Generalized metrizable spaces

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1 Introduction

Roughly speaking, I consider a class of spaces to be a class of generalized metrizable spaces if every metrizable space is in the class, and if the defining property of the class gives its members enough structure to make a reasonably rich and interesting theory. See my article [71] for basic information about many of these classes. In this article, I will survey results in this area from approximately 2001 to the present. This article can be considered a sequel to [74] and [77] which appeared in earlier volumes in this series.

Throughout this article, all spaces are assumed to be at least T_1; in some sections we will announce that more separation is assumed.

2 Around regular G_δ-diagonals

A space X has a (*regular*) G_δ*-diagonal* if the diagonal $\Delta = \{(x,x) : x \in X\}$ is a (regular) G_δ-set in X^2, where a subset H of a space Y is regular G_δ if there are open sets U_n, $n \in \omega$, such that $H = \bigcap_{n\in\omega} U_n = \bigcap_{n\in\omega} \overline{U}_n$. Also, X has a *zero-set diagonal* if $\Delta = f^{-1}(0)$ for some continuous $f : X^2 \to \mathbb{R}$. Finally, X is *submetrizable* if X has a weaker metrizable topology. Clearly

submetrizable $\Rightarrow$ zero-set diagonal $\Rightarrow$ regular G_δ-diagonal $\Rightarrow$ G_δ-diagonal.

If X is ccc and submetrizable, then X has a weaker separable metrizable topology and hence has cardinality not greater than 2^ω. This suggests the question, asked by Arhangel'skii [3], and by Ginsburg and Woods [66] specifically for G_δ-diagonal, whether the same might hold when submetrizable is replaced by a weaker diagonal condition.

Shakhmatov [133] (see also Uspenskii [154]) showed that regular ccc spaces with a G_δ-diagonal can be arbitrarily large. But Buzyakova [45] showed that the regular G_δ-diagonal case is different.

Theorem 2.1. *A ccc space with a regular G_δ-diagonal has cardinality at most 2^ω.*

In another paper, Buzyakova [43] proved the following related result:

Theorem 2.2. *If X has a zero-set diagonal and X^2 has countable extent (i.e., every uncountable subset of X^2 has a limit point), then X is submetrizable (with respect to a separable metrizable space).*

This theorem was motivated in part by an old theorem of Martin [108] stating that a separable space with a zero-set diagonal must be submetrizable. It is not known if Theorem 2.2 holds when X^2 has countable extent is replaced by X has countable extent, or when zero-set diagonal is replaced by regular

G_δ-diagonal. Buzyakova does show that if X^2 has countable extent and X has a regular G_δ-diagonal then X has a weaker 2^{nd}-countable Hausdorff topology.
Buzyakova [44] also constructed some relevant examples.

Theorem 2.3. *1. There is a hereditary realcompact locally compact locally countable separable Tychonoff space with countable extent and a G_δ-diagonal that fails to be submetrizable.*
2. Assuming the Continuum Hypothesis, there is a pseudocompact non-compact locally compact locally countable separable Tychonoff space that has countable extent and a G_δ - diagonal.

In the same paper, she asks the following questions suggested by the above examples:

Question 2.4. 1. Is there a ZFC example of a pseudocompact non-compact Tychonoff space with countable extent that has a G_δ-diagonal?
2. Let X be a countably paracompact Tychonoff space with countable extent and a G_δ- diagonal. Is then X submetrizable? What if X is first-countable (or locally compact)?

Recalling that X has a G_δ-diagonal iff there is a sequence $\mathcal{G}_n$, $n \in \omega$, of open covers of X such that $\{x\} = \cap\{st(x, \mathcal{G}_n) : n \in \omega\}$[1] for all $x \in X$, Arhangel'skii and Buzyakova [13] define X to have a *rank k diagonal* iff $\{x\} = \cap\{st^k(x, \mathcal{G}_n) : n \in \omega\}$ for all $x \in X$. Moore spaces have a rank 2 diagonal, submetrizable implies rank k for all k, and P. Zenor had shown that rank 3 diagonal implies regular G_δ-diagonal. In [13], the authors construct a separable Tychonoff space with a diagonal of exactly rank 3 (rank 3 but not higher) which does not have a zero-set diagonal (hence is not submetrizable). This seems to be the first known example of a Tychonoff space with a regular G_δ-diagonal with no zero-set diagonal, as well as the first known example of a separable Tychonoff space with a regular G_δ-diagonal which is not submetrizable.
The question of the existence of spaces having diagonals of exactly rank k for higher k was left open, but was subsequently answered by Y. Zuoming and Y. Ziqiu [159]:

Example 2.5. For all $k \geq 4$, there is a separable subparacompact Tychonoff space X with a diagonal of exactly rank k which does not have a zero-set diagonal.

We end this section with an interesting result of Burke and Arhangel'skii. A subset $A \subseteq X$ is said to be *bounded in* X if each locally finite family of open sets in X, all of which meet A, is finite. A space which is bounded in itself is usually called feebly compact. W. G. McArthur [111] proved that every regular feebly compact space with a regular G_δ-diagonal is compact and metrizable. Burke and Arhangel'skii [11] generalize this as follows:

[1] If $\mathcal{G}$ is a collection of subsets of X, and $P \subset X$, then $st(P, \mathcal{G}) = st^1(P, \mathcal{G}) = \bigcup\{G \in \mathcal{G} : G \cap P \neq \emptyset\}$. For $k > 1$, $st^k(P, \mathcal{G}) = st(st^{k-1}(P, \mathcal{G}), \mathcal{G})$. Also $st(\{x\}, \mathcal{G}) = st(x, \mathcal{G})$.

Theorem 2.6. *If X is a regular space with a regular G_δ-diagonal, then the closure of every bounded subset of X is metrizable.*

This result was improved in [13]: under the same assumptions, one may conclude that every closed and bounded subset of X is completely metrizable. See Section 23 for results in [11] in the area of topological algebra.

3 Small diagonal

Spaces in this section are assumed to be regular and T_1. A space X has a *small diagonal* provided that whenever an uncountable subset A of $X \times X$ is disjoint from the diagonal, there is an uncountable subset of A whose closure is disjoint from the diagonal. This condition is a natural weakening of the G_δ-diagonal property. As is well-known, (countably) compact spaces with a G_δ-diagonal are metrizable. An old problem of Hušek asks if a compact Hausdorff space with a small diagonal must be metrizable. Assuming CH, the answer is positive; this is due to Juhasz and Szentmiklossy [93], whose results also imply that every compact Hausdorff space with a small diagonal has countable tightness. But Hušek's problem is unsolved in ZFC. Here are some recent partial and/or related results:

Theorem 3.1. *1. PFA implies every compact space with a small diagonal is metrizable* [56]*;*
2. Assuming $\diamondsuit^+$, there is a perfect preimage of ω_1 with a small diagonal [56]*;*
3. No scattered perfect preimage of ω_1 has a small diagonal [57]*;*
4. If $2^\omega > \omega_1$, then there is a Lindelöf space with a small diagonal but no G_δ-diagonal [56]*;*
5. Assuming MA(Cohen)+$2^\omega = 2^{\omega_1}$, there is a countably compact space with a small diagonal which is not metrizable [56]*;*
6. If X is compact and has a small diagonal, then every ccc subspace of X has countable π-weight [53]*;*
7. If X is compact, has a small diagonal, and admits a continuous map onto a space of weight ω_1 with metric fibers, then X is metrizable [53]*;*
8. If there is a Luzin set, then every compact space with a small diagonal has points of countable character [53]*.*

The second result answers several questions in [75], and the fourth and fifth results show that some examples in [75] exist under weaker assumptions. The seventh result generalizes my result [75] that a metrizably fibered compact space with a small diagonal must be metrizable.

4 Continuously Urysohn, 2-Maltsev, and (P)

P. Zenor[158] defined the class of *weakly continuously Urysohn (wcU)* spaces as those spaces X admitting a continuous function $\phi : X^2 \setminus \Delta \times X \to \mathbb{R}$ such that $\phi(x, y, x) \neq \phi(x, y, y)$ for all $x \neq y \in X$. It is easy to show that zero-set diagonal implies wcU (if $f : X^2 \to [0, 1]$ is such that $\Delta = f^{-1}(0)$, let $\phi(x, y, z) = f(x, z)/(f(x, z) + f(y, z))$). Zenor proved that X being wcU is equivalent to the existence of a certain kind of continuous extender of real-valued functions defined on compact subsets of X.

Theorem 4.1. [158] *Let X be a Hausdorff space, and let $\mathcal{C}_K(X)$ be the space of all continuous partial functions into $\mathbb{R}$ whose domain is some compact subset of X, equipped with the Vietoris topology (identifying a partial function with its graph). Then the following are equivalent:*

1. *There is a continuous "extender" $e : \mathcal{C}_K(X) \times X \to \mathbb{R}$ such that $e(f, x) = f(x)$ for all $x \in$ dom f;*
2. *X is weakly continuously Urysohn.*

As the name implies, wcU spaces generalize the previously known class of *continuously Urysohn (cU)* spaces, i.e., spaces X admitting a continuous function $\phi : X^2 \setminus \Delta \to C^*(X)$, where $C^*(X)$ is the space of all bounded continuous real-valued functions with the topology of uniform convergence, such that $\phi(x, y)(x) \neq \phi(x, y)(y)$ for every $x \neq y \in X$. This class was first studied by Stepanova [145], who showed that a paracompact p-space which is cU must be metrizable. Zenor and I [85] showed:

Theorem 4.2. *1. A regular wcU $w\Delta$-space has a base of countable order;*
2. Monotonically normal wcU spaces are hereditarily paracompact;
3. Separable wcU spaces are submetrizable (hence cU);
4. Nonarchimedean spaces are cU.

