SEQUENTIAL PROPERTIES OF FUNCTION SPACES WITH THE COMPACT-OPEN TOPOLOGY

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Abstract. Let M be the countably infinite metric fan. We show that $C_k(M, 2)$ is sequential and contains a closed copy of Arens space S_2. It follows that if X is metrizable but not locally compact, then $C_k(X)$ contains a closed copy of S_2, and hence does not have the property AP.

We also show that, for any zero-dimensional Polish space X, $C_k(X, 2)$ is sequential if and only if X is either locally compact or the derived set X' is compact. In the case that X is a non-locally compact Polish space whose derived set is compact, we show that all spaces $C_k(X, 2)$ are homeomorphic, having the topology determined by an increasing sequence of Cantor subspaces, the nth one nowhere dense in the $(n+1)$st.

1. Introduction

Let $C_k(X)$ be the space of continuous real-valued functions on X with the compact-open topology. $C_k(X)$ for metrizable X is typically not a k-space, in particular not sequential. Indeed, by a theorem of R. Pol [8], for X paracompact first countable (in particular, metrizable), $C_k(X)$ is a k-space if and only if X is locally compact, in which case X is a topological sum of locally compact σ-compact spaces and $C_k(X)$ is a product of completely metrizable spaces. A similar result holds for $C_k(X, [0, 1])$: it is a k-space if and only if X is the topological sum of a discrete space and a locally compact σ-compact space, in which case $C_k(X)$ is the product of a compact space and a completely metrizable space. It follows that, for separable metric X, the following are equivalent:

(1) $C_k(X)$ is a k-space;
(2) $C_k(X)$ is first countable;
(3) $C_k(X)$ is a complete separable metrizable space, i.e., a Polish space;
(4) X is a locally compact Polish space.

The same equivalences hold for $C_k(X, [0, 1])$. On the other hand, for Polish X, $C_k(X)$ always has the (strong) Pytkeev property [9].

A space X has the property AP if whenever $x \in \overline{A} \setminus A$, there is some $B \subseteq A$ such that $x \in \overline{B} \subseteq A \cup \{x\}$. X has the property WAP when a subset A of X is closed if and only if there is no $B \subseteq A$ such that $|\overline{B} \setminus A| = 1$. Thus, every Fréchet space is AP and every sequential space is WAP. It was asked in [5] whether $C_k(\omega^\omega)$ is WAP.

In this note, we first show that if X is metrizable but not locally compact, then $C_k(X)$ contains a closed copy of Arens space S_2, and hence is not AP. In fact, such a closed copy of S_2 is contained in $C_k(M, 2)$, where M is the countable metric fan. We then show that $C_k(M, 2)$ is sequential, in contrast to the full function space $C_k(M)$. Next we show that for a zero-dimensional Polish space X, if $C_k(X, 2)$ is not metrizable (which is the case if and only if X is not locally compact), then $C_k(X,2)$ is sequential if and only if the derived set X' is compact. We obtain a complete description of $C_k(X,2)$ for a non-locally compact Polish X such that X' is compact: any such $C_k(X,2)$ is homeomorphic to the space $(2^\omega)^\infty$, which is the space with the topology determined by an increasing sequence of Cantor sets, the nth one nowhere dense in the $(n+1)$st.

2. When $C_k(X)$ contains S_2

Arens's space S_2 is the set

$$\{(0,0),(1/n,0),(1/n,1/nm) : n,m\in\omega\setminus\{0\}\}\subseteq\mathbb{R}^2$$

carrying the strongest topology inducing the original planar topology on the convergent sequences $C_0=\{(0,0),(\frac{1}{n},0):n>0\}$ and $C_n=\{(\frac{1}{n},0),(\frac{1}{n},\frac{1}{nm}):m>0\}$, $n>0$. The *sequential fan* is the quotient space $S_\omega=S_2/C_0$ obtained from the Arens space by identifying the points of the sequence C_0 [6]. S_ω is a non-metrizable Fréchet-Urysohn space, and S_2 is sequential and not Fréchet-Urysohn. In fact, any space which is sequential but not Fréchet-Urysohn contains S_2 as a subspace.

The *countably infinite metric fan* is the space $M=(\omega\times\omega)\cup\{\infty\}$, where points of $\omega\times\omega$ are isolated, and the basic neighborhoods of ∞ are $U(n)=\{\infty\}\cup((\omega\setminus n)\times\omega)$, $n\in\omega$. M is not locally compact at its non-isolated point ∞.

Lemma 2.1. *$C_k(M,2)$ contains a closed copy of S_2.*

Proof. For each $n>1$ and each k, let

$$U(n,k)=(\{0\}\times n)\ \cup\ ((n\setminus\{0\})\times k)\ \cup\ U(n),$$

and let $f_{n,k}$ be the member of $C_k(M)$ which is 0 on $U(n,k)$ and 1 otherwise (i.e., the characteristic function of $M\setminus U(n,k)$).

