MONOTONICALLY COMPACT AND MONOTONICALLY LINDELÖF SPACES

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Abstract. We answer questions of Bennett, Lutzer, and Matveev by showing that any monotonically compact $LOTS$ is metrizable, and any first-countable Lindelöf GO-space is monotoncically Lindelöf. We also show that any compact monotonically Lindelöf space is first-countable, and is metrizable if scattered, and that separable monotonically compact spaces are metrizable.

1. Introduction

A space X is *monotonically compact* (mC)(resp., *monotonically Lindelöf* (mL)) if one can assign to each open cover $\mathcal{U}$ a finite (resp., countable) open refinement $r(\mathcal{U})$ such that $r(\mathcal{V})$ refines $r(\mathcal{U})$ whenever $\mathcal{V}$ refines $\mathcal{U}$. It is easily seen that separable metric spaces are mL and compact metric spaces are mC.

What is not yet clear is just what nonmetric spaces are mC or mL. H. Bennett, D.J. Lutzer, and M. Matveev [2] showed that separable GO-spaces[1], the lexicographic square, and some Suslin lines are mL, while $\omega_1 + 1$ is not mL. R. Levy and Matveev [8],[9] showed that $\beta\omega$, the one-point compactification of an uncountable discrete space, and some countable spaces are not mL, while under CH, there is a countable nonmetrizable space which is mL.

In this paper, we show that any monotonically compact linearly ordered space ($LOTS$) is metrizable, and any first-countable Lindelöf GO-space is monotoncically Lindelöf. These results answer several questions in [2],[1], and [10]. Matveev asked in [7] whether every monotonically compact Hausdorff space is metrizable. This remains unsolved; we obtain the partial results that any compact monotonically Lindelöf space is first-countable and is metrizable if scattered, and that separable monotonically compact spaces are metrizable.

We refer the reader to [10] for an excellent survey of these properties which includes a long list of open questions.

All spaces are assumed to be regular and T_1.

2. Neighborhood-pair assignments

In Theorem 2.1 below, we show that mC and mL spaces have a certain property which can be phrased in terms of a kind of neighborhood-pair assignment. It may be interesting to note that every result on mC and mL spaces in this paper, with the exception of Theorem 5.1, can be derived from this neighborhood-pair assignment property.

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[1]A *GO-space* is a space which is homeomorphic to a subspace of a linearly ordered space.

Given a space X, let $\mathcal{P}_X$ denote the collection of all triples $p = (x^p, U_0^p, U_1^p)$ such that U_0^p and U_1^p are open and $x^p \in U_0^p \subset \overline{U_0^p} \subset U_1^p$.

Theorem 2.1. *Suppose X is mC (mL). Then to each $p \in \mathcal{P}_X$, one can assign an open V^p satisfying:*

(i) $x^p \in V^p \subset U_1^p$;

(ii) *Whenever $\mathcal{Q} \subset \mathcal{P}_X$, there is a finite (resp., countable) $\mathcal{Q}' \subset \mathcal{Q}$ such that, for any $q \in \mathcal{Q}$, there is some $q' \in \mathcal{Q}'$ with $V^q \subset U_1^{q'}$ or $V^q \cap U_0^{q'} = \emptyset$.*

Proof. For each $p \in \mathcal{P}_X$, let $\mathcal{U}_p = \{U_1^p, X \setminus \overline{U_0^p}\}$. Let V^p be any member of $r(\mathcal{U}_p)$, where r is the operator which witnesses mC (resp., mL), which contains x^p. Clearly $V^p \subset U_1^p$.

Suppose $\mathcal{Q} \subset \mathcal{P}_X$. Let $\mathcal{U}_\mathcal{Q} = \bigcup_{q \in \mathcal{Q}} \mathcal{U}_q$. Since $r(\mathcal{U}_\mathcal{Q})$ refines $\mathcal{U}_\mathcal{Q}$, for each $O \in r(\mathcal{U}_\mathcal{Q})$, there is some $q_O \in \mathcal{Q}$ such that $O \subset U_1^{q_O}$ or $O \cap U_0^{q_O} = \emptyset$. Let $\mathcal{Q}' = \{q_O : O \in r(\mathcal{U}_\mathcal{Q})\}$.

Now consider $q \in \mathcal{Q}$. Note that $\mathcal{U}_q$ refines $\mathcal{U}_\mathcal{Q}$. Thus V^q is a subset of some member O of $r(\mathcal{U}_\mathcal{Q})$. Then condition (ii) holds by taking $q' = q_O$. □

We illustrate the use of Theorem 2.1 by giving the following alternate proof of a result of Levy and Matveev.