The first part of the above theorem generalizes Stepanova's result (since paracompact p-spaces are $w\Delta$ and paracompact spaces with a base of countable order are metrizable), and the second part generalizes the same result for cU GO-spaces obtained by Bennett and Lutzer [28]. The fourth part was subsequently generalized by A. Guldurdek [87] by showing that protometrizable spaces are cU. Guldurdek also showed that the wcU and cU properties are not preserved by finite products or perfect images, but that wcU spaces are preserved by perfect open maps. Surprisingly, the following remains open:

Question 4.3. [85] Is every wcU space cU?

One may rephrase the wcU property as follows.[2] Given a space X and $i = 1, 2$, or 3, let $\Pi_i = \{(x_1, x_2, x_3) \in X^3 : x_j = x_k \text{ if } i \notin \{j, k\}\}$, and let Δ

[2] This was noted in J. Chaber's review [46] of [85].

be the diagonal of X^3. Then X is wcU iff $\Pi_1 \setminus \Delta$ and $\Pi_2 \setminus \Delta$ can be separated in $X^3 \setminus \Pi_3$ by a continuous function. Inspired by results in the theory of compact topological groups, Gartside and Reznichenko [65] define a space to be *generalized 2-Maltsev* if there is a G_δ subset of $X^3 \setminus \Pi_3$ containing $\Pi_1 \setminus \Delta$ and whose closure in $X^3 \setminus \Pi_3$ misses $\Pi_2 \setminus \Delta$.[3] Clearly wcU implies generalized 2-Maltsev. They note that X is 2-Maltsev if there is a weaker topology τ on X such that (X, τ) has a G_δ-diagonal or $(X, \tau)^3 \setminus \Delta$ is normal. They go on to define the following still weaker condition (P):

(P) To each $M \in [X]^{<\omega}$, one can assign an open cover $\gamma(M)$ of X such that, for any $A \subset X$ and $x \in \overline{A} \setminus A$, we have $\{x\} = \bigcap\{st(x, \gamma(M)) : M \in [A]^{<\omega}\}$.

Then they prove:

Theorem 4.4. *1. Generalized 2-Maltsev spaces have (P);*
2. A separable space with (P) has a G_δ-diagonal;
3. A regular M-space with (P) is metrizable;
4. A regular Σ-space with (P) is a σ-space.

The third and fourth items in the above theorem have results of M. Katetov, J. Chaber, J. Pelant, and myself as corollaries.

5 Gruenhage spaces and property $(*)$

We now mention a couple of properties related to G_δ-diagonals which have had some impact in functional analysis. All spaces in this section are assumed to be Hausdorff. Long ago [73] I introduced the following property, and showed that any compact space with this property has a dense metrizable subspace. Given a space X, I called a collection $\mathcal{U}$ of open subsets of X a *σ-distributively point-finite T_0-separating open cover* if $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$ such that, given $x \neq y \in X$, there is $n \in \omega$ such that

1. there exists $U \in \mathcal{U}_n$ containing exactly one of x and y;
2. $\mathrm{ord}(x,\mathcal{U}_n) < \omega$ or $\mathrm{ord}(y,\mathcal{U}_n) < \omega$.

(Here, $\mathrm{ord}(x, \mathcal{V}) = |\{V \in \mathcal{V} : x \in V\}|$.)

Subsequently, spaces (not necessarily compact) with such an open cover $\mathcal{U}$ were called *Gruenhage spaces* (see Def. 2.1 of [131]), and were shown to have a connection with renorming in Banach spaces. For example, R.J. Smith proved:

Theorem 5.1. [137] *If K is a compact Gruenhage space, then $C(K)^*$ admits a strictly convex dual norm.*

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[3] The definition of generalized 2-Maltsev given here is equivalent to that in [65]; we have only interchanged the roles of Π_2 and Π_3.

Later, Smith, Orihuela, and Troyanski defined a property weaker than both G_δ-diagonal and Gruenhage, which they called $(*)$. A space X has $(*)$ iff there is a sequence $\mathcal{U}_n$, $n \in \omega$, of collections of open subsets of X such that, given $x \neq y \in X$, there is $n \in \omega$ such that

1. $U \cap \{x, y\}$ is a singleton for some $U \in \mathcal{U}_n$;
2. No $U \in \mathcal{U}_n$ contains both x and y.

They proved:

Theorem 5.2. ([118])

1. *A compact scattered space K has $(*)$ iff $C(K)^*$ admits a strictly convex dual norm;*
2. *Every Gruenhage space and every space with a G_δ-diagonal has $(*)$*
3. *Assuming CH or $\mathfrak{b} = \omega_1$, there is a locally compact scattered space with a G_δ-diagonal which is not Gruenhage, and hence its one-point compactification is a compact space satisfying $(*)$ but not Gruenhage.*

Subsequently, Smith [138] obtained a ZFC example satisfying the conditions of 5.2(3). We list some other results about these properties in the following theorem.

Theorem 5.3. *1. X is Gruenhage iff there is a sequence $\mathcal{U}_n$, $n \in \omega$, of collections of open subsets of X, and sets R_n, $n \in \omega$, such that $U \cap V = R_n$ for every $U \neq V \in \mathcal{U}_n$, and, given $x \neq y \in X$, there is $n \in \omega$ and $U \in \mathcal{U}_n$ such that $U \cap \{x, y\}$ is a singleton* [137]*;*

2. *If $|X| \leq 2^\omega$, then X is Gruenhage iff there is a sequence U_n, $n \in \omega$, of open sets such that for any $x \neq y \in X$, $U_n \cap \{x, y\}$ is a singleton for some n* [139]*;*
3. *The perfect image of a Gruenhage space is Gruenhage* [137]*;*
4. *The continuous image of a compact scattered space satisfying $(*)$ also satisfies $(*)$* [118]
5. *A countably compact space with $(*)$ is compact* [118]*;*
6. *A tree with the interval topology has $(*)$ iff it is Gruenhage* [118].

It is unsolved whether or not $C(K)^*$ admits a strictly convex dual norm whenever K is compact and satisfies $(*)$, or if property $(*)$ is preserved by perfect mappings.

6 Stratifiable vs. M_1

Ceder [48] defined M_1*-spaces* (resp., M_2*-spaces*) as those regular spaces having a σ-closure-preserving base (resp., quasi-base), where $\mathcal{B}$ is a *quasi-base* for X if whenever $x \in U$, U open, there is some $B \in \mathcal{B}$ with $x \in B^o \subset B \subset U$. He also defined M_3-spaces, renamed by Borges as *stratifiable* spaces; these are now known to be the same as M_2-spaces. However, whether or not stratifiable and M_1 are the same is still an open question. The best partial result is the following, due to Mizokami, Shimane, and Kitamura [115]:

Theorem 6.1. *A stratifiable space X is M_1 if it has the following property:*

(δ) Whenever U is dense open in X and $x \in X \setminus U$, there is a closure-preserving collection $\mathcal{F}$ of closed subsets of X that is a network at x, and such that $\overline{F \cap U} = F$ for every $F \in \mathcal{F}$.

Sequential stratifiable spaces satisfy (δ); more generally, so do stratifiable spaces having the following property which has been called *WAP* (*weak approximation by points*) or *weakly Whyburn*:

(WAP) If A is not closed, there exists $B \subset A$ such that $|\overline{B} \setminus A| = 1$.

It is known that a stratifiable space is M_1 if each point has a closure-preserving neighborhood base. In his original paper, Ceder asked if every M_1-space has the property that each closed subset has a closure-preserving outer base[4]. This question was unsolved until Mizokami [114] finally gave an the following (difficult!) affirmative answer.

Theorem 6.2. *Every closed subset of an M_1-space has a closure-preserving outer base.*

Combining this with previous results, Mizokami obtains the following corollary.

Corollary 6.3. *1. An adjunction space of M_1-spaces is M_1;*
2. A stratifiable space which is the union of countably many closed M_1 subspaces is M_1.

The first part of this corollary answers another question of Ceder, while the second part answers a question I asked in [72].

7 Stratifiability of function spaces

Gartside and Reznichecko [64] proved that the space $C_k(X)$ of continuous real-valued functions on X with the compact open topology is stratifiable whenever X is a Polish space. Their proof did not determine whether or not such $C_k(X)$ were M_1, so for a time there was hope that perhaps $C_k(\omega^\omega)$ would be a counterexample solving Ceder's long-standing problem. First, Tamano and I [84] obtained the following partial result:

Theorem 7.1. *If X is σ-compact Polish, then $C_k(X)$ is a μ-space (and hence hereditarily M_1).*

[4] An *outer base* for a subset H of X is a collection $\mathcal{U}$ of open supersets of H such that any open set containing H contains a member of $\mathcal{U}$

Here, a *μ-space* is a space that can be embedded in the countable product of paracompact spaces which are the countable union of closed metrizable subspaces. It is known that the class of stratifiable μ-spaces is hereditary, and every member of the class is M_1.

