MARY ELLEN'S CONJECTURES

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Introduction

For a special issue like this, it would seem to be very appropriate for us, the editors, to write an article summarizing Mary Ellen's contributions to the field, and giving some account of her mathematical career. But there is an excellent article by Frank Tall [65] in the book The Work of Mary Ellen Rudin that does exactly this, covering her work and giving a sketch of her mathematical life up until shortly after her retirement in 1991. The article also includes a list of her students and their theses titles, her postdocs, and her publications. While she published a couple of papers on other topics after that time, she spent most of her post-retirement working on Nikiel's conjecture, that the images of compact ordered spaces are exactly the compact monotonically normal spaces. She eventually solved it (positively) in 2001 [63]; it took one of the deepest and most difficult arguments of her career to do so.

Instead of copying Frank's article, we would like to do something a bit different. In the book Open Problems in Topology, published in 1990, Mary Ellen has an article [62] in which she states 17 "conjectures".[1] This is the way she chose to phrase some unsolved problems, most of them due originally to others, that she was particularly interested in. We quote from her article to give a sense of her reasons for choosing these particular problems:

"The problems I list here have mostly been listed in Rudin [1988]; I have worked on all of them long enough to have some real respect for them; they have all stood the test of time and labor by various people; and I feel any solution to any of them would have applications at least to other problems exhibiting similar pathologies."

Here we go over Mary Ellen's list of conjectures, giving a brief summary of her contributions and where each conjecture stands today. Many of these problems had or could be expected to have consistent answers. But Mary Ellen clearly had a bias for ZFC results, so instead of asking "Is it consistent that P holds?" she would ask "Is it true (in ZFC) that "not P" holds?", or, as in this article, state "Conjecture: Not P holds.".

[1]In truth, only 14 are labeled "conjectures". Conjecture 9 concerns the shrinking property, after which Mary Ellen lists three related problems that she says she got from her student Beslagic. Unlike the other problems, she doesn't label these three problems "conjectures"; yet the next labeled conjecture is "Conjecture 13".

The first two conjectures are about non-metrizable Hausdorff manifolds.[2] Mary Ellen got this area started with her celebrated examples with Zenor (one assuming the Continuum Hypothesis (CH) and one assuming axiom $\diamondsuit$) of perfectly normal non-metrizable manifolds [55]. She later established the independence of the statement "Every perfectly normal manifold is metrizable" by showing that there are no such if Martin's Axiom plus $\neg CH$ holds [56].

Conjecture 1. Every normal manifold is collectionwise Hausdorff

This would be a very powerful theorem, if true. One consequence would be that PFA is enough to imply that every hereditarily normal manifold of dimension > 1 is metrizable. This is because Nyikos showed [45] that PFA implies every hereditarily normal, hereditarily collectionwise Hausdorff manifold of dimension > 1 is metrizable.

Unfortunately, Conjecture 1 remains unanswered even for hereditarily normal manifolds. There is a fairly short proof that every *perfectly* normal manifold is collectionwise Hausdorff, and more generally, that a normal manifold is collectionwise Hausdorff if and only if every closed discrete subset is a G_δ [44]. This may have been behind Mary Ellen's short comment [62] on Conjecture 1: "I believe strongly that [Conjecture 1] is true, but I have no idea how to begin a proof." Nobody else seems to have an idea either, but neither are there any ideas on how to show that it is false in any model. There are plenty of consistency results, even for just first countable spaces or just locally compact spaces for normal implying collectionwise Hausdorff. But there are none, for example, that are compatible with MA + not-CH.

However, as happens so often in mathematics, this tantalizing state of affairs spurred a very fruitful research effort spanning more than a decade. Led by Frank Tall, it culminated in a model of PFA(S)[S] where every hereditarily normal manifold of dimension > 1 is metrizable [18]. The proof is many times longer and more difficult than the one for Nyikos's weaker PFA theorem; however, it is broken up into several theorems of independent interest, so we can expect this research effort to yield many fine theorems in the future.

Conjecture 2. There is a normal nonmetrizable manifold with a countable point-separating open cover

Here, to say $\mathcal{U}$ is *(strongly) point-separating* means that for any $p \neq q$, there is some $U \in \mathcal{U}$ with $p \in U$ and $q \notin U$ (and $q \notin \overline{U}$. resp.). There are in ZFC nonnormal manifolds with countable point-separating open covers, e.g., the Moore space manifolds mentioned by the second author in [44]. Of course, a Moore space example satifying Conjecture 2 is impossible by the Reed-Zenor theorem that normal locally compact locally connected Moore spaces are metrizable [49].