Theorem 2.2. [8] *The one-point compactification of an uncountable discrete space is not mL.*

Proof. Let $X = \omega_1 \cup \{\infty\}$ be compact Hausdorff, where ω_1 has the discrete topology. For each $\alpha < \omega_1$, the triple $p_\alpha = (\infty, X \setminus \{\alpha\}, X \setminus \{\alpha\})$ is in $\mathcal{P}_X$. Let V^{p_α} be as in Theorem 2.1. Then $V^{p_\alpha} = X \setminus F_\alpha$ for some finite $F_\alpha \subset \omega_1$. There is an uncountable $A \subset \omega_1$ such that $\{F_\alpha : \alpha \in A\}$ is free, i.e., $\alpha \neq \beta \in A$ implies $\alpha \notin F_\beta$ (see, e.g., Theorem A3.5 of [5]).

Let $\mathcal{Q} = \{p_\alpha : \alpha \in A\}$, and suppose $\mathcal{Q}'$ is the countable subset of $\mathcal{Q}$ guaranteed by 2.1. Pick $\alpha \in A \setminus \{\beta : p_\beta \in \mathcal{Q}'\}$. By 2.1, there must be some $p_\beta \in \mathcal{Q}'$ such that $V^{p_\alpha} \subset X \setminus \{\beta\}$ or $V^{p_\alpha} \cap X \setminus \{\beta\} = \emptyset$. The latter clearly fails, so the former must hold. However, $\beta \in V^{p_\alpha}$ (since $\beta \notin F_\alpha$) and $\beta \notin X \setminus \{\beta\}$, contradiction. □

Remark. Levy and Matveev also showed that the one-point Lindelöfization of a discrete space of cardinality greater than ω_1 is not mL [8]. This can also be obtained by an argument similar to the above.

The following is a somewhat weaker form of Theorem 2.1 that is often useful. Recall that a collection $\mathcal{A}$ of sets is *linked* if for every $A_1, A_2 \in \mathcal{A}$, $A_1 \cap A_2 \neq \emptyset$.

Theorem 2.3. *Suppose X is mC (mL), let $Y \subset X$, and for each $y \in Y$, let $N(y)$ be an open neighborhood of y. Then there is an open neighborhood $M(y)$ of y contained in $N(y)$ such that the following holds:*

($*$) *If $Y' \subset Y$ and $\{M(y) : y \in Y'\}$ is linked, then there is a finite (resp., countable) $Y'' \subset Y'$ such that $Y' \subset \bigcup_{y \in Y''} N(y)$.*

Proof. We only show the proof for mC, as mL is similar. Let $N'(y)$ be an open neighborhood of y such that $\overline{N'(y)} \subset N(y)$. Then $p(y) = (y, N'(y), N(y)) \in \mathcal{P}_X$. Let $M(y) = V^{p(y)} \cap N'(y)$, where $V^{p(y)}$ is as in Theorem 2.1. Suppose $Y' \subset Y$ and $\{M(y) : y \in Y'\}$ is linked. Suppose there is no finite $Y'' \subset Y'$ satisfying the conclusion of the theorem. Then one can find $y_0, y_1, \ldots$ in Y' such that $y_n \notin \bigcup_{i<n} N(y_i)$. Let $\mathcal{Q} = \{(y_n, N'(y_n), N(y_n)) : n \in \omega\}$. Then there must be a

finite subset $\mathcal{Q}'$ of $\mathcal{Q}$ satisfying the conclusion of 2.1. Let $n \in \omega$ be such that $(y_i, N'(y_i), N(y_i)) \in \mathcal{Q}$ implies $i < n$. By 2.1, there must be $i < n$ such that $V^{p(y_n)} \subset N(y_i)$ or $V^{p(y_n)} \cap N'(y_i) = \emptyset$. But $y_n \in V^{p(y_n)} \setminus N(y_i)$, so the former does not hold, while $V^{p(y_n)} \cap N'(y_i) \supset M(y_n) \cap M(y_i) \neq \emptyset$, so the latter doesn't either. □

Recall that a space X has *property K* if every uncountable collection of open sets contains an uncountable linked subcollection. It is well-known and easy to see that separable implies property K implies CCC. It is also well-known that under $MA_{\aleph_1}$, every CCC space satisfies property K (see, e.g., Theorem 4.2 of [11]).

We illustrate the use of 2.3 with the following result.[2]

Theorem 2.4. *If X is mL and has property K, then X is hereditarily Lindelöf.*

Proof. If X is not hereditarily Lindelöf, it contains a right-separated subspace $\{x_\alpha : \alpha < \omega_1\}$. For each $\alpha < \omega_1$, let $N(x_\alpha)$ be an open neighborhood of x_α such that $N(x_\alpha) \cap \{x_\beta : \beta > \alpha\} = \emptyset$. Let $M(x_\alpha)$ be as in 2.3.