Gartside and Feng [62] obtained the following related result:

Theorem 7.2. *1. If X is a compact-covering image of a closed subspace of the product of a σ-compact Polish space and a compact space, then $C_k(X, M)$, the space of continuous maps from X into M with the compact-open topology, is stratifiable for any metric space M;*
2. If X is σ-compact Polish, K is compact and M metric then every point of $C_k(X \times K, M)$ has a closure-preserving local base, and hence this function space is M_1.

However, Tamano [147] later showed that $C_k(\omega^\omega)$ is not a counterexample to Ceder's problem:

Theorem 7.3. *If X is Polish, then $C_k(X)$ is M_1.*

It remains unsolved whether or not $C_k(X)$ is hereditarily M_1 or a μ-space for any Polish X.

Another question asked by Gartside and Reznichenko was whether for separable metric X, $C_k(X)$ stratifiable implies X must be complete and hence Polish. Reznichenko [130] gave an affirmative answer:

Theorem 7.4. *For a separable metric space X, $C_k(X)$ is stratifiable iff X is Polish.*

8 Local versions of M_i-spaces

Local versions of M_1 and M_2 were defined, first by R. Buck [39], who called them m_1-spaces and m_2-spaces, respectively, and later by Dow, Martinez, and Tkachuk [55], who, apparently unaware of Buck's work, named them Japanese and weakly Japanese, respectively. That is, a space is m_1 (or *Japanese*) if each point has a closure-preserving open base; replace "base" with "quasibase" for the definition of m_2 (or *weakly Japanese*). Here I will use Buck's notation.

Interestingly, local analogues of the M_2 vs. M_1 problem are also unsolved:

Question 8.1. 1. Is every regular m_2-space m_1?
2. Is the m_1 property (closed) hereditary?

Dow, Martinez, and Tkachuk also ask if the answer to the first question is positive for compact spaces. The following are some of their results about these properties:

Theorem 8.2. [55]

1. Every GO-space is m_1;
2. A dyadic compact space is m_2 iff it is metrizable;
3. Every scattered Corson compact is m_1, but there is an Eberlein compact space which is not m_2;
4. The m_2 property is not preserved by perfect mappings.

Buck also defines m_3*-spaces* and *monotonically Hausdorff* spaces, the former a local property inspired by Ceder's original definition of M_3-spaces, and the latter a weakening of monotonically normal which he shows to be equivalent to m_3 in Hausdorff spaces. It is also not known if m_3 implies m_2. Dow, Martinez, and Tkachuk ask the related question whether monotonically normal implies m_1.

Answering questions in [55] and [151], Feng and Gartside [58] recently constructed an uncountable compact Hausdorff space K such that the space $C_p(K)$ is m_1, where $C_p(X)$ is the space of continuous real-valued functions on X with the topology of pointwise convergence. This should be compared with the well-known result that if $C_p(X)$ is stratifiable, then X must be countable.

9 Quarter-stratifiable spaces

T. Banakh [21] introduced an interesting generalization of semi-stratifiable spaces which he named quarter-stratifiable. A space (X, τ) is *quarter-stratifiable* if there is a function $g : \omega \times X \to \tau$ such that

1. For each $n \in \omega$, $X = \bigcup\{g(n, x) : x \in X\}$;
2. If $x \in g(n, x_n)$ for each $n \in \omega$, then $x_n \to x$.

Further, if there is a weaker metric topology μ on X and a function g as above with $g(n, x) \in \mu$ always, then X is said to be *metrically quarter-stratifiable.*

One should notice here that x need not be a member of $g(n, x)$; if this were required in place of condition (1) in the definition of quarter-stratifiable, we would have a property equivalent to semi-stratifiable (see, e.g., Theorem 5.8 of [71]). An illuminating example is the Sorgenfrey line: it is not semi-stratifiable, but $g(n, x) = (x - 1/2^n, x)$ witnesses its (metrically) quarter-stratifiability.

Recall that Moore spaces are semi-stratifiable (see, e.g., [71]), and that there are Moore spaces that are not submetrizable (see, e.g., [117]); it follows (as is pointed out in [21]) that there are quarter-stratifiable spaces that are not metrically so. However, it is apparently not known whether every quarter-stratifiable space that has a weaker metric topology is metrically quarter-stratifiable.

The motivation for introducing quarter-stratifiability is the following result.

Theorem 9.1. [21] *Suppose X is metrically quarter-stratifiable, Y and Z are spaces, and $f : X \times Y \to Z$ a function. Then:*

1. *If Z is a locally convex equiconnected space (in particular, a locally convex topological vector space) and f is separately continuous, then f is of Baire class 1;*
2. *If every closed subset of Z is regular G_δ and f is continuous with respect to the first variable and Borel measurable of class α with respect to the second variable, then f is Borel measurable of class $\alpha + 1$.*

The first result generalizes a result of W. Rudin [132] and the second generalizes a theorem of Kuratowski [99] and Montgomery [116], who proved these results for X, Y, and Z metrizable. We list other properties of quarter-stratifiability proved by Banakh in the following theorem.

Theorem 9.2. [21]

1. *X is quarter-stratifiable iff there are open covers $\mathcal{U}_n$, $n \in \omega$, and functions $s_n : \mathcal{U}_n \to X$ such that $x \in U_n \in \mathcal{U}_n$ implies $s(U_n) \to x$;*
2. *(Metrically) quarter-stratifiability is preserved by open subspaces, retracts, and countable products;*
3. *Paracompact T_2 quarter-stratifiable spaces are metrically quarter-stratifiable;*
4. *Every T_2 quarter-stratifiable space has a G_δ-diagonal;*
5. *If X is quarter-stratifiable, then the density $d(X)$ is not greater than the Lindelöf degree $l(X)$, and every countably compact or paracompact Čech-complete subspace is metrizable;*
6. *Every space with a G_δ-diagonal is homeomorphic to a closed subspace of a quarter-stratifiable T_1-space;*

Bennett and Lutzer [29] examined this property in GO-spaces, and proved that every quarter-stratifiable GO-space is hereditarily metrically quarter-stratifiable and has a σ-closed-discrete dense subset. They also give an example of a separable perfect GO-space with a G_δ-diagonal that is not quarter-stratifiable.

10 Compact G_δ-sets and c-semistratifiable spaces

All spaces in this section are at least Hausdorff. In 1973, H.W. Martin [107] introduced the class of c-semistratifiable (CSS) spaces, which, roughly speaking, are spaces in which compact subsets are G_δ in a monotone way. More precisely, X is *c-semistratifiable* if for every compact set C, there are open sets $G(C, n)$, $n \in \omega$, satisfying:

1. $C = \bigcap_{n \in \omega} G(C, n)$;
2. $G(C, n + 1) \subset G(C, n)$ for all n; and

3. $C \subset D$ implies $G(C,n) \subset G(D,n)$ for all n.[5]

More recently, Bennett, Byerly, and Lutzer [25] studied spaces in which compact sets are G_δ and compared them to c-semistratifiable spaces, obtaining the following results:

Theorem 10.1. *1. A (countably) compact subset of a space X is metrizable and a G_δ-set in X if X has a $\delta\theta$-base, a point-countable T_1-separating open cover, or a quasi-G_δ-diagonal;*
2. Any compact subset of a space X having a base of countable order must be G_δ, but this does not hold for countably compact subsets;
3. A submetacompact locally CSS space is CSS;
4. Every compact subset of a space with a point-countable base must be G_δ, but there is a LOTS with a point-countable base which is not CSS;
5. Every monotonically normal CSS space is hereditarily paracompact;
6. Being CSS and having a G_δ-diagonal are equivalent in GO-spaces to having a σ-closed-discrete dense subset, but if there is a Souslin line, then they are not equivalent in the more general class of perfect GO-spaces.

11 Cosmic spaces

It is well-known that the covering dimension $\dim X$, the small inductive dimension $\operatorname{ind} X$, and the large inductive dimension $\operatorname{Ind} X$ of a separable metric space X coincide. Arhangel'skii asked whether they agree in the class of regular continuous images of separable metric spaces, or *cosmic spaces* as they are often called. Cosmic spaces are also characterized as regular spaces which have a countable network.

For a cosmic space X it is known that $\operatorname{ind} X = \operatorname{Ind} X$, so the question is whether $\dim X = \operatorname{ind} X$. G. Delistathis and W. S. Watson [51] claimed to construct, under the Continuum Hypothesis, a cosmic space X with $\dim X = 1$ and $\operatorname{ind} X > 1$. Unfortunately, that construction was incorrect (specifically, Lemmas 2.2 and 2.3 in their paper were incorrect). However, now an example with these properties has been constructed by M. Charalambous , in ZFC.