Let $f_n\in C_k(M)$ be the function which is 1 on $\{0\}\times(\omega\setminus n)$ and 0 otherwise, and let c_0 be the constant 0 function. For each n, $\lim_k f_{n,k}=f_n$, and $\lim_n f_n=c_0$. Thus, c_0 is a limit point of the set $A=\{f_{n,k}:n>1,k\in\omega\}$. Let $S=\{f_n:n>1\}$, and $X=\{c_0\}\cup S\cup A$.

We claim that X is homeomorphic to the Arens space S_2. It suffices to show that for each sequence $(k_n)_{n>1}$, c_0 is not in the closure of the set $\{f_{n,k}:k<k_n,\ n>1\}$. Given $(k_n)_{n>1}$, set $K=\{(n-1,k_n):n>1\}\cup\{\infty\}$. Then K is a sequence convergent to ∞, and for each $f_{n,k}\in\{f_{n,k}:k<k_n,\ n>1\}$ there exists $x\in K$, namely, $x=(n-1,k_n)$, such that $f(x)=1$. Therefore $\{f_{n,k}:k<k_n,\ n>1\}$ does not intersect the neighborhood $\{f\in C_k(M,2):f\restriction K\equiv 0\}$ of c_0, and hence does not contain c_0 in its closure.

By [6, Corollary 2.6], if every point z in a topological space Z is regular G_δ (i.e., $\{z\}$ is equal to $\bigcap_n\overline{U_n}$ for some open neighborhoods U_n of z), and Z contains a copy of S_2, then Z contains a closed copy of S_2. Since every point of $C_k(M,2)$ is regular G_δ, $C_k(M,2)$ contains a closed copy of S_2. In fact, the space X constructed above is closed, even in $C_p(M,2)$. $\square$

Theorem 2.2. *If X is metrizable and not locally compact, then $C_k(X)$ contains closed copies of S_2 and S_ω.* $\square$

Proof. By Lemma 8.3 of [4], a first countable space X contains a closed topological copy of the space M if and only if X is not locally compact. E.A. Michael [7, Theorem 7.1] observed that, for Y a closed subspace of a metrizable space X, the linear extender $e : C(Y) \to C(X)$ given by the Dugundji extension theorem is a homeomorphic embedding when both $C(Y)$ and $C(X)$ are given the compact-open topology (or the topology of uniform convergence, or pointwise convergence). Thus we have that for each metrizable space X which is not locally compact, $C_k(M,2)$ is closely embedded in $C_k(X)$, and hence $C_k(X)$ contains a closed copy of S_2. Finally, $C_k(X)$ also contains a closed copy of S_ω because for any topological group G, G contains a (closed) copy of S_2 if and only if it contains a closed copy of S_ω. $\square$

Remark. C.J.R. Borges [2] showed that the Dugundji extension theorem holds for the class of stratifiable spaces, and hence Theorem 2.2 holds more generally for first countable stratifiable spaces.

3. Sequentiality of $C_k(X,2)$

A topological space X *carries the inductive topology* with respect to a closed cover $\mathcal{C}$ of X, if for each $F \subseteq X$, F is closed whenever $F \cap C$ is closed in C for each $C \in \mathcal{C}$. A topological space is a *k-space* (respectively, *sequential space*) if it carries the inductive topology with respect to its cover by compact (respectively, compact metrizable) subspaces. X is sequential if and only if for every non-closed $A \subseteq X$, there exists a sequence in A converging to a point in $X \setminus A$.

Since the metric fan M is not locally compact, $C_k(M)$ and $C_k(M,[0,1])$ are not k-spaces [8]. However, we have the following.

Theorem 3.1. *$C_k(M,2)$ is sequential.*

Proof. Suppose not. Then there is $A \subseteq C_k(M,2)$ which is not closed and yet contains all limit points of convergent sequences of its elements. As M is zero-dimensional, $C_k(M,2)$ is homogeneous. Thus, without loss of generality, we may assume that $c_0 \in \overline{A} \setminus A$, where c_0 is the constant 0 function. We may additionally assume that $f(\infty) = 0$ for all $f \in A$. Let $A_n = \{f \in A : f(U(n)) = \{0\}\}$.

Note that the sets A_n are increasing with n, and their union is A.