Since X is has property K, there must be an uncountable $A \subset \omega_1$ such that $\{M(x_\alpha) : \alpha \in A\}$ is linked. Theorem 2.3 implies there is a countable $A' \subset A$ such that $\{x_\alpha : \alpha \in A\} \subset \bigcup_{\beta \in A'} N(x_\beta)$, which is clearly impossible. □

Corollary 2.5. *Separable mL spaces, and under MA_{ω_1}, CCC mL spaces, are hereditarily Lindelöf.*

Corollary 2.6. [8] *$\beta\omega$ and $\beta\omega \setminus \omega$ are not mL.*

Proof. $\beta\omega$ is separable and not hereditarily Lindelöf, and $\beta\omega \setminus \omega$ contains a copy of $\beta\omega$. □

The next result improves an unpublished result of J. Vaughan, who showed that monotonically compact spaces are Fréchet.

Theorem 2.7. *Compact monotonically Lindelöf spaces are first-countable.*

Proof. Suppose X is compact and mL but not first-countable at p. We will define a decreasing sequence H_α, $\alpha < \omega_1$, of closed G_δ-sets containing p, and $x_\alpha \in H_\alpha$, satisfying:

(i) $x_\alpha \in H_\alpha \setminus H_{\alpha+1}$;
(ii) α a limit $\Rightarrow H_\alpha = \bigcap_{\beta<\alpha} H_\beta$;
(iii) If α is a limit and there is $x \in H_\alpha$, $x \neq p$, such that $x \in \overline{\{x_\beta : \beta < \alpha\}}$, then $x_\alpha \in \overline{\{x_\beta : \beta < \alpha\}}$.

To start, let $H_0 = X$ and let $x_0 \in H_0 \setminus \{p\}$. Suppose H_β and x_β have been defined for all $\beta < \alpha$. If $\alpha = \gamma + 1$, let K be any closed G_δ-set with $p \in K \subset X \setminus \{x_\gamma\}$, let $H_\alpha = H_\gamma \cap K$, and pick $x_\alpha \in H_\alpha \setminus \{p\}$. If α is a limit, let $H_\alpha = \bigcap_{\beta<\alpha} H_\beta$. If there is $x \in H_\alpha$, $x \neq p$, such that $x \in \overline{\{x_\beta : \beta < \alpha\}}$, then choose $x_\alpha \in H_\alpha \cap \overline{\{x_\beta : \beta < \alpha\}}$. Otherwise, let x_α be any point of $H_\alpha \setminus \{p\}$.

Now that we have x_α and H_α defined for all $\alpha < \omega_1$, let $N(x_\alpha) = X \setminus H_{\alpha+1}$, and let $M(x_\alpha)$ be as in Theorem 2.3. Let

$$S = \{\alpha < \omega_1 : x_\alpha \in \overline{\{x_\beta : \beta < \alpha\}}\}.$$

[2]The author thanks the referee for comments which led to generalizing Theorems 2.4 and 3.1 from separable spaces to spaces having property K.

Case 1. S is stationary. Then by the Pressing Down Lemma, there is $\gamma < \omega_1$ and an uncountable $W \subset S$ such that $x_\gamma \in M(x_\alpha)$ for all $\alpha \in W$. By Theorem 2.3, there is a countable $W' \subset W$ such that $\{x_\alpha\}_{\alpha \in W} \subset \bigcup_{\beta \in W'} N(x_\beta)$. But if $\alpha > sup\ W'$, then $x_\alpha \in H_{\beta+1}$ for all $\beta \in W'$, hence $x_\alpha \notin N(x_\beta)$ for any $\beta \in W'$, contradiction.

Case 2. S is nonstationary. Let C be a closed unbounded set missing S. We claim that for any neighborhood U of p, we have $\{x_\alpha : \alpha \in C\} \setminus U$ is finite. Note that, once we prove this claim, we are done, since then $\{x_\alpha : \alpha \in C\} \cup \{p\}$ is the one-point compactification of an uncountable discrete space, which is not mL.

Suppose U is an open neighborhood of p such that $\{x_\alpha : \alpha \in C\} \setminus U$ is infinite. Let $\delta < \omega_1$ be least such that $x_\alpha \notin U$ for infinitely many $\alpha \in C \cap \delta$. Then there are $\alpha_n \in C$ with $x_{\alpha_n} \notin U$, $\alpha_0 < \alpha_1 < \dots$, and $\delta = sup_{n \in \omega} \alpha_n$. Since $x_{\alpha_n} \in H_{\alpha_n}$ and $H_{\alpha_0}, H_{\alpha_1}, \dots$ is a decreasing sequence of closed sets whose intersection is H_δ, there must be a point x in $H_\delta \cap \overline{\{x_{\alpha_n} : n \in \omega\}}$. Since $\delta \in C$ and $C \cap S = \emptyset$, the only limit point of $\{x_{\alpha_n} : n \in \omega\}$ is p, contradicting $\{x_{\alpha_n} : n \in \omega\} \cap U = \emptyset$. □

Since first-countable compact scattered spaces are metrizable, we have the following corollary.