Example 11.1. ([47]) There is a regular continuous image X of a separable metrizable space such that $\dim X = 1$ and $\operatorname{ind} X = 2$. Furthermore, X is a countable union of separable metrizable subspaces.

Dow and Hart [52] independently obtained a similar example assuming Martin's Axiom for σ-centered partial orders (though they only showed ind $X > 1$). These examples also answer a question of S. Oka, who asked if dim

[5] Semistratifiable spaces are characterized by the existence of an operator $G(C,n)$ satisfying these same conditions for all *closed* sets C.

$X = \mathrm{Ind}\ X$ for paracompact perfectly normal spaces which are a countable union of metrizable subspaces.

Incidentally, these examples give new examples of cosmic spaces that are not μ-spaces, where X is a *μ-space* if it can be embedded in $\Pi_{n\in\omega} Y_n$, where each Y_n is paracompact and a countable union of closed metrizable subspaces. It is known that the standard dimensions agree for μ-spaces, so the above examples are not μ-spaces. The first known ZFC example of a cosmic space which is not μ was given by Tamano [146] in 2001; in 2005, Tamano and Todorcevic [148] show that certain function spaces are also of this kind.

Theorem 11.2. [148]

1. *If $C_p(X, \mathbb{R})$ is a μ-space, then X is a countable union of compact metrizable subspaces;*
2. *For a zero-dimensional space X, $C_p(X, \{0, 1\})$ is a μ-space if and only if X is a countable union of compact, metrizable subspaces.*

Hence $C_p(\omega^\omega, \{0, 1\})$ is a cosmic space that is not a μ-space; it is apparently not known whether or not $C_p([0, 1], \mathbb{R})$ is a μ-space.

12 k^*-metrizable spaces

All spaces in this section are assumed to be Hausdorff. In [23], Banakh, Bogachev, and Kolesnikov introduce and study a new class of generalized metrizable spaces. They define a space X to be *k^*-metrizable* if there is a metric space M, a continuous surjection $\pi : M \to X$, and a (not necessarily continuous) function $s : X \to M$ such that $\pi \circ s = id_X$ and for every compact subset K of X, $\overline{s(K)}$ is compact in M. (The map π with these properties is called *subproper.*)

The motivation for this class of spaces comes from probability; see [23] for details. Regarding properties of these spaces, an easy observation is that compact subsets of k^*-metrizable spaces are metrizable. Here are a couple of alternate characterizations of this class:

Theorem 12.1. *The following are equivalent for a Hausdorff space X:*

1. *X is k^*-metrizable;*
2. *Every compact subset of X is metrizable, and X has the following property: there is a metric space M, a continuous surjection $\pi : M \to X$, and a (not necessarily continuous) function $s : X \to M$ such that $\pi \circ s = id_X$ and the image under s of any convergent sequence in X has a convergent subsequence in M;*
3. *There is a metric ρ on X such that (i) each ρ-convergent sequence converges in X, (ii) a ρ-Cauchy sequence converges in X iff it contains a convergent subsequence in X, and (iii) compact subsets of X are totally bounded with respect to ρ.*

Spaces with the property given in item 2 are called *cs^*-metrizable*.

The next characterization is useful in relating k^*-metrizable spaces to other generalized metrizable spaces. Recall that a collection $\mathcal{F}$ of subsets of X is a *k-network* for X if, given a compact set K and an open set U containing K, there is a finite subset $\mathcal{F}'$ of $\mathcal{F}$ with $K \subset \cup\mathcal{F}' \subset U$. Also, $\mathcal{F}$ is *compact-finite* if every compact set meets only finitely many members of $\mathcal{F}$, and is *σ-compact-finite* if it is the union of countably many compact-finite subcollections. An *$\aleph$-space* (resp., *$\aleph_0$-space*) is a space with a σ-locally-finite (resp., countable) k-network.

Theorem 12.2. *A regular space X is k^*-metrizable iff X has a σ-compact-finite k-network.*

Theorem 12.3. *Lasnev spaces (i.e., closed images of metrizable spaces), $\aleph_0$-spaces, and $\aleph$-spaces are k^*-metrizable. Furthermore, a regular space is Lasnev iff it is a k^*-metrizable Fréchet space, and is an $\aleph_0$-space iff it is k^*-metrizable and cosmic.*

The class also has some nice preservation properties.

Theorem 12.4. *k^*-metrizable spaces are preserved by arbitrary subspaces, countable products, box products, subproper (hence perfect) maps, and the hyperspace of nonempty compact subsets with the Vietoris topology.*

See [23] for cardinal characteristics of these spaces, connections to Banach spaces and spaces of probability measures, a discussion of the related classes of k-metrizable spaces and cs^*-metrizable spaces, and more.

13 D-spaces

A space X is a *D-space* if whenever $N(x)$ is a neighborhood of x for each $x \in X$, there is a closed discrete set D such that $X = \bigcup_{x \in D} N(x)$. It is a long-standing open question whether or not every regular Lindelöf or paracompact Hausdorff space is a D-space, though there is a recent example, due to D. Soukup and P. Szeptycki[140], assuming $\diamondsuit$ of a Hausdorff (but not regular) hereditarily Lindelöf non-D-space. For a fairly recent survey of D-spaces, see [82].

Many base properties and generalized metric properties imply D. That semi-stratifiable spaces, hence Moore, semi-metric, stratifiable, and σ-spaces, are D has long been known ([38]; see also [59]), but in the last decade the following new results have been obtained:

Theorem 13.1. *The following are D-spaces:*

1. *Spaces having a point-countable base* [12]*;*

2. *Strong Σ-spaces (hence paracompact p-spaces, as well as countable products of σ-compact spaces)* [42]*;*
3. *Spaces having a point-countable weak base* [40] [120]*, and sequential spaces with a point-countable W-system* [40] *or point-countable k-network* [120]*;*
4. *Subspaces of symmetrizable spaces* [40]*;*
5. *Spaces having a σ-cushioned (mod k) pair-network, hence $\Sigma^{\#}$-spaces* [104]*;*
6. *t-metrizable spaces* [88]*;*
7. *Subspaces of $C_p(X)$, where X is a Lindelöf Σ-space* [78]*;*
8. *Spaces satisfying Collins-Roscoe conditions (G) or well-ordered (A), linearly semistratifiable spaces, and elastic spaces* [141]*;*
9. *Base-base paracompact (hence totally paracompact) spaces* [128] *(see also* [125]*) and Menger spaces* [16]*;*
10. *(Weakly) monotonically monolithic spaces* [150] [121].

Lin's result (item 5) simultaneously generalizes semi-stratifiable implies D and strong Σ implies D. Re item 9, Aurichi's result about Menger spurred much activity, in spite of the fact that it could be considered a corollary of the previously known result that totally paracompact spaces are D. See Section 20 for more about base-base paracompactness. Re items 8 and 10, we discuss monotonically monolithic spaces and the Collins-Roscoe condition (G) in Section 15.

14 Monotone normality and resolvability

A space X is said to be *κ-resolvable* (resp., *almost κ-resolvable*) if there is a pairwise-disjoint (resp., almost disjoint modulo a nowhere-dense set) collection of κ-many dense subsets. A space is *maximally resolvable* if and only if it is κ-resolvable for $\kappa = \Delta(X)$, where $\Delta(X)$ the minimum cardinality of a nonempty open set in X. Metrizable spaces and linearly ordered spaces are maximally resolvable. Since these two classes are the most important subclasses of monotonically normal spaces, it was thus natural to consider the resolvability of this more general class. Juhasz, Soukup, and Szentmiklossy showed:

Theorem 14.1. [92]

1. *Every crowded monotonically normal space is ω-resolvable, and almost μ-resolvable for $\mu = inf\{2^\omega, \omega_2\}$;*
2. *Every monotonically normal space of cardinality less than $\aleph_\omega$ is maximally resolvable;*
3. *If κ is a measurable cardinal, then there is a monotonically normal space X with $\Delta(X) = \kappa$ which has no ω_1-resolvable subspace ;*
4. *If there is a supercompact cardinal, then it is consistent that there is a monotonically normal space X with $|X| = \Delta(X) = \aleph_\omega$ having no ω_2-resolvable subspace.*

Note that the examples of (3) and (4) cannot be maximally resolvable. Recently, Juhasz and Magigor [90] extended some of these results as follows.

Theorem 14.2. *1. The existence of a monotonically normal space which is not maximally resolvable is equicontinuous with the existence of a measurable cardinal;*
2. It is consistent modulo a measurable cardinal that there is a monotonically normal space X with $\Delta(X) = \aleph_\omega$ which is not ω_1-resolvable.

A question from [92] which is still open is whether every crowded monotonically normal space is almost $\mathfrak{c}$-resolvable.