Claim 3.2. *There exists a sequence $(k_n)_{n\in\omega}$ such that for each n with $f \in A_{n+1}$, $1 \in f(\bigcup_{i\le n}\{i\} \times k_i)$.*

Proof. By induction. Assume that for all $i < n$, there are k_i such that $f \in A_{i+1}$ implies $1 \in f(\bigcup_{j\le i}\{j\} \times k_j)$, but that for each k, there is $f_k \in A_{n+1}$ such that $f_k((\bigcup_{i<n}\{i\} \times k_i) \cup (\{n\} \times k)) = \{0\}$. Let $f'_k = f_k \restriction (n+1) \times \omega$. As $2^{(n+1)\times\omega}$ is homeomorphic to the Cantor space, there is a subsequence $\{f'_{k_i}\}$ of $\{f'_k\}$, converging to an element $f' \in 2^{(n+1)\times\omega}$. As $f_k \in A_{n+1}$, $f_k(U(n+1)) = \{0\}$. Define $g \in C_k(M)$ by $g(U(n+1)) = \{0\}$ and $g \restriction (n+1) \times \omega = f'$. Then in $C_k(M)$, $g = \lim_i f_{k_i}$, and therefore $g \in A$. As $f'_k(\{n\} \times k) = \{0\}$, $g(\{n\} \times \omega) = \{0\}$. As $g(U(n+1)) = \{0\}$, $g(U(n)) = \{0\}$, and thus $g \in A_n$. But $g(\bigcup_{i\le n-1}\{i\} \times k_i) = \{0\}$ (indeed, this holds for all f_k's), contradicting the induction hypothesis. $\square$

Let

$$K = (\bigcup_{i\in\omega}\{i\} \times k_i) \cup \{\infty\}.$$

Let V be the set of all functions which map K into the interval $(-1/2, 1/2)$. Then V is a neighborhood of c_0 which misses A, a contradiction. $\square$

We proceed to characterize the zero-dimensional Polish spaces X such that $C_k(X,2)$ is sequential.

A topological space Y has the *strong Pytkeev property* [9] (respectively, *countable* cs**-character*) if for each $y \in Y$, there is a *countable* family $\mathcal{N}$ of subsets of Y, such that for each neighborhood U of y and each $A \subseteq Y$ with $y \in \overline{A} \setminus A$ (respectively, each sequence A in $Y \setminus \{y\}$ converging to y), there is $N \in \mathcal{N}$ such that $N \subseteq U$ and $N \cap A$ is infinite.

For every Polish space X the space $C_k(X)$ has the strong Pytkeev property [9, Corollary 8]. Thus, any subspace of $C_k(X)$ has the strong Pytkeev property, and therefore has countable cs*-character.

An *mk_ω-space* is a topological space which carries the inductive topology with respect to a countable cover of compact metrizable subspaces. A topological group G is an *mk_ω-group* if G is an mk_ω-space.

$C_k(X,2)$ has a natural structure of a topological group.

Theorem 3.3 ([3]). *Let G be a sequential non-metrizable topological group with countable* cs**-character. Then G contains an open mk_ω-subgroup H and thus is homeomorphic to the product $H \times D$ for some discrete space D.*

Corollary 3.4. *Let G be a sequential separable topological group with countable* cs**-character. If G is not metrizable, then G is σ-compact.*

Lemma 3.5. *Let X be a zero-dimentional first countable space. Then $C_k(X,2)$ is metrizable if and only if X is locally compact and σ-compact.*

Proof. ($\Leftarrow$) As $C_k(X,2)$ is a topological group, its metrizability is equivalent to its first countability at c_0, the constant zero function.

($\Rightarrow$) Assume that $C_k(X,2)$ is metrizable and fix a countable base $\{W_n : n \in \omega\}$ at c_0. Without loss of generality, $W_n = \{f \in C_k(X,2) : f \restriction K_n \equiv 0\}$ for some compact $K_n \subseteq X$, and $K_n \subseteq K_{n+1}$ for all n. It suffices to prove that for every $x \in X$ there are a neighborhood U of x and $n \in \omega$, such that $U \subseteq K_n$. If not, we can find $x \in X$ and a sequence $(x_n)_{n\in\omega}$ of elements of X such that $x_n \in U_n \setminus K_n$, where $\{U_n : n \in \omega\}$ is a decreasing base at x. Set $K = \{x\} \cup \{x_n : n \in \omega\}$ and $W = \{f \in C_k(X,2) : f \restriction K \equiv 0\}$. Since $K_n \cap K$ is finite for every $n \in \omega$, there exists a function $f \in C_k(X,2)$ such that $f \restriction K_n \equiv 0$ but $f \restriction K \not\equiv 0$, and hence $W_n \not\subseteq W$ for all $n \in \omega$. This contradicts our assumption that $\{W_n\}$ is a local base at c_0. □

For a topological space X, X' is the set of all non-isolated points of X.