Corollary 2.8. *A compact scattered monotonically Lindelöf space is metrizable.*

3. mC spaces with property K are metrizable

Theorem 3.1. *Every monotonically compact space having property K is metrizable.*

Proof. Let X be monotonically compact and have property K, and suppose X is not metrizable. Choose $x_0 \neq y_0 \in X$, and let U_0 be an open neighborhood of x_0 with $y_0 \notin \overline{U_0}$.

Suppose $\alpha < \omega_1$ and x_β, y_β and U_β have been chosen for each $\beta < \alpha$. No countable collection of open sets in X can separate points even in the T_0 sense, for otherwise, since X is perfectly normal by Theorem 2.4, one sees that there would be a countable collection which separates points in the T_1 sense, making X metrizable (see, e.g., Theorem 7.6 in [4]). Thus there are points $x_\alpha \neq y_a$ in X such that if $\beta < \alpha$, then $U_\beta \cap \{x_\alpha, y_\alpha\} = \emptyset$ or $U_\beta \supset \{x_\alpha, y_\alpha\}$, . Now let U_α be an open neighborhood of x_α with $y_\alpha \notin \overline{U_\alpha}$. This defines x_α, y_α,and U_α for each $\alpha < \omega_1$.

Let U'_α be open such that $x_\alpha \in U'_\alpha \subset \overline{U'_\alpha} \subset U_\alpha$. Then $p_\alpha = (x_\alpha, U'_\alpha, U_\alpha) \in \mathcal{P}_X$. Let $V^{p_\alpha} = V_\alpha$ be as in Theorem 2.1.

By property K, there is an uncountable $W \subset \omega_1$ such that $\{V_\alpha \cap U'_\alpha : \alpha \in W\}$ is linked. For $\beta < \alpha \in W$, put $\{\alpha, \beta\}$ in Pot I if $V_\alpha \not\subset U_\beta$; otherwise, put $\{\alpha, \beta\}$ in Pot II.

We claim that there is no infinite subset A of W which is homogeneous for Pot I. Suppose there were. We may assume A has order type ω. By Theorem 2.1, there is a finite $A' \subset A$ such that, for any $\alpha \in A$, there is some $\beta \in A'$ such that $V_\alpha \subset U_\beta$ or $V_\alpha \cap U'_\beta = \emptyset$. Since the latter never holds for $\alpha, \beta \in W$, $V_\alpha \subset U_b$ must hold. But then choosing $\alpha \in A$ with $\alpha > \beta$ for every $\beta \in A'$ yields a contradiction.

Thus, by Erdos' theorem $\omega_1 \to (\omega, \omega_1)^2$ [3], there is an uncountable $W' \subset W$ that is homogeneous for Pot II; that is, $\beta < \alpha \in W'$ implies $U_\beta \supset V_\alpha$. Applying hereditarily Lindelöf again, there must be $\gamma < \omega_1$ such that $\forall \mu, \nu \in \omega_1 \setminus \gamma$,

$$\overline{\cup\{x_\alpha, y_\alpha\}_{\alpha \in W' \setminus \mu}} = \overline{\cup\{x_\alpha, y_\alpha\}_{\alpha \in W' \setminus \nu}}.$$

Let β be the least element of $W' \setminus \gamma$. Then

$$y_\beta \in \overline{\cup\{x_\alpha, y_\alpha\}_{\alpha \in W' \setminus \beta}} = \overline{\cup\{x_\alpha, y_\alpha\}_{\alpha \in W' \setminus (\beta+1)}}.$$

Now for each $\alpha \in W' \setminus (\beta+1)$, we have $U_\beta \supset V_\alpha \ni x_\alpha$, hence $\{x_\alpha, y_\alpha\} \subset U_\beta$. Thus $U_\beta \supset \cup\{x_\alpha, y_\alpha\}_{\alpha \in W' \setminus (\beta+1)}$, and so $\overline{U_\beta} \supset \overline{\cup\{x_\alpha, y_\alpha\}_{\alpha \in W' \setminus (\beta+1)}}$, which puts $y_\beta \in \overline{U_\beta}$, contradiction. □

Corollary 3.2. *Separable mC spaces, and under MA_{ω_1}, CCC mC spaces, are metrizable.*

4. Monotonically compact *LOTS*

The following result answers Question 10 in [1].