15 Monotonically monolithic and Collins-Roscoe condition (G)

All spaces in this section are assumed to be regular. Recall that a space X is *κ-monolithic* if for any $A \subset X$ of cardinality not greater than κ, $\overline{A}$ has a network of cardinality not greater than κ, and X is *monolithic* if it is κ-monolithic for every cardinal κ. Monotonically monolithic spaces were recently introduced by V.V. Tkachuk [150], and monotonically κ-monolithic spaces by O.Alas, Tkachuk, and R. Wilson [1]. A space X is *monotonically monolithic* if one can assign to each $A \subset X$ a collection $\mathcal{N}(A)$ of subsets of X such that

1. $|\mathcal{N}(A)| \leq |A| + \omega$;
2. $A \subset B \Rightarrow \mathcal{N}(A) \subset \mathcal{N}(B)$;
3. If $\{A_\alpha : \alpha < \delta\}$ is an increasing collection of subsets of X, and $A = \bigcup_{\alpha<\delta} A_\alpha$, then $\mathcal{N}(A) = \bigcup_{\alpha<\delta} \mathcal{N}(A_\alpha)$;
4. If U is open and $x \in \overline{A} \cap U$, then there is $N \in \mathcal{N}(A)$ with $x \in N \subset U$.

The operator $\mathcal{N}$ is called a monotonically monolithic operator for X.

Further, for an infinite cardinal κ, X is said to be *monotonically κ-monolithic* if $\mathcal{N}(A)$ is defined for all sets A with $|A| \leq \kappa$ and satisfies the above conditions.

Condition (4) may be rephrased by declaring that $\mathcal{N}(A)$ contains a network at every point of $\overline{A}$.[6] L.-X. Peng[121] called a space X *weakly monotonically monolithic* if it has an operator satisfying the above conditions but with condition (4) replaced by

4.′ If A is not closed, then $\mathcal{N}(A)$ contains a network at some point $x \in \overline{A} \setminus A$.

Theorem 15.1. *1. Any space with a point-countable base is monotonically monolithic* [150]*;*

[6] A collection $\mathcal{F}$ of subsets of a space X is a *network* at $x \in X$ if, given any open neighborhood U of x, there is some $F \in \mathcal{F}uu$ with $x \in F \subset U$.

2. *If X is a Lindelöf Σ-space, then $C_p(X)$ is monotonically monolithic* [150];[7]
3. *Monotonically monolithic spaces are hereditarily D-spaces* [150].
4. *Monotonically (κ-)monolithic spaces are preserved by countable products, subspaces, and closed mappings* [150][1].

The first three results served as motivation for introducing the class of monotonically monolithic spaces, because they generalized simultaneously the results of A.V. Arhangel'skii and R. Buzyakova [12] that spaces with a point-countable base are (hereditarily) D, and our result [78] that $C_p(X)$ is hereditarily D whenever X is a Lindelöf Σ-space. Subsequently, Peng [121] showed:

Theorem 15.2. *Weakly monotonically monolithic spaces are D-spaces.*

From this result, many known results about base properties or generalized metric properties implying D can be recovered–see [121] for details.

In [49], Collins and Roscoe introduced the following condition:

(G) For each $x \in X$, there is assigned a countable collection $\mathcal{G}(x)$ of subsets of X such that, whenever $x \in U$, U open, there is an open V with $x \in V \subset U$ such that, whenever $y \in V$, then $x \in N \subset U$ for some $N \in \mathcal{G}(y)$.

It is easy to see that any space with a point-countable base satisfies (G), where $\mathcal{G}(x)$ is simply the collection of all members of a point-countable base which contain x. Indeed, the question whether or not (G) being witnessed by a collection of open sets (i.e., the property called "open (G)") is equivalent to having a point-countable base is a well-known open question.

As mentioned in [74], it is straightforward to check that (G) is equivalent to the following:

(G$'$) For each $x \in X$, one can assign a countable collection $\mathcal{G}(x)$ of subsets of X such that, for any $A \subset X$, $\bigcup_{a\in A} \mathcal{G}(a)$ contains a network at every point of $\overline{A}$.

If we let $\mathcal{N}(A) = \bigcup_{x\in A} G(x)$, where $\mathcal{G}(x)$ satisfies G$'$, then it is easy to check that this is a monotonically monolithic operator. So (G) implies monotonically monolithic. In particular, this means that stratifiable spaces, which satisfy (G) [50], are monotonically monolithic.

In [83] we proved the following result, which shows a close connection between the monotonically monolithic property and the Collins-Roscoe condition (G).

Theorem 15.3. *A space X is monotonically monolithic (resp., weakly monotonically monolithic) iff one can assign to each finite subset F of X a countable collection $\mathcal{N}(F)$ of subsets of X such that, for each $A \subset X$, $\bigcup_{F\in[A]^{<\omega}} \mathcal{N}(F)$ contains a network at each point of $\overline{A}$ (resp., at some point of $\overline{A} \setminus A$, if A is not closed).*

[7] A space X is a *Lindelöf Σ-space* if it is the continuous image of closed subspace of the product of a separable metric space with a compact space, and $C_p(X)$ denotes the space of continuous real-valued functions on X with the topology of pointwise convergence.

There is an interesting connection between these properties and certain classes of compacta. Recall that a compact space X is *Corson compact* if it embeds in a Σ-product of real lines, and is *Gul'ko compact* if $C_p(X)$ is a Lindelöf Σ-space. Every Gul'ko compact is Corson compact.

Theorem 15.4. [83]

1. *Every monotonically ω-monolithic compact space is Corson compact;*
2. *There is a Corson compact space which is not monotonically ω-monolithic;*
3. *Every Gul'ko compact space satisfies (G).*

Subsequently, Tkachuk obtained a number of other interesting results on these properties.

Theorem 15.5. *1. If D is an uncountable discrete space, then $C_p(\beta D)$ is monotonically monolithic but does not satisfy (G)* [152]*;*
2. *If X is a Lindelöf Σ-space and $nw(X) \leq \omega_1$ then $C_p(X)$ satisfies (G)* [152]*;*
3. *Any space X satisfying (G) is cosmic whenever ω_1 is a caliber of X* [152]*;*
4. *For any Tychonoff space X, if $C_p(X)$ is a Lindelöf Σ-space then X satisfies (G)* [152]*;*
5. *Property (G) is preserved by closed maps, countable products, and σ-products* [153]*;*
6. *X has (G) if X is a Lindelöf Σ-space and has a weakly σ-point-finite T_0-separating family of cozero sets* [153]*;*
7. *If X is monotonically κ-monolithic and $t(X) \leq \kappa$, then X is monotonically monolithic* [153]*;*
8. *If X is perfectly normal, Corson compact, and monotonically (ω-)monolithic, then X is metrizable* [153]*;*
9. *There is a Corson compact space satisfying (G) which is not Gul'ko compact* [153]*;*
10. *A hereditarily Lindelöf space satisfying open (G) has a point-countable base* [153].

Item 1 of Theorem 15.5 shows that (G) and monotonically monolithic are distinct properties, answering a question I asked in [83], while 15.5(9) answers another question of mine. Theorem 15.5(2) generalizes the respective result of Dow, Junnila, and Pelant [54] for compact spaces X. Regarding 15.5(3), the respective question for monotone monolithity, formulated in [150], remains open.

16 Monotonically compact and monotonically Lindelöf

All spaces in this section are assumed to be regular. A space X is *monotonically Lindelöf* (resp., *monotonically compact*) if to every open cover $\mathcal{U}$ one can assign a countable (resp., finite) open refinement $r(\mathcal{U})$ covering X

such that $r(\mathcal{U})$ refines $r(\mathcal{V})$ whenever $\mathcal{U}$ refines $\mathcal{V}$. Monotonically compact and monotonically Lindelöf spaces were defined by M. Matveev, and were first studied in print by Bennett, Lutzer, and Matveev in [36]. One may similarly define other monotonic covering properties, and some of these have been studied too. Gartside and Moody [63] defined monotonically paracompact as above with $r(\mathcal{U})$ being a star refinement of $\mathcal{U}$ and showed that monotonically paracompact spaces are exactly the class of protometrizable spaces. Stares [144] remarks that it is not known if one gets the same class by defining the property so that $r(\mathcal{U})$ is a locally finite refinement of $\mathcal{U}$.[8]

It is easily seen that compact metrizable spaces are monotonically compact and second countable spaces are monotonically Lindelöf.

Theorem 16.1. *1. Monotonically compact Hausdorff spaces are metrizable* [80]*;*
2. Every Lindelöf first countable GO-space is monotonically Lindelöf [79]*;*
3. Compact monotonically Lindelöf spaces are first countable [79]*;*
4. Monotonically Lindelöf spaces having property K[9] *(e.g., separable) are hereditarily Lindelöf* [79]*;*
5. $\beta\omega$ and ω^* are not monotonically Lindelöf [100].
6. There are countable spaces which are not monotonically Lindelöf, and under CH, there is a countable space which is monotonically Lindelöf but not second countable [101].

Theorem 16.1(1), which shows that the only monotonically compact Hausdorff spaces are compact metrizable spaces, answers a question of Matveev, while 16.1(2) answers some questions in [36]. A.-J. Xu and W.-X. Shi [155] obtained a kind of converse to 16.1(2) by showing that if X is a monotonically Lindelöf GO-space, then the character of X is $\leq \omega_1$. L.-X. Peng and H. Li [122] improved 16.1(3) by showing that every compact monotonically metalindelöf space is first countable.