There seems to have been a flurry of activity in the late 1980's around Conjecture 2 and related properties of manifolds. In 1989, Mary Ellen used $\diamondsuit$ to construct a hereditarily separable Dowker manifold which has a countable point-separating

[2]A *manifold* is a space M such that there is a positive integer n such that each point of M has an open neighborhood homeomorphic to $\mathbb{R}^n$.

open cover [59].[3] On the other hand, Bennett and Balogh [6] showed that even "point-countable" instead of "countable" is enough to imply metrizabilty of manifolds with strongly point-separating open covers. They also prove that there are no hereditarily normal counterexamples to Conjecture 2. In another 1989 paper [8], they show that assuming $2^\omega < 2^{\omega_1}$, every hereditarily normal quasi-developable manifold is metrizable. Earlier, Balogh [2] showed that under MA, any counterexample to Conjecture 2 must have weight $\mathfrak{c}$.

Mary Ellen says that an example satisfying Conjecture 2 would "amaze" her; the conjecture seems to still be wide open.

Mary Ellen's famous example of a Dowker space (i.e., a normal space whose product with the unit interval is not normal) [53] was the beginning of a lifetime interest in related problems. The next three conjectures are all related to the Dowker pathology.

Conjecture 3. There is a Dowker filter

A *Dowker filter* is a filter $\mathcal{F}$ on a set S satisfying:

a) If $F : S \to \mathcal{F}$, then there is $x \neq y$ in S with $y \in F(x)$ and $x \in F(y)$, but
b) If $X \subset S$, there is an $F : S \to \mathcal{F}$ such that, for all $x \in X$ and $y \in (S \setminus X)$, either $y \notin F(x)$ or $x \notin F(y)$.

This notion is due to Dowker [17], who proves that any Dowker filter must have $|S| = \omega_2$. Mary Ellen admits that this problem sounds rather esoteric, but she is interested in it because "after forty years we essentially know only one way to construct a normal, not collectionwise normal space; use Bing's G". If such a filter existed, it would provide a new way of constructing such spaces. Gruenhage amd Zoltán ("Zoli") Balogh [7] used a forcing argument to show that Dowker filters exist in some models of set theory, but Mary Ellen was not particularly interested in this. By Conjecture 3, she means there is one in ZFC.

Mary Ellen does obtain a weaker result: she shows that it is possible to assign a filter $\mathcal{F}_x$ for each $x \in S$ satifying (a) and (b) for functions F such that $F(x) \in \mathcal{F}_x$ for each $x \in S$. She uses this to construct a simplicial complex with interesting properties (normal and not collectionwise-normal, in particular). Balogh and Gruenhage were both visiting Madison in the Fall semester of 1989, and at one point she handed them her short paper [58] on this. Gruenhage recalls: "We both read it and finished it at the same time, looked up at each other and asked: did you understand it? (Answer: no.) A little later we obtained our afore-mentioned consistency result, and I went on to other things. However, Zoli played around with her technique and sought to really understand it."

It paid off for him in spades! Mary Ellen's method here served as the basis for Balogh's powerful technique for constructing ZFC examples of normal not countably paracompact spaces, e.g., a Dowker space of cardinality $\mathfrak{c}$ [4] (the first "new" Dowker space since Mary Ellen's orginal!) and a normal screenable Dowker space [5]. So Mary Ellen was correct about the strong implications a Dowker filter would

[3] She mentions this paper after proposing Conjecture 2, but mistakenly says she used the Continuum Hypothesis, which is weaker than $\diamondsuit$, to construct the manifold.

have; but she didn't realize she already had a technique strong enough to do many of the things that she wanted to do with it! Nevertheless, Conjecture 3 remains unsettled in ZFC.

Conjecture 4. There is a Dowker space of cardinality ω_1.