Theorem 4.1. *A monotonically compact LOTS is metrizable.*

Proof. Suppose X is a *LOTS* which is monotonically compact. Let $\mathcal{I}$ be the collection of closed intervals of X. Inductively define a map $C : 2^{<\omega_1} \to \mathcal{I}$ as follows: Let $C(\emptyset) = X$. Suppose $C(\sigma)$ has been defined for all $\sigma \in 2^{<\alpha}$, where $\alpha < \omega_1$. If α is a limit ordinal and $\tau \in 2^\alpha$, let $C(\tau) = \bigcap_{\beta<\alpha} C(\tau \restriction \beta)$. If $\alpha = \beta + 1$, let $I = C(\tau \restriction \beta)$. If I is a singleton or doubleton, let $C(\tau) = I$. Otherwise, let $a < b$ be the endpoints of I, and choose a point c between a and b. Then let $C(\tau) = [a, c]$ if $\tau(\beta) = 0$ and $C(\tau) = [c, b]$ if $\tau(\beta) = 1$.

Let T be all elements t of the range of C that have more than 2 points, or exactly two points but every predecessor has more than two points. Then T is a tree under reverse inclusion, and the αth level of T is all members of $C(2^\alpha)$ in T. Note that any two members of T which are incomparable in the tree order are either disjoint or meet at an endpoint.

Claim 1. Every branch of T is countable. If T had an uncountable branch, then X would contain a well-ordered increasing or decreasing sequence in type ω_1, hence a closed copy of $\omega_1 + 1$. But $\omega_1 + 1$ is not monotonically compact [2].

Claim 2. Every level of T is countable. Suppose not, and let α be least such that the αth level L_α of T is uncountable. Note that α must be a limit ordinal. For each $I = [a_I, b_I] \in L_\alpha$, let $N(a_I) = (-\infty, b_I)$ and $N(b_I) = (a_I, \infty)$. Let $M(a_I)$ and $M(b_I)$ be as guaranteed by Theorem 2.3.

Let $\sigma_I \in 2^\alpha$ be such that $I = C(\sigma_I)$. Since $C(\sigma_I) = \bigcap_{\beta<\alpha} C(\sigma_I \restriction \beta)$, there is $\beta_I < \alpha$ such that $C(\sigma_I \restriction \beta_I) \subset M(a_I) \cup I \cup M(b_I)$. Note that $C(\sigma_I \restriction \beta_I)$ is a member of T below level α. Since by assumption there are only countably many such, there are $t \in T$ and an infinite subset $\mathcal{J}$ of L_α such that $C(\sigma_I \restriction \beta_I) = t$ for every $I \in \mathcal{J}$. For $J, J' \in \mathcal{J}$, define $J < J'$ iff $a_J < a_{J'}$. There is a sequence $J_0, J_1, \ldots$ in $\mathcal{J}$ which is either increasing or decreasing. W.l.o.g., suppose it is increasing. Then $\bigcap_{i\in\omega} M(a_{J_i})$ contains the left endpoint of t, but no finite subcollection of $\{N(a_{J_i}) : i \in \omega\}$ covers $\{a_{J_i} : i \in \omega\}$. This contradicts 2.3.

Claim 3. T is countable. Suppose T is uncountable. By Claim 2, it has uncountable height. For each countable limit ordinal α, choose $t_\alpha = [a_\alpha, b_\alpha]$ in the αth level of T, let $N(a_\alpha) = (-\infty, b_\alpha)$ and $N(b_\alpha) = (a_\alpha, \infty)$, and let $M(a_\alpha), M(b_\alpha)$ be as guaranteed by Theorem 2.3.

As in the proof of Claim 2, there is a predecessor s_α of t_α in T such that $s_\alpha \subset M(a_\alpha) \cup t_\alpha \cup M(b_\alpha)$. By the Pressing Down Lemma, there is an uncountable $W' \subset \omega_1$ such that the level of s_α is the same ordinal for every $\alpha \in W'$. Then

since each level is countable, there are $s \in T$ and uncountable $W \subset W'$ such that $s_\alpha = s$ for each $\alpha \in W$. For $\alpha < \beta \in W$, put $\{\alpha, \beta\}$ in Pot II if $t_\beta \subset t_\alpha$, otherwise put $\{\alpha, \beta\}$ in Pot I. Since there is no uncountable chain in T, it follows from the partition relation $\omega_1 \to (\omega, \omega_1)^2$ that there is an infinite $A \subset W$ homogeneous for Pot I; i.e., for distinct α, β in A, t_α and t_β are incomparable, and hence viewed as intervals are either disjoint or meet at an endpoint. As in the proof of Claim 2, there is a sequence $t_{\alpha_0}, t_{\alpha_1}, \dots$ that is either increasing or decreasing in the order $s < t$ iff $\inf s < \inf t$, and the rest of the argument is finished as in Claim 2.