Popvassilev [127] subsequently defined a space to be *monotonically (countably) metacompact* if to every (countable) open cover $\mathcal{U}$ one can assign a point-finite open refinement $r(\mathcal{U})$ covering X such that $r(\mathcal{U})$ refines $r(\mathcal{V})$ whenever $\mathcal{U}$ refines $\mathcal{V}$. He proved that the ordinal space $\omega_1 + 1$ is not monotonically countably metacompact. The property was further studied by Bennett, Hart, and Lutzer [26], who showed:

Theorem 16.2. *1. Every metacompact Moore space is monotonically metacompact;*
2. A monotonically (countably) metacompact GO-space is hereditarily paracompact;

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[8] Recently, S. Popvassilev and T. Chase in as yet unpublished work independently found examples showing these classes are different, while J.E. Porter proved that protometrizable spaces do satisfy the locally finite version of monotonic paracompactness.

[9] A space X has *property K* if every uncountable collection of open sets contains an uncountable subcollection with nonempty intersection.

3. *A GO-space with a σ-closed-discrete dense subset is metrizable if and only if it is monotonically (countably) metacompact;*
4. *A compact GO-space is metrizable if and only if it is monotonically (countably) metacompact;*
5. *There is a non-metrizable LOTS that is monotonically metacompact.*

A key lemma used in the argument for item 3 has an erroneous proof in [26]. This was fixed by Peng and Li [122], who also answered a question in [26] by showing that a monotonically normal space that is monotonically countably metacompact (or monotonically metalindelöf) must be hereditarily paracompact. Recently, T. Chase and I have answered a question mentioned in [127] and [26] by proving that every compact monotonically countably metacompact space is metrizable.

17 Montonically countably paracompact (MCP)

All spaces in this section are assumed to be regular. Monotonic versions of countable paracompactness and countable metacompactness that are quite different in spirit from the monotonic properties mentioned in the previous section were introduced independently in [70], [119], and [149]. There[10], a space X is defined to be *monotonically countably metacompact (MCM)* if and only if there is an operator U assigning to each $n \in \omega$ and each closed set D an open set $U(n, D)$ such that

1. $D \subset U(n, D)$;
2. If $E \subset D$, then $U(n, E) \subset U(n, D)$; and
3. If $\{D_n : n \in \omega\}$ is a decreasing sequence of closed sets with empty intersection, then $\bigcap_{n\in\omega} U(n, D_n) = \emptyset$.

X is *monotonically countably paracompact (MCP)* if U also satisfies

3′. If $\{D_n : n \in\}$ is a decreasing sequence of closed sets with empty intersection, then $\bigcap_{n\in\omega} \overline{U(n, D_n)} = \emptyset$.

Note that countably compact spaces are trivially MCP. Properties MCM and MCP should be considered monotonic separation properties, more related to monotonically normal spaces than the monotonic covering properties we discussed in the previous section. It turns out that MCM spaces are precisely the β-spaces [70] and MCP is closely related to the wN (weak Nagata) property of Hodel [89], which can be characterized (see [157]) by conditions (1), (2) and

3″. If $\{D_n : n \in \omega\}$ is a decreasing sequence of closed sets, then $\bigcap_{n\in\omega} \overline{U(n, D_n)} = \bigcap_{n\in\omega} D_n$.

[10] The definition given here is not precisely the one given in [70], but was shown to be equivalent to it in [157].

Theorem 17.1. *1. For q-spaces (in particular, first countable spaces or locally compact spaces), MCP is equivalent to wN* [70]*, and hence (see* [144]*) implies collectionwise Hausdorff, and metrizability if Moore ;*
2. Monotonically normal MCM spaces are MCP [70]*;*
3. The Sorgenfrey line is monotonically normal but not MCM [70]*;*
4. Any space with at most one non-isolated point is MCP [70]*, but there is a countable regular space which is not MCP* [144]*;*
5. X is MCP iff $X \times [0,1]$ is MCP [67]*;*
6. If there is a space X which is MCP but not collectionwise Hausdorff, then there is a measurable cardinal, and if there are two measurable cardinals, then there is a non-collectionwise Hausdorff MCP space [68].

18 β and strong β-spaces

Spaces in this section are assumed to be Hausdorff. Y. Yajima [156] introduced the following new class of spaces: A space (X, τ) is called a *strong β-space* if there is a function $g : X \times \omega \to \tau$ satisfying:

1. $x \in \bigcap_{n\in\omega} g(x, n)$;
2. If $\bigcap_{n\in\omega} g(x_n, n)$ is nonempty, then $\bigcap_{k\in\omega} \overline{\{x_n : n \geq k\}}$ is nonempty and compact.

If the phrase "and compact" is omitted in the conclusion of condition 2, or equivalently, the conclusion of condition 2 is changed to "then $\{x_n : n \in \omega\}$ has a cluster point", then we have the definition of a β-space. Trivially, countably compact spaces are β-spaces. Recall (see, e.g., [71]) that many classes of generalized metrizable spaces, e.g., Σ-spaces, semi-stratifiable spaces, etc., are β-spaces, and in the previous section we mentioned that the MCM property is equivalent to being a β-space. Yajima proves the following about strong β-spaces:

Theorem 18.1. *1. Every semi-stratifiable space, strong Σ-space, and strict p-space is a strong β-space;*
2. The class of strong β-spaces is countably productive, and preserved by perfect mappings in both directions;
3. The product of a β-space with a strong β-space is a β-space;
4. Every normal isocompact β-space is a strong β-space;
5. There is a countably compact (hence β) dense subset of ω^* which is not a strong β-space.

The second item gives some important advantages of strong β over β, e.g., strong β is countably productive, while it is not known if the product of two β-spaces has to be a β-space.

See [96] for further conditions on when a β -space is strong β.

Regarding β-spaces, Bennett and Lutzer [29] studied the property in ordered spaces and monotonically normal spaces, obtaining the following result.

Theorem 18.2. *1. A GO-space X is metrizable iff X is a β-space and either has a G_δ-diagonal or is quasi-developable, or X is perfect and hereditarily a β-space;*
2. Every monotonically normal hereditarily β-space is hereditarily paracompact.

They also ask if a compact first-countable LOTS which is hereditarily a β-space must be metrizable.

19 Noetherian type

In this section and the two following, we discuss properties defined by a condition on a base.

Peregudov [123] defines the *Noetherian type* $Nt(X)$ of a space X to be the least cardinal κ such that X has a base $\mathcal{B}$ such that each member of $\mathcal{B}$ is contained in fewer than κ-many other members of $\mathcal{B}$. D. Milovich [112] calls such a base $\mathcal{B}$ *κ^{op}-like.* So, e.g., $Nt(X) = \omega$ iff X has a base $\mathcal{B}$ which is ω^{op}-like, i.e., each member of $\mathcal{B}$ is contained in at most finitely many other members of $\mathcal{B}$. An ω^{op}-like base was called a *Noetherian base* by Peregudov and Shapirovskii in [124] and an *OIF base* (or *open-in-finite base*) by Bennett and Lutzer in [27] and by Balogh, Bennett, Burke, Gruenhage, Lutzer, and Mashburn in [17]. It is easy to see that any metric space or metacompact Moore space has Noetherian type ω, as does 2^κ for any κ. We collect in the following theorem a number of recent results on this topic.

Theorem 19.1. *1. The Noetherian type of ω^* is at least the splitting number $\mathfrak{s}$ and is consistently less than the additivity of the meager ideal. It can be $\omega_1, \mathfrak{c}, \mathfrak{c}^+$, or strictly between ω_1 and $\mathfrak{c}$* [113]*;*
2. Every homogeneous dyadic compactum has Noetherian type ω [112]*;*
3. If X is compact and its weight $w(X)$ is regular, then $Nt(X) = Nt(X^n)$ for each $n \in \omega$ if either X is hereditarily normal, or homogeneous, or $\beta\omega$ does not embed in X, or $|X| < 2^{w(X)}$; also $Nt(X) = Nt(X^n)$ if X is compact homogeneous and GCH holds [97]*;*
4. There are spaces X and Y such that $\omega = Nt(X \times Y) < min\{Nt(X), Nt(Y)\}$ [97]

19.1(3) and 19.1(4) are related to a still unsolved problem of Balogh, Bennett, et al whether there exists a space X such that $Nt(X^2) = \omega < Nt(X)$. It is also not known if there are compact spaces X and Y satisfying the conditions of 19.1(4).

I should probably mention a somewhat different notion of Noetherian base. In an unpublished note, van Douwen called a base $\mathcal{B}$ for X *Noetherian* if every increasing sequence of elements of $\mathcal{B}$ is finite. In [74] I stated that van Douwen showed that if κ is a strongly inaccessible cardinal, then κ with the order topology does not have a Noetherian base. I also stated that Szentmiklossy in an unpublished note proved that every regular space of cardinality less than the first strongly inaccessible cardinal has a Noetherian base. But Szentmiklossy has since withdrawn his claim, so this is now an open question.

20 Base paracompact

A topological space is *totally paracompact* [61] if every open base has a locally finite subcover. This is a strong property–even the space of irrationals does not satisfy it. J. Porter [129] defined a property which is much weaker but in the same spirit as follows: a space is *base-paracompact* if there is a base of cardinality equal to its weight such that every open cover has a locally finite refinement by members of the base. All metrizable spaces are base-paracompact, but it is unknown if every paracompact space is base-paracompact.