Theorem 3.6. *Let X be a zero-dimensional Polish space which is not locally compact. Then $C_k(X,2)$ is sequential if and only if the derived set X' is compact.*

Proof. Assume that X' is compact and consider the subgroup $H = \{f \in C_k(X,2) : f \restriction X' \equiv 0\}$. H is an open subgroup of $C_k(X,2)$, and thus it suffices to prove that H is sequential. Since X is not locally compact, there is a clopen outer base $\{U_n : n \in \omega\}$ of X' such that $U_0 = X$, $U_{n+1} \subseteq U_n$, and $U_n \setminus U_{n+1}$ is infinite for all $n \in \omega$. Let $f : X \to M$ be a map such that $f(X') = \{\infty\}$ and $f \restriction (U_n \setminus U_{n+1})$ is an injective map onto $\{n\} \times \omega$. Then the map

$$f^* : \{g \in C_k(M,2) : g(\infty) = 0\} \to H$$

assigning to g the composition $g \circ f$ is easily seen to be a homeomorphism, and hence H is sequential.

Now assume that X' is not compact. Then there exists a countable closed discrete subspace $T \subseteq X'$, and hence there exists a discrete family $\{U_t : t \in T\}$ of clopen subsets of X such that $t \in U_t$ for all $t \in T$. $C_k(X, 2)$ contains a closed copy of the product $\Pi_{t\in T} C_k(U_t, 2)$.

Claim 3.7. *Let Z be a non-discrete metrizable separable zero-dimensional space. Then $C_k(Z, 2)$ is not compact.*

Proof. If Z is locally compact, then it contains a clopen infinite compact subset C. Then $C_k(C, 2)$ is a closed subset of $C_k(Z, 2)$ homeomorphic to ω, and hence $C_k(Z, 2)$ is not compact.

If Z is not locally compact, then Z contains a closed copy Y of M. By Lemma 2.1, $C_k(Y, 2)$ contains a closed copy of S_2, and is thus not compact. As restriction to Y is a continuous map from $C_k(Z, 2)$ onto $C_k(Y, 2)$, $C_k(Z, 2)$ is not compact. □

Claim 3.8. *If none of the spaces X_i, $i \in \omega$, is compact, then the product $\Pi_{i\in\omega} X_i$ is not σ-compact.*

Proof. A simple diagonalization argument. □

Since $T \subseteq X'$, U_t is not discrete for all $t \in T$. By Claims 3.8 and 3.7, the product $\Pi_{t\in T} C_k(U_t, 2)$ is not σ-compact. Thus, $C_k(X, 2)$ is not σ-compact.

As X is Polish, $C_k(X)$ has the strong Pytkeev property [9], and thus has countable cs*-character. Consequently, so does its subspace $C_k(X, 2)$. As $C_k(X, 2)$ is separable and X is not locally compact, $C_k(X, 2)$ is not first countable, and hence it is not metrizable. Apply Corollary 3.4. □

Corollary 3.9. *$C_k(\omega \times M, 2)$ is not sequential.* □

Let $(0) \in 2^\omega$ be the constant zero sequence. Following [1], let $(2^\omega)^\infty$ be the space $\bigcup_{n\in\omega}(2^\omega)^n$, where $(2^\omega)^n$ is identified with the subspace $(2^\omega)^n \times \{(0)\}$ of $(2^\omega)^{n+1}$, with the inductive topology with respect to the cover $\{(2^\omega)^n : n \in \omega\}$.

Theorem 3.10 (Banakh [1])**.** *Every non-metrizable uncountable zero-dimensional mk_ω-group is homeomorphic to $(2^\omega)^\infty$.*

Corollary 3.11. *For zero-dimensional Polish spaces X, the following are equivalent:*

(1) *$C_k(X, 2)$ is sequential but not metrizable;*
(2) *$C_k(X, 2)$ is homeomorphic to $(2^\omega)^\infty$;*
(3) *X is not locally compact but X' is compact.*

Proof. (1) → (3). Since $C_k(X, 2)$ is not metrizable, X is not locally compact (Lemma 3.5). By Theorem 3.6, X' is compact.

(3) → (1). Since X is not locally compact, $C_k(X, 2)$ is not metrizable (Lemma 3.5). By Theorem 3.6, the compactness of X' implies that $C_k(X, 2)$ is sequential.

(1) → (2). By [9, Corollary 8], $C_k(X, 2)$ has countable cs*-character. Applying Theorem 3.3, we have that $C_k(X, 2)$ contains an open mk_ω-subgroup. Since $C_k(X, 2)$ is separable, it is an mk_ω-group. Apply Theorem 3.10. □

$C_k(\omega \times M, 2)$ is homeomorphic to $C_k(M, 2)^\omega$, and hence to $((2^\omega)^\infty)^\omega$. Thus, a negative answer to the following question would imply that $C_k(P)$ is not WAP for "most" Polish spaces, including ω^ω and some σ-compact ones.

Question 3.12. *Does the space $((2^\omega)^\infty)^\omega$ have the WAP property? What about S_2^ω?*

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