This conjecture was unusual in that it was just a sample of one from a bevy of problems about "small Dowker spaces." In fact, Mary Ellen had an even wider project in mind concerning this problem:

"What I want is *another* Dowker space. After 20 years it is time. In any problem where Dowker spaces are required, we can only give a consistency example unless our one peculiar example can be modified to give an answer. It seems a ridiculous state of affairs." [62]

This state of affairs was particularly unsatisfactory because, on the one hand, the concept of a Dowker space is so simple (a normal space whose product with [0,1] is not normal) and on the other hand, the one ZFC example Mary Ellen had constructed in 1969 was so big and awkward: its cardinality is $\aleph_\omega^{\aleph_0}$, and it is a subspace of the box product $\square_{n=1}^{\infty}\omega_n+1$. In contrast, there was a big supply of Dowker spaces that were "small" by various standards (including being of cardinality ω_1) from such axioms as CH, $\clubsuit$, and the existence of a Souslin tree.

It was probably this state of affairs that led Mary Ellen to write, "I would be equally happy to see a Dowker space of cardinality $\mathfrak{c}$" [62] despite her knowledge of numerous contrasts between spaces of cardinality $\mathfrak{c}$ and those of cardinality ω_1 in so many models of set theory.

Five years later, Zoli constructed the first of his Dowker spaces of size $\mathfrak{c}$, using a method developed by him [4]. All these spaces begin with the cofinite topology on a set of cardinality $\mathfrak{c}$, and refine it in $2^{\mathfrak{c}}$ steps, starting with an ascending open cover $\mathcal{U} = \{U_n : n \in \omega\}$, and building in various other properties (including normality) while guarding against a locally finite refinement of $\mathcal{U}$.

Zoli's method has been fruitful in building a variety of Dowker spaces of cardinality $\mathfrak{c}$, and remains a promising avenue to solving the other Dowker space problem Mary Ellen explicitly stated in connection to Conjecture 4: "or one could ask for a Dowker square of two countably paracompact spaces". Amer Bešlagič had three papers about consistent examples of this kind, referenced in [64], which is pretty much up to date on various problems and theorems involving Dowker spaces. One minor exception is that Szeptycki and Hernandez-Hernandez had some ideas about using club-guessing to produce a model of MA + not-CH $+\exists$ a Dowker space of cardinality ω_1 [28], but so far have not been successful in producing one.

Conjecture 5. Every normal space with a σ-disjoint base is paracompact

The second author suggested this problem to Mary Ellen at a conference in the mid-seventies, and she immediately became interested in it. [She had been complaining about more technical problems which did not excite her interest.] A year or two later she announced at a conference that she had a very complicated counterexample using "Shelah's 1/2 MA $+1/2\diamondsuit$ axiom." She withdrew that claim, later announcing at the Helsinki ICM in 1978 that she had an example using $\diamondsuit^+$. However, in a letter to the second author in October of that year, she wrote, "...but those airport proofs have a way of not being all there and now I have no proof using anything. It's a good problem however and I keep coming back with different ideas. I should stick with it for awhile." One outcome, three years later, was her screenable Dowker space using $\diamondsuit^{++}$ [60], one of the most original constructions in all of set-theretic topology.

Zoli also thought in 1993 that he had a $\diamondsuit^+$ counterexample. It looked so much like a minor simplification of Mary Ellen's screenable Dowker space that, had it panned out, it could have been called "the big one that got away from Mary Ellen". However, it had an irreparable hole, and Zoli may just have rediscovered Mary Ellen's failed $\diamondsuit^+$ example. And perhaps this failure was what spurred Zoli on to come up with his Dowker space construction technique, and specifically his screenable Dowker space [5].

By the time she wrote [62], Mary Ellen's opinion on Conjecture 5 had shifted, and she began her comments on it with "This would be a beautiful theorem and I think it is true." A counterexample would be another example of a Dowker space, because of a 1955 theorem of Nagami that every normal, screenable, countably paracompact space is paracompact, and a σ-disjoint base is an obvious strengthening of screenability (the property that every open cover has a σ-disjoint refinement).

However, Conjecture 5 remains completely open: no consistency results in either direction, let alone a ZFC theorem. The same is true of a problem due to G.M. Reed [48].

Question 0.1. *If a normal space is a union of a countable family of open metrizable subspaces, must it be metrizable?*

Since every metrizable space has a σ-discrete base, a counterexample to this question would be a non-paracompact space with a σ-disjoint base. Szeptycki's chapter in OPIT II [64] is still the state of the art on it and on Conjecture 5.

Back in 1984 [61], Mary Ellen had a neat division of Dowker space problems that still holds today: "The basic open questions in this area seem to be of two kinds: Can we find 'real' Dowker spaces with small cardinal functions? And can we construct Dowker spaces with strong global countable structures under any assumptions?" Conjecture 5 and Reed's variation on it, and Conjecture 13 below, fit into the latter category, and so does another problem Mary Ellen mentioned in the same article: Is there a symmetrizable Dowker space?

