

METRIZABILITY NUMBER AND PERFECT MAPS

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ABSTRACT. The metrizable number, $m(X)$, of a space X is the least cardinality of a cover of X by metrizable subspaces. We show that if $f : X \rightarrow Y$ is a perfect map, and $m(X) < \omega$, then $m(Y) \leq \omega$, and we give an example where $2 = m(X) < m(Y) = \omega$. The example shows that, in the finite case, the metrizable number can be increased by a perfect mapping; this was shown by Ismail and Szymanski not to be the case in the realm of locally compact spaces. It remains an open question whether or not an infinite metrizable number can be increased by a perfect mapping; in particular, the case $m(X) = \omega$ is unsolved.

INTRODUCTION.

The study of compact spaces which are unions of a “small” number of metrizable subspaces was initiated by A. V. Arhangel’skii [A₁],[A₂],[A₃]. More recently, in a series of related papers, M. Ismail and A. Szymanski [IS₁], [IS₂], [IS₃] introduce and study the notion of *metrizable number*, $m(X)$, of an arbitrary space X , defined to be the least cardinal of a cover of X by metrizable subspaces. They obtain theorems showing that locally compact spaces with finite metrizable number are “nice” in well-defined ways. E.g., such spaces have a dense open metrizable subspace whose complement has metrizable number lower than the whole space. This implies that if X is locally compact and $m(X) < \omega$, then the metrizable number of any perfect image of X is not greater than $m(X)$.

In this note, we show that the above result fails in the non-locally compact case by giving an example of a space X which is the union of two metrizable subspaces, and a perfect mapping $f : X \rightarrow Y$ such that Y is not a finite union of metrizable subspaces. On the other hand, we also show that if $m(X) < \omega$, then a perfect image of X must in any case be σ -metrizable, i.e., its metrizable number must be countable.

A problem left open by this note is whether the metrizable number can be raised by a perfect map in the infinite case. In particular, we do not know the answer to the following:

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Question. *Is the perfect image of a σ -metrizable space always σ -metrizable?*

We do not know the answer even if the domain is assumed to be compact; this particular form of the question was asked by A. Szymanski at the 2001 Summer Conference on Topology and its Applications at the City College of New York.

All spaces are assumed to be regular and T_1 .

THE EXAMPLE.

First we describe an example Z due to Dennis Burke[B] of a screenable non-metrizable Moore space which admits a perfect mapping onto a non-screenable Moore space. A key property we use about Z is that it is the union of two dense metrizable subspaces. The set for Z is $I \cup \mathbb{R}_0 \cup \mathbb{R}_1$, where I is the set of points in the plane with positive y -coördinate, and \mathbb{R}_e , $e < 2$, are two copies of the real line. The points of I are isolated. For $a \in \mathbb{R}$, let a_e denote its copy in \mathbb{R}_e . Then let the k^{th} basic open set around a_e be $B_k(a_e) = \{a_e\} \cup \{(x, y) \in I : y = (-1)^e(x - a), 0 < y < 1/2^k\}$.

Note that for $e < 2$, the collections $\mathcal{B}_e = \{B_k(a_e) : a \in \mathbb{R}\}$ are pairwise-disjoint. It follows that the dense subspaces $I \cup \mathbb{R}_e$, $e < 2$, are metrizable. As Burke notes, the map which identifies a_0 and a_1 for each $a \in \mathbb{R}$ is perfect, and its range is naturally homeomorphic to Heath's "tangent V " space H , i.e., the space H is the set I above, together with the x -axis, where I is isolated and neighborhoods of points on the x -axis have neighborhoods which consist of two line segments reaching up into I of slope ± 1 . A simple fact about H that we will use is that no subspace of H which contains all of I together with uncountably many points on the x -axis is metrizable.

But Z won't do for our example; any perfect image of Z is also the union of two metrizable (even discrete) subspaces: the image under any perfect map of the closed discrete subspace $\mathbb{R}_0 \cup \mathbb{R}_1$ of Z will be discrete, and the set of points of the image whose fibers are contained in I will also be discrete.

What we do instead is to build from Z an associated space $S(Z)$ which is the union of two dense metrizable subspaces, but such that no non-empty open subspace is metrizable. $S(Z)$ will be a certain set of finite sequences s , i.e., each s is a function whose domain $\text{dom}(s)$ is a natural number $n_s = \{0, 1, 2, \dots, n_s - 1\}$. We may also denote s by $\langle s(0), s(1), \dots, s(n_s - 1) \rangle$. Let x_s denote the last term of s , i.e., $x_s = s(n_s - 1)$. Then set for $S(Z)$ is the set of all s such that

- (i) $x_s \in \mathbb{R}_0 \cup \mathbb{R}_1$;
- (ii) $s(i) \in I$ for all $i < n_s - 1$.

