(x) \mid Y : \exists \alpha (x \in D_{\alpha + 1} \cap Y)\}$ which are contained in some member of $U$.

(II) For each $t = (a_t, b_t)$ in $T$, let $r_t = \sup\{r \in t : (a_t, r) \cap Y < U\}$, and put $(a_t, r_t) \cap Y$ in $r'(U)$.

(III) Similarly define $l_t = \inf\{l \in t : (l, b_t) \prec U\}$ and and put each $(l_t, b_t) \cap Y$ in $r'(U)$.

Note that each member of $r'(U)$ is convex, and hence can be represented by an open interval $(a, b)$, where $a, b \in X \cup \{\pm \infty\}$.

Let $r_m(U)$ be the maximal members (under set inclusion) of $r'(U)$. Suppose $(a, b) \in r_m(U)$. If $a$ has no immediate successor, choose points $a(n)$ in $(a, b)$ converting to $a$ from the right, else let $a(n) = a$ for all $n$. Choose $b(n)$ converging to $b$ similarly. We can do this so that $a(n) < b(n)$ for all $n$. Finally, let $r(U) = \{(a(n), b(n)) : n \in \omega, (a, b) \in r_m(U)\}$.

**Claim 1.** If $x \neq y \in Y$, $x \in D_{\alpha + 1}$, $y \in D_{\beta + 1}$, $B_x \subset B(x)$, $B_y \subset B(y)$, and $B_x \cap Y \subset B_y \cap Y$, then $\beta < \alpha$. This holds because if $\alpha = \beta$ then $B_x \cap B_y = \emptyset$ by the construction, and if $\alpha < \beta$ then $B_y \cap D_{\alpha + 1} = \emptyset$ so $x \notin B_y$.

**Claim 2.** If $(a_t, r_t) \cap Y$ is a nonempty proper subset of $(a_s, r_s) \cap Y$ or if $(l_t, b_t) \cap Y$ is a nonempty proper subset of $(l_s, b_s) \cap Y$, then $s <_T t$. We show this if the former case holds, the latter being analogous. Clearly $s$ and $t$ must be comparable. Suppose $t <_T s$. Then $a_t < a_s < r_s \leq b_s \leq b_t$. Then the only way $(a_t, r_t) \cap Y$ could be a nonempty proper subset of $(a_s, r_s) \cap Y$ is if $(a_t, a_s) \cap Y = \emptyset$. But then $(a_s, r_s) \cap Y = (a_t, r_t) \cap Y$ for all $r$, so $r_t = r_s$ and $(a_t, r_t) \cap Y = (a_s, r_s) \cap Y$, contradiction.

**Claim 3.** Every member of $r'(U)$ is contained in a maximal member of $r'(U)$. Immediate from Claims 1 and 2.

**Claim 4.** $\cup r'(U) = \cup r_m(U) = \cup r'(U)$. The second equality is immediate from Claim 3, and the first is easy to see from the way $r(U)$ was defined from $r_m(U)$. 
Claim 5. $r(U)$ is a refinement of $U$ covering $X$. Suppose $(a(n), b(n)) \in r(U)$. Then $(a, b) \in r_m(U)$. If $(a, b)$ got in $r'(U)$ via (I), then $(a, b)$ and hence also $(a(n), b(n))$ is contained in some member of $U$. Suppose $(a, b) = (a_t, r_t)$ for some $t \in T$. If $b(n) < b$, then by the definition of $r_t$ we have that $(a(n), b(n))$ is a subset of some member of $U$. If $b(n) = b$, it has to be because $b = r_t$ has an immediate predecessor, which implies that $r_t$ is in the set it is defined to be the supremum of, and so $(a(n), b(n)) \subset (a_t, r_t) \subset U$.

To finish the proof of Claim 5, we need to show $r(U)$ covers $X$. Let $x \in X$. If $x \in D_{\alpha+1}$ for some $\alpha$, then $x$ is in some member of $r'(U)$ via I, and then by Claim 4, $x \in \cup r(U)$.