Conjecture 6. M_3 spaces are M_1

This is a well-known problem asked by Ceder [11] in 1961. No M_2 appears in this conjecture because Gruenhage [24] and Junnila [29] independently showed $M_3 = M_2$. A space is M_1 if it is regular and has a σ-closure-preserving base $\mathcal{B}$, i.e., $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ where each $\mathcal{B}_n$ is closure-preserving. Note that if you replace "closure-preserving" with "locally finite", then you have a well-known characterization of metrizability. If instead you replace "base" with "quasi-base", you have the definition of M_3; $\mathcal{B}$ is a quasi-base if whenever $x \in U$ with U open, then there is $B \in \mathcal{B}$ with $x \in B^o \subset B \subset U$. An important observation is that if $B \subset \overline{B}^o$ for every $B \in \mathcal{B}$, then $\{B^o : B \in \mathcal{B}\}$ is a σ-closure-preserving base. So if an M_3 space which is not M_1 exists, the closure-presrvingness of the $\mathcal{B}_n$'s must depend in some essential way on "stickers" or "outliers" in $B \setminus \overline{B^o}$.

The reason Ceder introduced the more technical M_3-spaces[4] is because M_3-spaces could be shown to be closed under arbitrary subspaces and closed images, while the truth of either of these statements for M_1-spaces is unsolved; the statements are in fact known to be equivalent to $M_3 \to M_1$.[5]

The best partial result is the following, due to Mizokami, Shimane, and Kitamura [39]:

Theorem 0.2. *A stratifiable space X is M_1 if it has the following property:*

(δ) Whenever U is dense open in X and $x \in X \setminus U$, there is a closure-preserving collection $\mathcal{F}$ of closed subsets of X that is a network at x, and such that $\overline{F \cap U} = F$ for every $F \in \mathcal{F}$.

If x and U are as in (δ), and $(x_n)_{n=1}^{\infty}$ is a sequence of points in U converging to x, then note that $\mathcal{F} = \{\{x_n : n \geq k\} : k \in \omega\}$ satisfies (δ). Hence Fréchet spaces satisfy (δ); more generally, so do sequential spaces, and more generally yet, so do stratifiable spaces having the following property which has been called *WAP* (*weak approximation by points*) or *weakly Whyburn*:

(WAP) If A is not closed, there exists $B \subset A$ such that $|\overline{B} \setminus A| = 1$.

It has long been known that a stratifiable space is M_1 if each point has a closure-preserving neighborhood base. A another relevant result of Mizokomi [38] is that each closed subset of an M_1-space has a closure-preserving open outer base; whether or not this is true for M_3-spaces is also equivalent to $M_3 \to M_1$. Mizokami's result has as a corollary that any stratifiable space which is the union of countably many closed M_1 subspaces is M_1.

For a time it was unknown whether the space $C_k(\mathbb{P})$ of continuous real-valued functions on the irrationals, with the compact open topology, was M_1; Gartside and Reznichenko [23] had proven it stratifiable. But Tamano [67] used some techniques of Mizokami to prove that it is M_1; however, it is still not known if every subspace of $C_k(\mathbb{P})$ is M_1.

This problem is one that stymied Mary Ellen (along with the rest of us). She once thought that one of her students had a proof that $M_3 = M_1$, but a gap was found and the best that student could do was to show that every stratifiable space of cardinality less than $\mathfrak{b}$ is M_1 [72].

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[4]Actually, the definition we have given for M_3-spaces is Ceder's definition of M_2-spaces.

[5]Even the question whether M_1-spaces are closed hereditary is unsolved and equivalent to $M_3 \to M_1$. Related to this, it is known [27] that every stratifiable space is homeomorphic to a closed subset of an M_1-space.

Conjecture 7. Every Collins space has a point-countable base

"Collins space" is Mary Ellen's term for a "space satisfying open (G)"; Conjecture 7 has been called "the point-countable base problem". The particulars: In [15], Collins and Roscoe introduce the following condition:

G) For each $x \in X$, there is assigned a countable collection $\mathcal{G}(x)$ of subsets of X such that, whenever $x \in U$, U open, there is an open V with $x \in V \subset U$ such that, whenever $y \in V$, then $x \in N \subset U$ for some $N \in \mathcal{G}(y)$.