The following notions lie between metrizability and base-paracompactness. A space is *base-base paracompact* [128] if it has an open base such that every subfamily which is still a base contains a locally finite subcover. A space is *base-cover paracompact* [125] if it has an open base every subcover of which contains a locally finite subcover, and is *base-family paracompact* [126] if it has an open base every subfamily of which has a subfamily with the same union, such that the latter subfamily is locally finite at each point of its union.

Totally paracompact spaces are clearly base-base paracompact. It is also easy to see that base-family paracompactness is hereditary, and that it implies base-cover paracompactness, which implies base-base paracompactness.

Theorem 20.1. *1. Proto-metrizable spaces (hence metrizable and non-archimedean spaces) are base-family paracompact* [126]*;*
2. A T_1-space X is metrizable if and only if $X \times (\omega + 1)$ is base-family paracompact [126]*;*
3. A paracompact Hausdorff space is locally compact if and only if its product with every compact space is base-cover paracompact [125]*;*
4. Every base-base paracompact space is a D-space [128]*;*
5. A subspace of the Sorgenfrey line is base-cover paracompact if and only if it is F_σ [125] *(hence the Sorgenfrey line is not base-family paracompact);*
6. There is a Nagata space (i.e., first countable and stratifiable) which is base-family paracompact but not metrizable [126]*;*
7. The sequential fan S_ω is totally paracompact but not base-cover paracompact [125].

It is an open question whether there is a paracompact space which is not base-base paracompact. It is not even known whether or not every subspace of the Sorgenfrey line (in particular, the subspace of irrationals) is base-base paracompact.

21 Sharp base

A base $\mathcal{B}$ of a space X is said to be a *sharp base* if for every injective sequence $(B_i\colon\ i < \omega)$ in $\mathcal{B}$ with $x \in \bigcap_{i<\omega} B_i$, the family $\{\bigcap_{i\leq n} B_i\colon\ n < \omega\}$ is a base at x. In my article in the previous book in this series, I mentioned a construction of C. Good, R. W. Knight and A. M. Mohamad [69] of a pseudocompact non-compact non-developable space with a sharp base whose product with the unit interval does not have a sharp base. This example answered questions in [2] and [15]. Unfortunately, it turned out not to be regular. Subsequently, B. Bailey and I [18] showed how to modify the construction to obtain a regular space with the same properties.

Also, Z. Balogh and D. Burke [19] obtained the following results on sharp bases:

Theorem 21.1. *1. There is a space X with a sharp base and a perfect mapping $f : X \to Y$ such that Y does not have a sharp base (in fact, Y is not a p-space);*
2. If X has a sharp base, then it has a point-countable sharp base which is point-finite on the set of isolated points.

The first statement answers a question in [69]; regarding the second statement, it was known that a space with a sharp base has a point-countable sharp base [15].

22 $dis(X)$ and $m(X)$

For a space X, $dis(X)$ (resp., $m(X)$) is the least cardinal such that X can be covered by κ-many discrete (resp., metrizable) subspaces.

Theorem 22.1. (Gruenhage [81]) *Let κ be an infinite cardinal. If X is the union of κ-many discrete subspaces, then so is any perfect image of X. I.e., if $f : X \to Y$ is a perfect surjection, then $dis(Y) \leq dis(X) + \omega$.*

This result generalized a result of Burke and Hansell [41], who proved it for the case $\kappa = \omega$. Since any compact Hausdorff space with no isolated points admits a perfect mapping onto $I = [0, 1]$, and $dis(I) = \mathfrak{c}$, we have the following corollary, which answered a question of Juhasz and van Mill [91]:

Corollary 22.2. *If X is a compact Hausdorff space with no isolated points, then $dis(X) \geq \mathfrak{c}$.*

Bella [24] recently proved the following extension of this corollary.

Theorem 22.3. *If X is a Čech complete, compactly rooted space with no isolated points, then $dis(X) \geq \mathfrak{c}$.*

The class of compactly rooted spaces is a class defined by Arhangel'skii which contains all p-spaces and all perfect preimages of spaces having a G_δ-diagonal; see [5] for more details. Bella showed that it is consistent that "Čech complete" cannot be weakened to "Baire" in his theorem, but it is not known if that is so in ZFC.

Juhasz and Szentmiklossy [94] improved Corollary 22.2 with the following result:

Theorem 22.4. *If X is compact Hausdorff with no isolated points, and $\chi(x, X) \geq \kappa$ for all $x \in X$, then $dis(X) \geq 2^\kappa$.*

Let $\Delta(X)$ denote the least cardinal of a nonempty open set in X. By the well-known Čech-Pospišil theorem, if X is compact Hausdorff with no isolated points, and $\chi(x, X) \geq \kappa$ for all $x \in X$, then $\Delta(X) \geq 2^\kappa$. This led Juhasz and Szentmiklossy to ask:

Question 22.5. [94] Is $dis(X) \geq \Delta(X)$ for any compact Hausdorff space X?

This question is still open. We mention here some partial results of S. Spadaro. Call a space ω_1-*expandable* if every closed discrete set D expands to an open collection $\{U_d : d \in D\}$ such that, for each $x \in X$, we have $|\{d \in D : x \in U_d\}| \leq \omega_1$.

Theorem 22.6. *1. $dis(X) \geq \Delta(X)$ if X is Baire and ω_1-expandable, and either developable or a regular σ-space; in particular, $dis(X) \geq \Delta(X)$ for any Baire metrizable space* [142]*;*
2. There is a regular Baire σ-space X with $dis(X) < \Delta(X)$ [142]*;*
3. It is consistent that there is a normal Moore Baire space X with $dis(X) < \Delta(X)$ [142]*;*
4. For a compact Hausdorff space X, $dis(X) \geq \Delta(X)$ if X is polyadic, or Gul'ko compact, or homogeneous, or hereditarily collectionwise-normal, or hereditarily normal and $2^\kappa < 2^{\kappa^+}$ for all κ. [143]

That $dis(X) \geq \Delta(X)$ for any Baire metrizable space seems to be new even for completely metrizable spaces.

The analogue of Theorem 22.1 for $m(X)$ is open:

Question 22.7. Let κ be an infinite cardinal. If X is the union of κ-many metrizable subspaces, is the same true of any perfect image of X? What if X is compact?

This question for the case $\kappa = \omega$ for compact X is due to A. Szymanski, and even that is still open, i.e., if a compact Hausdorff space X is a countable union of metrizable subspaces, is the same true for any continuous image of X? I showed [76] that the conclusion holds if either X is a finite union of metrizable subspaces, or if X is the union of countably many metrizable subspaces that are G_δ-sets in their closures.

Regarding compacta of finite metrizability number, it is an old result of Michael and Rudin that a compact space which is union of two metrizable subspaces must be Eberlein compact. Juhasz, Szentmiklossy, and Szymanski [95] obtained the following interesting extension of this result:

Theorem 22.8. *Let X be a compact Hausdorff space with $m(X) < \omega$. Then:*

1. *X is Eberlein compact iff X is hereditarily σ-metacompact;*
2. *X is uniform Eberlein compact iff X is hereditarily uniformly σ-metacompact;*
3. *X is Corson compact iff X is hereditarily metalindelöf.*

It seems not to be known if these results are sharp. In particular, the authors ask:

Question 22.9. Is there a hereditarily metalindelöf (resp., hereditarily σ-metacompact) compact Hausdorff space X with $m(X) = \omega$ which is not Corson compact (resp., Eberlein compact)?

23 Generalized metrizable spaces and topological algebra

In this section we mention some results in topological algebra concerning generalized metrizable spaces. Recall that *semitopological group* (resp., a *paratopological group*) G is a group $(G, \circ)$ with a topology such that the map $\circ : G \times G \mapsto G$ is separately (resp., jointly) continuous. A paratopological group in which the inverse operation is continuous is a *topological group.*

Burke and Arhangel'skii [11] discussed the regular G_δ-diagonal property in the setting of semitopological and paratopological groups, obtaining the following:

Theorem 23.1. *1. Every Hausdorff first-countable Abelian paratopological group has a regular G_δ-diagonal;*

2. *Every Tychonoff separable semitopological group with countable pseudocharacter has a weaker separable metric topology;*
3. *Every Tychonoff semitopological group with a countable π-base is submetrizable;*
4. *There is a countable Tychonoff (therefore, Lindelöf and normal) paratopological group G with a countable π-base which is not first countable (therefore, not metrizable), and not Fréchet-Urysohn;*

Subsequently, C. Liu [105] and, independently, Arhangel'skii and Bella [10] showed that the "Abelian" assumption in the first result is superfluous. The fourth result should be compared to the result that a topological group with a countable π-base must be metrizable.

Some related results were recently obtained by F. Lin and Liu [103]:

Theorem 23.2. *1. Every regular first-countable Abelian paratopological group is submetrizable;*
2. There is a Hausdorff paratopological gro5up with countable pseudocharacter which is not submetrizable;
3. There is a nonmetrizable separable Moore paratopological group.