Suppose $x \notin \bigcup_{\alpha<\omega_1} D_{\alpha+1}$. Let $C_x$ be the collection of all elements of $T$ containing $x$; of course this is a well-ordered chain of some length $\alpha < \omega_1$. Note that $\alpha$ is a limit ordinal. If $C_x$ had a $<_T$-maximal element $t$ at level $\beta = \alpha - 1$, then since $x \notin D_t$, $x$ would be in some convex component $s$ of $t \setminus D_t$, contradicting maximality of $t \in C_x$. Then either $x$ is the unique element of $\cap C_x$ and the elements of $C_x$ are a base at $x$, or $\cap C_x$ is a nontrivial closed interval, and $x$ is one of the endpoints. Let $x \in U \in U$. If the former case holds, or the latter case holds and $x$ is the left endpoint of $\cap C_x$, there is some $t \in C_x$ and some $r > x$ with $(a_t, r) \in U$. It follows that $r_t > x$, and hence $x$ is in some member of $r'(U)$ via II. If $x$ were the right endpoint, a similar argument shows that $x$ is in some member of $r'(U)$ via III.

Claim 6. $V < U$ implies $r(V) < r(U)$. Let $(a(n), b(n)) \cap Y \in r(U)$. Then $(a, b) \cap Y \in r_m(U)$ with $(a, b) \in (c, d)$, because then for some $m$, $(a(n), b(n)) \cap Y \subset (c(m), d(m)) \cap Y \in r(U)$. Now if $(a, b) \cap Y$ got in $r'(U)$ via I, then $(a, b) \cap Y \subset V < U$, so $(a, b) \cap Y \in r'(U)$. Thus there is $(c, d) \cap Y \in r_m(U)$ with $(a, b) \cap Y \subset (c, d) \cap Y$.

Suppose $(a, b) \cap Y = (a_t, r_t) \cap Y$ for some $t \in T$, where the superscript $V$ means $r_t$ has been defined as in II with respect to the open cover $V$. Clearly $r_t^U \geq r_t^V$. Thus $(a_t, r_t) \cap Y \subset (a_t, r_t^U) \cap Y \in r'(U)$, and so there is some $(c, d) \cap Y \in r_m(U)$ containing $(a, b) \cap Y$. The case where $(a, b) = (l_t^U, b_t)$ is analogous.

Claim 7. $r(U)$ is countable. It suffices to prove $r_m(U)$ is countable. Suppose $(a_\alpha, b_\alpha) \cap Y$, $\alpha < \omega_1$, are distinct elements of $r_m(U)$. Since $Y$ is Lindelöf, there is $x \in Y$ such that every neighborhood of $x$ meets uncountably many $(a_\alpha, b_\alpha) \cap Y$’s.

There is $(a, b) \cap Y \in r_m(U)$ with $x \in (a, b)$. If $(a, b) \cap Y$ meets $(a_\alpha, b_\alpha) \cap Y$, then by maximality of these sets, it must be that either $a$ or $b$ is in $(a_\alpha, b_\alpha)$. If $(a_\alpha, b_\alpha)$ is in $B(x)$ for some $x \in D_{\alpha+1}$, let $t_\alpha$ be such that $x \in D_{t_\alpha}$. If $(a_\alpha, b_\alpha)$ is equal to $(a_t, r_t)$ or $(l_t, b_t)$, let $t_\alpha = t$. Note that $(a_\alpha, b_\alpha) \subset t_\alpha$ for each $\alpha$, and since $D_t$ is countable, for each $t \in T$, there are only countably many $\alpha < \omega_1$ such that $t_\alpha = t$. Hence we conclude that either $a$ or $b$ must be in uncountably many distinct $t \in T$, a contradiction since $T$ has no uncountable chains. 

References


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