A space is said to satisfy *open (G)* if each member of $\mathcal{G}(x)$ in the definition of (G) is open. It is easy to check that if $\mathcal{B}$ is a point-countable base for X, then setting $\mathcal{G}(x) = \{B \in \mathcal{B} : x \in B\}$ witnesses open (G) for X. So Conjecture 7, which is still wide open, holds iff every space satisfying open (G) has a point-countable base. Mary Ellen states "In truth I feel that there is a counterexample to (7)", so her real conjecture is that Conjecture 7 is false!

In a 1985 paper with Collins, Reed, and Roscoe [14], Mary Ellen proved that if in (G) we have that each $\mathcal{G}(x)$ is a decreasing open base at x, then the space must be metrizable. This result generated considerable excitement, and elicited a flurry of activity focused on Conjecture 7. Some of those early partial results are that a space X satisfying open (G) is hereditarily metalindelöf [13], and has a point-countable base if the density $d(X) \leq \omega_1$ [42], or if X is GO-space, semistratifiable, or a submetacompact β-space [25]. But after this early activity, the topic seemed to lay dormant for a long time.

As mentioned in [26], it is straightforward to check that (G) is equivalent to the following:

(G′) For each $x \in X$, one can assign a countable collection $\mathcal{G}(x)$ of subsets of X such that, for any $A \subset X$, $\bigcup_{a \in A} \mathcal{G}(a)$ contains a network at every point of $\overline{A}$.

Indeed, $\mathcal{G}(x)$, $x \in X$, satisfies (G) iff it satisfies (G′). In [26], it was shown that the notion of "monotonically monolithic" introduced in 2009 by Tkachuk [68] was equivalent to a form of condition (G′) with points replaced by finite sets. Subsequently, Tkachuk published two papers studying (G) (with and without "open"), which he called the "Collins-Roscoe condition", in which he demonstrated some interesting uses of condition (G) in C_p-theory [69], and obtained new partial results to Conjecture 7: Every space satisfying open (G) has a point-countable π-base[6], and Conjecture 7 holds for hereditarily Lindelöf spaces [70].

Conjecture 8. Every normal, nonparacompact para-Lindelöf space fails to be collectionwise normal.

A space is *para-Lindelöf* if every open cover has a locally countable open refinement. Conjecture 8, first stated in print by Fleissner and Reed [22] is equivalent to "Every collectionwise normal (CWN) para-Lindelöf space is paracompact." We suppose Mary Ellen stated it the way she did because she was always interested in the normal, non-CWN pathology, and her former student Caryn Navy [43] obtained a (unfortunately still unpublished) series of ingenious normal para-Lindelöf spaces

[6] This was likely known to Collins and others, though it had not been explicitly stated in print.

which were not paracompact (answering in the negative the question whether all para-Lindelóf spaces were paracompact); none of her examples were CWN. Fleissner [21] exploited Navy's technique in his solution to the normal Moore space problem, and Mary Ellen herself [60] used the idea to construct the first example of a normal screenable non-paracompact space. Another partial result on Conjecture 8 is due to another of Mary Ellen's students, Diana Palenz, who proved that monotonically normal para-Lindelöf spaces are paracompact [46].

Conjecture 8 remains wide open. Anyone interested in the problem, besides looking at Navy's examples, might check out Steve Watson's method [71] for constructing regular para-Lindelöf nonparacompact spaces in which the para-Lindelöf property is coded in. Watson also asks if regular para-Lindelöf spaces are countably paracompact or if there is a para-Lindelöf Dowker space.

Conjecture 9. There is an uncountable monotone open cover without a shrinking of space X such that $X \times M$ is normal for all metric M.

In 1976, K. Morita stated three basic conjectures about normality in products. The first one, that X is discrete iff $X \times Y$ is normal for all normal Y, was solved in the affirmative by Mary Ellen in 1978. To explain the solution: define an open cover $\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$ to be a *shrinking* of the open cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ if $\overline{V}_\alpha \subset U_\alpha$ for every $\alpha < \kappa$. A space is normal iff every two-element open cover has a shrinking, and is countably paracompact iff every countable open cover has a shrinking (and thus is a Dowker space iff ω is the least cardinal of an open cover with no shrinking). A space is κ-*Dowker* if κ is the least cardinal of an open cover with no shrinking. Morita's first conjecture is true iff there is a κ-Dowker space for each infinite cardinal κ, and that is exactly what Mary Ellen constructs in [57].