Now for $k \in \omega$, we define a basic neighborhood $B_k(s)$ of s as follows:

$$B_k(s) = \{t \in S(Z) : n_t \geq n_s, t \upharpoonright (n_s - 1) = s \upharpoonright (n_s - 1), t(n_s - 1) \in B_k(x_s)\}.$$

That is, $B_k(s)$ is all those finite sequences t in $S(Z)$ that are at least as long as s , agree with s prior to the last term x_s of s , and whose $(n_s - 1)^{\text{th}}$ term is in the k^{th} neighborhood $B_k(x_s)$ in the space Z .

Let $S_e = \{s \in S(Z) : x_s \in \mathbb{R}_e\}$. Then $S(Z) = S_0 \cup S_1$. We claim that S_0 and S_1 are metrizable. We will show this for S_0 , S_1 being similar.

Given $s \in S_0$, let

$$d_s = \max\{m : (\exists i < n_s - 1)(\exists x \in \mathbb{R})(s(i) \in B_m(x))\}$$

if there are such m ; otherwise, let $d_s = -1$. Then, given $d, k, n \in \omega$, let

$$\mathcal{B}(d, k, n) = \{B_k(s) : s \in S_0, n_s = n, d_s = d\}.$$

Clearly, $\bigcup\{\mathcal{B}(d, k, n) : d, k, n \in \omega\}$ contains a base at all points of S_0 . It remains to prove the trace of $\mathcal{B}(d, k, n)$ on S_0 is discrete in S_0 . To this end, suppose every neighborhood of $t \in S_0$ meets infinitely many members of $\mathcal{B}(d, k, n)$. Since $\{B_k(x_0) : x \in \mathbb{R}\}$ is a disjoint collection in Z , it easily follows that $n_t < n$. Suppose $B_{d+1}(t) \cap B_k(s) \neq \emptyset$ for some s with $n_s > n_t$. Then $s(n_t - 1) \in B_{d+1}(x_t)$, so $d_s > d$. It follows that $B_{d+1}(t) \cap \bigcup\mathcal{B}(d, k, n) = \emptyset$, contradiction.

For sequences s and t , we let $s \frown t$ denote the sequence that is s followed by t . Define the equivalence \sim on $S(Z)$ by declaring $s \frown \langle x_0 \rangle \sim s \frown \langle x_1 \rangle$ for any finite sequence s of points of I and $x \in \mathbb{R}$. (Recall that for $x \in \mathbb{R}$ and $e < 2$, we let x_e denote the copy of x in \mathbb{R}_e .) Let $X = S(Z)$, $Y = S(Z)/\sim$, and let $f : X \rightarrow Y$ be the quotient map. We claim that this works.

f is perfect. For $s \in I^n$, and $x \in \mathbb{R}$, $B_k(s \frown \langle x_0 \rangle) \cup B_k(s \frown \langle x_1 \rangle)$ is easily seen to be a saturated open set, and their images form a base at the point $\{s \frown \langle x_0 \rangle, s \frown \langle x_1 \rangle\}$ in Y . So f is closed, and two-to-one, hence perfect.

Y is not a union of finitely many metrizable subspaces.

The following facts are easy to verify:

(1) The set of all extensions of a fixed $s \in I^n$ is closed and homeomorphic to X and its image is homeomorphic to Y .

(2) If for each $i \in I$, we choose a point $s^i \in X$ with $s^i(0) = i$, then the subspace $\{\langle x_e \rangle : x \in \mathbb{R}, e < 2\} \cup \{s^i : i \in I\}$ of X is homeomorphic to Z and its image under f is homeomorphic to H .

Now suppose $Y = \bigcup_{k < n} M_k$, where each M_k is metrizable. Suppose also n is the smallest possible. Without loss of generality, uncountably many points in Y of the form $\{\langle x_0 \rangle, \langle x_1 \rangle\}$, $x \in \mathbb{R}$, are in M_0 . Since the set of all extensions of $\langle i \rangle$, where $i \in I$, is by (1) above homeomorphic to X and its image to Y , by minimality of n we have that each $\langle i \rangle$ has some extension $s^i \in X$ whose image is in M_0 . But then by (2) we have that M_0 contains a copy of a subspace of H containing I together with uncountably many points on the x -axis, contradicting metrizability. \square

POSITIVE RESULTS.