The first result answers a question in [11], though it is not known if the Abelian assumption is necessary. The other two results answer questions of Arhangel'skii and Tkacenko [14]. The third result was obtained independently by P.Y. Li, L. Mou, and S.Z. Wang [102]. We also mention here that Liu and S. Lin [106] proved that every first countable paratopological group which is a β-space is developable.

Arhangel'skii published a series of papers on remainders, both in the topological group setting and in general spaces. The theme is to determine what having a "nice" remainder in a compactification implies about the group or space. Here is a small sample of his results on topological groups. Recall that the paracompact (Lindelöf) p-spaces are precisely the perfect pre-images of (separable) metrizable spaces.

Theorem 23.3. *Let G be a topological group. Then:*

1. *G is Lindelöf at ∞ (i.e., the remainder of every, or equivalently one, compactification is Lindelöf) iff G is a paracompact p-space [5];*
2. *G has a paracompact p-space remainder implies G is a paracompact p-space; if in addition G is not locally compact, then both G and its remainder are Lindelöf p-spaces; [5];*
3. *If G is not locally compact and has a remainder with a G_δ-diagonal or a point-countable base, then G and its remainder are separable metrizable spaces [6].*

See the papers [4, 5, 6, 7, 8, 9] for many more results on this theme. A pretty result for arbitrary spaces is that if a metrizable space X has a remainder with a G_δ-diagonal, then both X and its remainder are separable metrizable spaces.

Banakh and Zdomskyy have obtained some interesting results on $\mathcal{M}_\omega$-groups and $\mathcal{MK}_\omega$-groups. Here, a topological group G is $\mathcal{M}_\omega$ (resp., $\mathcal{MK}_\omega$) if there is a countable collection $\mathcal{K}$ of closed metrizable (resp., compact metrizable) subspaces such that U is open in G iff $U \cap K$ is open in K for each $K \in \mathcal{K}$.

Recall that a space is *punctiform* if it has no non-degenerate compact connected subspaces, and that a punctiform σ-compact space is 0-dimensional.

Also, the *compact scatteredness rank* of a space is the least upper bound of the Cantor-Bendixon ranks of its scattered compact subspaces. A space X has *countable cs*-character* if for each $x \in X$, there is a countable family $\mathcal{N}$ of subsets of X such that, for each neighborhood U of x and each sequence A in $X \setminus \{x\}$ converging to x, there is $N \in \mathcal{N}$ such that $N \subset U$ and $N \cap A$ is infinite. Finally, let $(2^\omega)^\infty$ denote the space whose topology is determined by an increasing sequence of Cantor sets, the nth one nowhere dense in the $(n+1)$st; define $\mathbb{R}^\infty$ similarly.

Theorem 23.4. (Banakh [20]) *The topology of a nonmetrizable punctiform $\mathcal{M}_\omega$-group is completely determined by its density and compact scatteredness rank; if separable and uncountable, it is homeomorphic to* $(2^\omega)^\infty$ *(and hence is 0-dimensional and $\mathcal{MK}_\omega$).*

Theorem 23.5. (Banakh and Zdomskyy [22])

1. *A topological group G is an $\mathcal{M}_\omega$-group iff G is sequential and has countable cs*-character;*
2. *Every nonmetrizable $\mathcal{M}_\omega$-group G contains an open $\mathcal{MK}_\omega$-subgroup H, and hence is homeomorphic to the product $H \times D$ for some discrete space D;*
3. *Each nonmetrizable sequential k^*-metrizable (see section 12) locally convex space is homeomorphic to* $\mathbb{R}^\infty$ *or* $\mathbb{R}^\infty \times [0,1]^\omega$*.*

These results were used in [86] to show that if X is a 0-dimensional non-locally compact Polish space whose derived set is compact, then $C_k(X, \{0,1\})$ is homeomorphic to $(2^\omega)^\infty$.

Finally, in the following theorem, we list some results in topological algebra involving stratifiability.

Theorem 23.6. *1. Monotonically normal topological vector spaces are stratifiable* (Shkarin [135])*;*
2. *The free locally convex space of a stratifiable space is stratifiable* (Sipacheva [136])*;*
3. *Suppose G is a topological group and H a locally compact metrizable subgroup. If the quotient space G/H is stratifiable (resp., semi-stratifiable, k-semi-stratifiable, a σ-space), then so is G* (R. Shen and S. Lin [134]).

Item 3 answered a question of Arhangel'skii and V. V. Uspenskij in the affirmative.

24 Domain representability

A partial order $(P, \leq)$ is called a *directed complete partial order (dcpo)* if every directed subset of P has a supremum in P. For $a, b \in P$, call $a \ll b$ if whenever $D \subset P$ is directed and $b \leq sup\ D$, then $a \leq d$ for some $d \in D$. P is said to be *continuous* if every set of the form $\{b : b \ll a\}$ is directed and has a as its supremum. A continuous dcpo P is called a *domain*, and if also every bounded subset of P has a supremum, then it is called a *Scott domain*. The topology on P generated by sets of the form $\{b : a \ll b\}$ is called the *Scott topology*. A space X is called *domain representable* (resp., *Scott domain representable*) if there is a domain (resp., Scott domain) P such that X is homeomorphic to the subspace of maximal elements of P (which is always nonempty by Zorn's lemma).

Domain representability is a kind of completeness property. Domain representable spaces are Baire, and every locally compact Hausdorff space and every Čech complete Moore space, hence every completely metrizable space, is domain representable. See the survey of Martin, Mislove, and Reed [110] for these and other results known at that time, as well as their motivation in theoretical computer science.

Now we recall the definition of the strong Choquet game $Ch(X)$. To start, Player E chooses a pair (x_0, U_0), where U_0 is open and $x_0 \in U_0$. Player NE then chooses an open V_0 with $x_0 \in V_0 \subset U_0$. E responds with (x_1, U_1) where $U_1 \subset V_0$ is open and $x_1 \in U_1$. NE then plays an open V_1 with $x_1 \in V_1 \subset U_1$, and so on. NE *wins* if $\bigcap_{n\in\omega} V_n \neq \emptyset$. For a metrizable space X , it is well-known that X is completely metrizable iff NE has a winning strategy in $Ch(X)$.

Theorem 24.1. (Martin [109])

1. *If X is domain representable, then NE has a winning strategy in $Ch(X)$;*
2. *If X is a Scott domain-representable Moore space, then X is Moore complete (= Čech complete for completely regular spaces);*
3. *If X is metrizable, then X is domain representable iff X is completely metrizable.*

Kopperman, Kunzi, and Waszkiewicz [98] subsequently showed that any completely metrizable space is Scott domain representable, hence for metrizable spaces, domain representability is equivalent to Scott domain representability. However, Bennett and Lutzer showed that this equivalence does not carry over to Moore spaces:

Theorem 24.2. *1. A G_δ subspace of a domain representable space is domain representable* [31]*;*

2. *There is a Scott domain representable Moore space with a closed (hence G_δ) subspace which is not Scott domain representable* [32].

Since compact Hausdorff spaces are domain representable, the first part of the above theorem shows that all Čech complete spaces are domain representable. For metrizable spaces, as seen in Martin's theorem above, domain representability is equivalent to Čech completeness. The situation is again different in Moore spaces, however.

Theorem 24.3. [11] (Bennett, Lutzer, Reed [37]) *The following are equivalent for a Moore space X:*

1. *X is domain representable*
2. *NE has a winning strategy in the strong Choquet game;*
3. *NE has a stationary winning strategy in the strong Choquet game;*
4. *X is subcompact;*
5. *X is Rudin complete.*

Subcompactness is one of the so-called Amsterdam completeness properties– see [35] for more information about such properties. Since Rudin completeness is strictly weaker than Moore completeness or Čech completeness, this shows that in Moore spaces, Čech completeness is strictly stronger than domain representability. What about Scott domain representability? As noted above, Martin showed that every Scott domain representable Moore space is Moore complete, hence Čech complete if completely regular. Martin also asked if the reverse were true, as it is in the class of metrizable spaces. However, Bennett, Lutzer, and Reed [37] showed that this is not the case in Moore spaces by giving examples of completely regular Čech complete Moore spaces that are either separable or metacompact but not Scott domain representable. They also showed that the statement that every countably paracompact separable Čech complete Moore space is Scott domain representable is consistent with and independent of ZFC. At this time, there seems to be no known satisfactory characterization of Scott domain representability in the class of Moore spaces [34].

Finally, we mention two more results of Bennett and Lutzer on this topic.

Theorem 24.4. [33]

1. *Any regular subcompact space is domain representable;*
2. *If X is regular and has a G_δ-diagonal, then X is domain representable if NE has a stationary winning strategy in the strong Choquet game.*

The second result can be used to show that spaces such as the Sorgenfrey line, Michael line, and others are domain representable. It is apparently not known if the word "stationary" can be omitted in this result. Concerning the first result, it seems not to be known whether or not subcompactness is equivalent to domain representability. Recently, Fleissner and Yengulalp [60] proved that, for completely regular X, $C_p(X)$ is subcompact iff $C_p(X)$ is domain representable iff X is discrete.

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[11] While this theorem is stated for Moore spaces, the authors show that all conditions except the last are equivalent in the more general category of spaces having a base of countable order.

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