The second conjecture states that X is metrizable iff $X \times Y$ is normal for every space Y such that $Y \times M$ is normal for every metric space M. The third, which implies the first and is implied by the second, states that X is metrizable and σ-locally compact iff $X \times Y$ is normal for all normal countably paracompact spaces Y. With Chiba and Przymusinski, Mary Ellen [12] showed that the statement of Conjecture 9 is equivalent to Morita's second conjecture, and so if shown to be true would finally settle the conjectures of Morita in the affirmative.

And that is exactly what happened. In 1985, Mary Ellen and her student A. Beslagic [9] had constructed assuming axiom $\diamondsuit$ a space satisfying the conditions of Conjecture 9. This occurred prior to her "Conjectures" article, so she was certainly asking for a ZFC example here. In 2001, Balogh [3], using the same technique as for his Dowker space of size $\mathfrak{c}$ (see the discussion of Conjecture 4), was able to obtain a space with the same properties in ZFC.

Conjectures 10-12: Perfect preimages

After Conjecture 9, Mary Ellen states three problems, labeled (A), (B), and (C), that were apparently communicated to her by her former student Amer Beslagic. Presumably these were meant to be Conjectures 10-12. They were not labeled as such, but the next conjecture in the paper after these three is Conjecture 13. In any case, here are the problems:

(A) Given $X \times Y$ normal, Y compact, and every open cover of X has shrinking. Does every open cover of $X \times Y$ have a shrinking?
(B) Given every open cover of X has a shrinking, and Y is a normal perfect preimage of X. Does every open cover of Y have a shrinking?
(C) Given X is normal and Y is a collectionwise normal perfect image of X. Is X collectionwise normal?

Recall that a space is normal iff every two-element open cover has a shrinking. Thus all three of these conjectures have the following form: Suppose $f : X \to Y$ is a perfect surjection, and Y has a strong normality property. If X is normal, does it follow that X also has the strong normality property? Of course, in (A) the map is the projection map, in (A) and (B) the strong normality property is "every open cover has a shrinking", and in (C) it is "collectionwise normal". A positive answer to (B) implies the same for (A).

As far as we know, all three of these problems are still open. Problem (C) is originally due to Peg Daniels [16, Question 5], who showed that if there is a counterexample, the space X must be Dowker, and must be collectionwise normal with respect to closed sets not containing an uncountable closed discrete set (e.g., closed Lindelöf sets).

13. The Linearly-Lindelóf Conjecture: There is a normal, non-Lindelöf space, every monotone open cover of which has a countable subcover

In other words, there is a normal, linearly Lindelöf, non-Lindelöf space: "monotone" here means "totally ordered by inclusion."

This would be another Dowker space: Miščenko [37] showed that every linearly Lindelöf, regular, non-Lindelöf space fails to be countably metacompact. (In normal spaces, countable metacompactness is equivalent to countable paracompactness and, as Dowker showed, to having a normal product with $[0, 1]$.)

We have no consistency results either way. The best partial result so far seems to be Nyikos's result [20] that if it is consistent that there is a supercompact cardinal, it is consistent that every locally compact, normal, linearly Lindelöf space is Lindelöf. Elliott Pearl's article [47] in OPIT II gives a wealth of information on this and other results and problems on linearly Lindelöf, non-Lindelöf spaces, including descriptions of the original Tychonoff example by Miščenko, a topological group due independently to Buzyakova and Gruenhage, and a pair of locally compact examples due to Kunen [30], [31].