In this section, we prove that the perfect image of a space having finite metrizable number must have countable metrizable number. The example of the previous section demonstrates that the conclusion is sharp. We also prove a partial result (Lemma 2) related to the question of preservation of σ -metrizable under perfect maps.

First we state without proof the following elementary result.

Lemma 0. *Let D be a dense subset of a regular space, and for each relatively open subset U of D , choose an open set U^* in X such that $U^* \cap D = U$. Then*

- (i) $U^* \cap V^* = \emptyset \iff U \cap V = \emptyset$, for any relatively open subsets U and V of D .
- (ii) If \mathcal{B} contains a local base for a point $p \in D$ in the relative topology of D , then $\{B^* : B \in \mathcal{B}\}$ contains a local base for p in X .

Next we establish the following structural result.

Lemma 1. *If $m(X) < \omega$, then $X = \bigcup_{i \in \omega} M_i$, where each M_i is metrizable, and G_δ in its closure $\overline{M_i}$.*

Proof. If $m(X) = 1$, the result is clear. Suppose $m(X) = k > 1$. and the result holds if $m(X) < k$. Let $X = \bigcup_{i < k} N_i$, where N_i is metrizable. Let $Y_i = X \setminus \overline{N_i}$. By the induction hypothesis, Y_i is the union of countably many metrizable subspaces M such that M is G_δ in \overline{M}^{Y_i} . Since $\overline{M}^{Y_i} = \overline{M}^X \cap Y_i$, such M are G_δ in the open subspace $Y_i \cap \overline{M}^X$ of \overline{M}^X , and hence G_δ in \overline{M}^X .

It remains to show that $Z = \bigcap_{i < k} \overline{N_i}$ is the union of countably many metrizable subspaces which are G_δ in their closures. Let $\{\mathcal{U}_{i,n}\}_{n \in \omega}$ be a sequences of open in N_i covers of N_i satisfying:

- (i) Each $\mathcal{U}_{i,n}$ is locally finite in N_i ;
- (ii) $\mathcal{U}_{i,n+1}$ is a refinement of $\mathcal{U}_{i,n}$;
- (iii) Each $U \in \mathcal{U}_{i,n+1}$ meets only finitely many members of $\mathcal{U}_{i,n}$.
- (iv) $\bigcup\{\mathcal{U}_{i,n} : n \in \omega\}$ is a base for the space N_i .

For each $U \in \mathcal{U}_{i,n}$, let U^* be open in $\overline{N_i}$ such that $U^* \cap \overline{N_i} = U$. Let $\mathcal{U}_{i,n}^* = \{U^* : U \in \mathcal{U}_{i,n}\}$. Let $O_{i,n} = \bigcup \mathcal{U}_{i,n}^*$. Let $Z_{i,n} = Z \setminus O_{i,n}$. Since $Z_{i,n} \cap N_i = \emptyset$, by the induction hypothesis, the closed subspace $Z_{i,n}$ of X is a countable union of metrizable spaces which are G_δ in their closure.

Let $G = Z \cap \bigcap \{O_{i,n} : i < k, n < \omega\}$. Then G is relatively G_δ in Z , and hence in \overline{G} . We finish the proof by noting that G is metrizable. Since $G \subset Z \subset \overline{N_i}$, the trace of each collection $\mathcal{U}_{i,n}^*$ is a collection of relatively open sets in G . Also note that each member of $\mathcal{U}_{i,n+1}^*$ meets only finitely many members of $\mathcal{U}_{i,n}^*$ (by (iii) above and Lemma 0). It follows that $\mathcal{U}_{i,n}^*$ is locally finite at each point of G . By (iv) and Lemma 0, we have that $\bigcup\{\mathcal{U}_{i,n}^* : n \in \omega\}$ contains a base in $\overline{N_i}$ for each point of N_i . It follows that the trace of $\bigcup\{\mathcal{U}_{i,n}^* : i < k, n \in \omega\}$ on G is a σ -locally finite base for G , so G is metrizable. \square

Lemma 2. *Suppose $X = \bigcup_{i \in \omega} M_i$, where each M_i is metrizable and G_δ in its closure $\overline{M_i}$. Then every perfect image of X is σ -metrizable.*

Proof. Let X satisfy the hypotheses, and let $f : X \rightarrow Y$ be perfect. Let $M_i = \bigcap_{n \in \omega} O_{in}$, where O_{in} is relatively open in $\overline{M_i}$.