Now we complete the proof of the theorem. Note that for each $x \in X$, there is a branch of T consisting of a decreasing well-ordered sequence of intervals containing x which either intersect to x, or there is a last member of the branch which has exactly two points, one of which is x. Thus, if to T we add the points of the 2-element members of T, we obtain a countable network for X. But a compact space with the countable network is metrizable. □

5. First countable Lindelöf GO-spaces

The following result improves the result in [2] that separable GO-spaces are mL, and answers several questions in [2],[1], and [10]. In particular, it shows that any Suslin line is mL.

Theorem 5.1. *Every first countable Lindelöf GO-space is monotonically Lindelöf.*

Proof. Let Y be a first-countable Lindelöf GO-space. Then Y is a dense subspace of a compact $LOTS$ X[6]. Note that X is first-countable. To see this, if not, then X would contain a copy of $\omega_1 + 1$. The point in X corresponding to the point ω_1 could not be in Y by first countability of Y. But on the other hand, this point must be in Y else Y would not be Lindelöf.

Construct a tree of open intervals of X as follows.

Let $X = [a, b]$, $D_0 = \{a, b\}$, and $L_0 = \{(a, b)\}$. Let $t = (a, b)$. Let $D_t(a)$ be $\{a'\}$, where a' is the immediate successor of a if such exists, else $D_t(a)$ is a sequence in t converging to a from the right. Define $D_t(b)$ similarly, making sure $sup(D_t(a)) < inf(D_t(b))$ if possible, which it is unless t contains only one point c, in which case $D_t(a) = D_t(b) = \{c\}$. Then let $D_t = D_t(a) \cup D_t(b)$.

Let $D_1 = D_t$ where $t = (a, b)$ is the only member of L_0, and let L_1 be the (nonempty) convex components of $X \setminus D_1 \cup \{a, b\}$.

Suppose a disjoint collection L_β of open intervals of X, and a set D_β of points of X, has been constructed for $\beta < \alpha$, where $\alpha < \omega_1$, satisfying:

(i) If $\beta < \gamma < \alpha$ and $t \in L_\gamma$, there is $s \in L_\beta$ with $\bar{t} \subset s$;
(ii) If $t = (a_t, b_t) \in L_\beta$, then $D_{\beta+1} \cap t = D_t(a_t) \cup D_t(b_t)$, where $D_t(a_t)$ and $D_t(b_t)$ are countable relatively closed discrete subsets (a_t, b_t) defined similarly to the case $t = (a, b) \in L_0$ above. $D_{\beta+1} = \bigcup_{t \in L_\beta} D_t$.
(iii) $L_{\beta+1} = \{s : \exists t \in L_\beta (s \text{ is a convex component of } \mathrm{t} \setminus \mathrm{D_t})\}$;
(iv) $X \setminus \bigcup L_\beta = \bigcup_{\gamma \le \beta} D_\gamma \cup \{x : |\cap B_x| \le 2\}$, where B_x is the unique branch of $\bigcup_{\gamma < \beta} L_\gamma$ consisting of those intervals which contain $x\}$.

Note that $T_\alpha = \bigcup_{\gamma < \alpha} L_\gamma$ is a tree under reverse inclusion, with L_β the βth level. Define L_α and D_α as follows. If $\alpha = \beta + 1$, then define D_α and L_α following (ii) above. If α is a limit, then for every branch B in T_α such that $\cap B$ is an interval $[a, b]$ with $(a, b) \neq \emptyset$, put (a, b) in L_α and a, b in D_α. This completes the construction

of the tree $T = \bigcup_{\alpha<\omega_1} L_\alpha$. Note that T has no uncountable chains (else X would contain a copy of $\omega_1 + 1$).

We will use $<_T$ for the tree order; so, if $s, t \in T$, then $s <_T t$ iff s contains t when viewed as subsets of X.

Now we describe a base at points in the D_α's, for α a successor. Let $x \in D_\alpha$, $\alpha = \beta + 1$. Then there is some $t = (a, b) \in L_\beta$ such that $x \in (a, b)$. Suppose $x \in D_t(a)$. If x has an immediate predecessor y (it could be, but is not necessarily, the case that $y = a$), let $P(x) = \{y\}$. Otherwise, there is a point p in $D_t(a)$ just to the left of x, $s = (p, x) \in L_{\alpha+1}$, and $D_s(x)$ is a sequence in (p, x) converging to x. In this case, let $P(x) = D_s(x)$. Define $R(x)$, a set of points to the right of x, similarly. Let $\mathcal{B}(x)$ be all open intervals containing x with endpoints in $P(x) \cup R(x)$. Then $\mathcal{B}(x)$ is clearly a base at x. Define $\mathcal{B}(x)$ for $x \in D_t(b)$ similarly. Note that it follows from this definition that if $x \neq y$, $x, y \in D_{\alpha+1}$, $B_x \in \mathcal{B}(x)$, and $B_y \in \mathcal{B}(y)$, then $B_x \cap B_y = \emptyset$.