14. Michael's Conjecture: There is a Michael space

A *Michael space* is a space whose product with the irrationals ω^ω is not Lindelöf; it is equivalent to say "not normal", so this is a natural question for Mary Ellen to be interested in, given her long-standing interest in normality of products. The conjecture is not quite as specialized as it sounds–a space has Lindelöf product with the irrationals iff it has Lindelöf product with every analytic set. In 1971, Michael [36] constructed, assuming the Continuum Hypothesis, a subspace of the Michael line (the reals with the irrationals isolated) which is a Michael space, and wondered about a ZFC example. Not much was done on the problem until Alster [1] in 1990 showed that MA is enough to construct a Michael space. Both Michael's and Alster's examples can be thought of as being a finer topology on a subspace of $\mathbb{R}$ containing the rationals $\mathbb{Q}$, with points of $\mathbb{Q}$ having their usual Euclidean neighborhoods, and the irrationals isolated in Michael's case and given a certain locally Lindelöf topology in Alster's case. The set Y of irrationals in Michael's example is *concentrated* about $\mathbb{Q}$, i.e., every open set containing $\mathbb{Q}$ contains all but countably many points of Y. Lawrence [32] showed that there is a concentrated Michael space iff $\mathfrak{b} = \omega_1$, and proved that the weight of any Michael space is at least $\min\{\mathfrak{b}, \omega_\omega\}$. Fleissner noted that Alster's example is also closely related to cardinal characteristics of the continuum; namely, $\mathfrak{b} = \mathfrak{d} = \mathrm{cov}(\mathcal{M})$ suffices to construct his example, where $\mathrm{cov}(\mathcal{M})$ is the least cardinal of a cover of $\mathbb{R}$ by meager sets. J. Moore [40] later improved this to $\mathfrak{d} = \mathrm{cov}(\mathcal{M})$. Moore also constructs, assuming $\mathfrak{b} = \mathfrak{d} = \mathrm{cov}(\mathcal{M}) = \aleph_{\omega+1}$, a Michael space X of weight $\aleph_\omega$ such that $X \times \omega^\omega$ has Lindelöf degree $\aleph_\omega$. Note that $X \times \omega^\omega$ must be linearly Lindelöf but not Lindelöf, providing an interesting connection with Mary Ellen's previous conjecture. In the same paper, Moore shows that there is a model of $\mathrm{cov}(\mathcal{M}) < \mathfrak{b} < \mathfrak{d}$ in which there is a Michael space.

There is a Michael-like space if one goes a step up in cardinality. In 1996, Lawrence [33] constructed a Lindelöf space X of weight ω_1 such that $X \times \omega_1^\omega$ is not normal (where ω_1 carries the discrete topology). Later, Burke and Pol [10] showed that there is such an example of the form $A \cup C \subset [0,1]$, where points of C are isolated and points of A have their usual Euclidean neighborhoods.

More recently, close connections to the "productively Lindelöf" property have been found, where a space is *productively Lindelöf* if its product with any Lindelöf space is Lindelöf. Tall and Tsaban [66] proved that there is a Michael space iff every analytic productively Lindelöf (metrizable?) space is σ-compact, and Repovs and Zdomskyy [50] showed that the existence of a Michael space implies that every regular productively Lindelöf space is Menger. Finally, we mention that a lemma due to Duanmu, Tall, and Zdomskyy [19] implies that if there is a Lindelöf space, regular or not, whose product with ω^ω is not Lindelöf, then there is a regular example and hence a Michael space.

Conjecture 15. There is an L-space.

An *L-space* is a regular space that is hereditarily Lindelöf but not hereditarily separable, while an *S-space* is one that is hereditarily separable but not hereditarily Lindelöf. It may be noteworthy that Mary Ellen worded Conjecture 15 this way despite it being consistent that there are no S-spaces and despite all the "dualities" between S and L spaces, especially the theorem that there is a strong S-space iff there is a strong L-space (a *strong S-space [respṡtrong L-space]* is a space, all of whose finite powers are S-spaces [respL̇-spaces]). See [51] for these and many other facts about S and L spaces.

Mary Ellen's Conjecture 15 turned out to be correct. Justin Tatch Moore, using some sophisticated functions associated with walks on ordinals, constructed an L-space using only the usual (ZFC) axioms of set theory [41].

The last two conjectures involve box products, a topic that once again Mary Ellen got started by proving that, assuming CH, the box product of countably many real lines (or more generally, σ-compact locally compact metric spaces) is paracompact [54].

Conjecture 16. Every box product of ω_1 copies of $\omega+1$ is normal (or paracompact)

This one turned out to be false, as Brian Lawrence showed [35]. This also settled an old problem by A.H. Stone: is the box product of uncountably many real lines normal? As far as we know, this is one of only three discoveries in set-theoretic topology to have a research announcement appear in the Proceedings of the National Academy of Sciences [34]. The other two were Michael Wage's disproof of the logarithmic inequality $dim(X\times Y)\leq dim(X)+dim(Y)$ and Fleissner's more or less final solution of the Normal Moore Space Problem.

Conjecture 17. Every box product of ω copies of $\omega+1$ is normal (or paracompact)

This old classic problem is still unsolved. For an up to date account of research on it, we refer the reader to the survey article by Judy Roitman and Scott Williams in this issue [52].

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