Let \mathcal{H} be the collection of all finite intersections of the collection $\{\overline{M_i} : i \in \omega\} \cup \{\overline{M_i} \setminus O_{in} : i, n \in \omega\}$. For each $j \in \omega$ and $H \in \mathcal{H}$, let $M(j, H) = \{y \in Y : (f \upharpoonright H)^{-1}(y) \subset M_j\}$. Then $M(j, H)$ is the perfect image of a metrizable space, hence is metrizable. We will finish the proof by showing $Y = \bigcup\{M(j, H) : j \in \omega, H \in \mathcal{H}\}$.

Pick $y \in Y$. We aim to show that y is in some $M(j, H)$. Let j_0 be least such that $M_{j_0} \cap f^{-1}(y) \neq \emptyset$. Let $X_0 = \overline{M_{j_0}}$ and $f_0 = f \upharpoonright X_0$. If $f_0^{-1}(y) \subset M_{j_0}$, we are done. Otherwise, there is $n_0 \in \omega$ with $f_0^{-1}(y) \cap (X_0 \setminus O_{j_0 n_0}) \neq \emptyset$. Let $X_1 = X_0 \setminus O_{j_0 n_0}$ and $f_1 = f \upharpoonright X_1$. Note that $X_1 \in \mathcal{H}$. Let j_1 be least such that $M_{j_1} \cap f_1^{-1}(y) \neq \emptyset$. Since $f_1^{-1}(y) \subset X_1$ and $X_1 \cap M_{j_0} = \emptyset$, we must have $j_1 > j_0$. Let $X_2 = X_1 \cap \overline{M_{j_1}}$ and $f_2 = f \upharpoonright X_2$. Note $X_2 \in \mathcal{H}$. Thus if $f_2^{-1}(y) \subset M_{j_1}$, we are done. Otherwise, there is $n_1 \in \omega$ with $f_2^{-1}(y) \cap (X_2 \setminus O_{j_1 n_1}) \neq \emptyset$. Let $X_3 = X_2 \setminus O_{j_1 n_1}$ and $f_3 = f \upharpoonright X_3$, and so on.

The argument will be complete if we show that the above process must terminate at some finite stage. Suppose it did not. Then there is some point x in $f^{-1}(y) \cap \bigcap_{i \in \omega} X_i$. Then $x \in M_j$ for some j . Since $O_{j_i n_i} \supset M_{j_i}$, it follows that the sequence j_0, j_1, \dots defined in the above construction is strictly increasing. So $j < j_i$ for some (least) i , contradicting minimality of j_i . \square

Our theorem now follows directly from Lemma 1 and Lemma 2.

Theorem. *Suppose X is a regular space having finite metrizable number. Then any perfect image of X is σ -metrizable.*

Our example in the previous section was two-to-one. The following corollary shows that if there is a counterexample to the σ -metrizable question, it must involve a more complicated mapping.

Corollary. *If X is σ -metrizable, and $f : X \rightarrow Y$ is perfect and finite-to-one, then Y is σ -metrizable.*

Proof. Let $X = \bigcup_{i \in \omega} M_i$, where each M_i is metrizable. Let $[\omega]^{<\omega}$ denote the collection of all finite subsets of ω . For each $F \in [\omega]^{<\omega}$, let

$$Y_F = \{y \in Y : f^{-1}(y) \subset \bigcup_{i \in F} M_i\}.$$

Then $Y = \bigcup_{F \in [\omega]^{<\omega}} Y_F$, and each Y_F is σ -metrizable by the Theorem. Hence Y is σ -metrizable. \square

Remarks. We do not know if the conclusion of Lemma 1 (which is the same as the hypothesis of Lemma 2) holds when $m(X) = \omega$, even in the compact case. If it does hold in general, then Lemma 2 would imply that σ -metrizability is preserved by perfect maps. We mention here that we have checked two of the “interesting” σ -metrizable spaces appearing in the literature, and found them to satisfy the conclusion of Lemma 1. One of these examples is van Mill’s example [vM] of a homogeneous non-metrizable Eberlein compact space. The other is the countable power of a perfect “ladder” space, shown to be σ -metrizable in [G].

We also note that any σ -discrete X satisfies the conclusion of Lemma 1, and thus perfect images of σ -discrete spaces are σ -metrizable. In fact, Burke and Hansell [BH] show the stronger result that perfect images of σ -discrete spaces are σ -discrete. This result could also be obtained by the argument of Lemma 2. It is also shown in [BH] that the perfect image of a space with a σ -relatively discrete network has a σ -relatively discrete network. But their argument does not seem to extend in an

obvious way to the σ -metrizable case. Finally, let us recall that metrizable scattered spaces are σ -discrete, so any σ -metrizable scattered space is too. Hence there are no scattered counterexamples to our question.

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