If $\mathcal{A}$ is a collection of subsets of X, and $Z \subset X$, then we let $\mathcal{A} \restriction Z$ denote the collection $\{A \cap Z : A \in \mathcal{A}\}$. If $\mathcal{A}'$ is another collection of subsets of X, we write $\mathcal{A}' \prec \mathcal{A}$ to mean $\mathcal{A}'$ refines $\mathcal{A}$. Also, if $C \subset X$, we write $C \prec \mathcal{A}$ to mean $C \subset A$ for some $A \in \mathcal{A}$.

Now we proceed to define $r(\mathcal{U})$.

Given an open cover $\mathcal{U}$ of Y, first define $r'(\mathcal{U})$ as follows.

(I) Put in $r'(\mathcal{U})$ all members of $\bigcup\{\mathcal{B}(x) \restriction Y : \exists\alpha(x \in D_{\alpha+1} \cap Y)\}$ which are contained in some member of $\mathcal{U}$.

(II) For each $t = (a_t, b_t)$ in T, let $r_t = sup\{r \in t : (a_t, r) \cap Y \prec \mathcal{U}\}$, and put $(a_t, r_t) \cap Y$ in $r'(\mathcal{U})$.

(III) Similarly define $l_t = inf\{l \in t : (l, b_t) \prec \mathcal{U}\}$ and and put each $(l_t, b_t) \cap Y$ in $r'(\mathcal{U})$.

Note that each member of $r'(\mathcal{U})$ is convex, and hence can be represented by an open interval (a, b), where $a, b \in X \cup \{\pm\infty\}$.

Let $r'_m(\mathcal{U})$ be the maximal members (under set inclusion) of $r'(\mathcal{U})$. Suppose $(a, b) \in r'_m(\mathcal{U})$. If a has no immediate successor, choose points $a(n)$ in (a, b) converging to a from the right, else let $a(n) = a$ for all n. Choose $b(n)$ converging to b similarly. We can do this so that $a(n) < b(n)$ for all n. Finally, let

$$r(\mathcal{U}) = \{(a(n), b(n)) : n \in \omega, (a, b) \in r'_m(\mathcal{U})\}.$$

Claim 1. If $x \neq y \in Y$, $x \in D_{\alpha+1}$, $y \in D_{\beta+1}$, $B_x \in \mathcal{B}(x)$, $B_y \in \mathcal{B}(y)$, and $B_x \cap Y \subsetneq B_y \cap Y$, then $\beta < \alpha$. This holds because if $\alpha = \beta$ then $B_x \cap B_y = \emptyset$ by the construction, and if $\alpha < \beta$ then $B_y \cap D_{\alpha+1} = \emptyset$ so $x \notin B_y$.

Claim 2. If $(a_t, r_t) \cap Y$ is a nonempty proper subset of $(a_s, r_s) \cap Y$ or if $(l_t, b_t) \cap Y$ is a nonempty proper subset of $(l_s, b_s) \cap Y$, then $s <_T t$. We show this if the former case holds, the latter being analogous. Clearly s and t must be comparable. Suppose $t <_T s$. Then $a_t < a_s < r_s \leq b_s \leq b_t$. Then the only way $(a_t, r_t) \cap Y$ could be a nonempty proper subset of $(a_s, r_s) \cap Y$ is if $(a_t, a_s] \cap Y = \emptyset$. But then $(a_s, r) \cap Y = (a_t, r) \cap Y$ for all r, so $r_t = r_s$ and $(a_t, r_t) \cap Y = (a_s, r_s) \cap Y$, contradiction.

Claim 3. Every member of $r'(\mathcal{U})$ is contained in a maximal member of $r'(\mathcal{U})$. Immediate from Claims 1 and 2.

Claim 4. $\cup r(\mathcal{U}) = \cup r'_m(\mathcal{U}) = \cup r'(\mathcal{U})$. The second equality is immediate from Claim 3, and the first is easy to see from the way $r(\mathcal{U})$ was defined from $r'_m(\mathcal{U})$.

Claim 5. $r(\mathcal{U})$ is a refinement of $\mathcal{U}$ covering X. Suppose $(a(n), b(n)) \in r(\mathcal{U})$. Then $(a, b) \in r'_m(\mathcal{U})$. If (a, b) got in $r'(\mathcal{U})$ via (I), then (a, b) and hence also $(a(n), b(n))$ is contained in some member of $\mathcal{U}$. Suppose $(a, b) = (a_t, r_t)$ for some $t \in T$. If $b(n) < b$, then by the definition of r_t we have that $(a(n), b(n))$ is a subset of some member of $\mathcal{U}$. If $b(n) = b$, it has to be because $b = r_t$ has an immediate predecessor, which implies that r_t is in the set it is defined to be the supremum of, and so $(a(n), b(n)) \subset (a_t, r_t) \prec \mathcal{U}$.

To finish the proof of Claim 5, we need to show $r(\mathcal{U})$ covers X. Let $x \in X$. If $x \in D_{\alpha+1}$ for some α, then x is in some member of $r'(\mathcal{U})$ via I, and then by Claim 4, $x \in \cup r(\mathcal{U})$.

Suppose $x \notin \bigcup_{\alpha<\omega_1} D_{\alpha+1}$. Let C_x be the collection of all elements of T containing x; of course this is a well-ordered chain of some length $\alpha < \omega_1$. Note that α is a limit ordinal. [If C_x had a $<_T$-maximal element t at level $\beta = \alpha - 1$, then since $x \notin D_t$, x would be in some convex component s of $t \setminus D_t$, contradicting maximality of $t \in C_x$.] Then either x is the unique element of $\cap C_x$ and the elements of C_x are a base at x, or $\cap C_x$ is a nontrivial closed interval, and x is one of the endpoints. Let $x \in U \in \mathcal{U}$. If the former case holds, or the latter case holds and x is the left endpoint of $\cap C_x$, there is some $t \in C_x$ and some $r > x$ with $(a_t, r) \subset U$. It follows that $r_t > x$, and hence x is in some member of $r'(\mathcal{U})$ via II. If x were the right endpoint, a similar argument shows that x is in some member of $r'(\mathcal{U})$ via III.

Claim 6. $\mathcal{V} \prec \mathcal{U}$ implies $r(\mathcal{V}) \prec r(\mathcal{U})$. Let $(a(n), b(n)) \cap Y \in r(\mathcal{V})$. Then $(a, b) \cap Y \in r'_m(\mathcal{V})$. It suffices to show that there is some $(c, d) \cap Y \in r'_m(\mathcal{U})$ with $(a, b) \subset (c, d)$, because then for some m, $(a(n), b(n)) \cap Y \subset (c(m), d(m)) \cap Y \in r(\mathcal{U})$. Now if $(a, b) \cap Y$ got in $r'(\mathcal{V})$ via I, then $(a, b) \cap Y \prec \mathcal{V} \prec \mathcal{U}$, so $(a, b) \cap Y \in r'(\mathcal{U})$. Thus there is $(c, d) \cap Y \in r'_m(\mathcal{U})$ with $(a, b) \cap Y \subset (c, d) \cap Y$.

Suppose $(a, b) \cap Y = (a_t, r_t^{\mathcal{V}}) \cap Y$ for some $t \in T$, where the superscript $\mathcal{V}$ means r_t has been defined as in II with respect to the open cover $\mathcal{V}$. Clearly $r_t^{\mathcal{U}} \geq r_t^{\mathcal{V}}$. Thus $(a_t, r_t^{\mathcal{V}}) \cap Y \subset (a_t, r_t^{\mathcal{U}}) \cap Y \in r'(\mathcal{U})$, and so there is some $(c, d) \cap Y \in r'_m(\mathcal{U})$ containing $(a, b) \cap Y$. The case where $(a, b) = (l_t^{\mathcal{V}}, b_t)$ is analogous.

Claim 7. $r(\mathcal{U})$ is countable. It suffices to prove $r'_m(\mathcal{U})$ is countable. Suppose $(a_\alpha, b_\alpha) \cap Y$, $\alpha < \omega_1$, are distinct elements of $r'_m(\mathcal{U})$. Since Y is Lindelöf, there is $x \in Y$ such that every neighborhood of x meets uncountably many $(a_\alpha, b_\alpha) \cap Y$'s. There is $(a, b) \cap Y \in r'_m(\mathcal{U})$ with $x \in (a, b)$. If $(a, b) \cap Y$ meets $(a_\alpha, b_\alpha) \cap Y$, then by maximality of these sets, it must be that either a or b is in (a_α, b_α). If (a_α, b_α) is in $\mathcal{B}(x)$ for some $x \in D_{\alpha+1}$, let t_α be such that $x \in D_{t_\alpha}$. If (a_α, b_α) is equal to (a_t, r_t) or (l_t, b_t), let $t_\alpha = t$. Note that $(a_\alpha, b_\alpha) \subset t_\alpha$ for each α, and since D_t is countable, for each $t \in T$, there are only countably many $\alpha < \omega_1$ such that $t_\alpha = t$. Hence we conclude that either a or b must be in uncountably many distinct $t \in T$, a contradiction since T has no uncountable chains. □

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