WHEN THE COLLECTION OF $\epsilon$-BALLS IS LOCALLY FINITE

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5/24/01

Abstract. Consider the class $\mathcal{N}$ of metrizable spaces which admit a metric $d$ such that, for every $\epsilon > 0$, the collection $\{B(x, \epsilon) : x \in X\}$ of all $\epsilon$-balls is locally finite. We show that $\mathcal{N}$ is precisely the class of strongly metrizable spaces, i.e., $X \in \mathcal{N}$ iff $X$ is homeomorphic to a subspace of $\kappa^\omega \times [0, 1]^\omega$ for some cardinal $\kappa$ (where $\kappa$ carries the discrete topology). In particular, this shows that not every metrizable space admits such a metric, thereby answering a question of Nagata.

1. Introduction

Let $(X, d)$ be a metric space. For $\epsilon > 0$, we let $B_d(x, \epsilon)$ denote the $\epsilon$-ball $\{y \in X : d(x, y) < \epsilon\}$ about $x$, and we let $B_d(\epsilon)$ denote the collection $\{B_d(\epsilon) : x \in X\}$ of all $\epsilon$-balls in $X$. We may delete the subscript $d$ in case the metric is understood.

In [N1], J. Nagata showed that every metrizable space $X$ admits a metric $d$ such that, for every $\epsilon > 0$, $B_d(\epsilon)$ is closure-preserving. Indeed, implicit in his paper is the fact that every separable metric space admits a metric $d$ such that $B_d(\epsilon)$ is finite for every $\epsilon > 0$. For the metric he builds has the property that, for each $\epsilon > 0$, there is a locally finite open cover $G_\epsilon$ of $X$ such that $B_d(\epsilon) = \{st(x, G_\epsilon) : x \in X\}$, where $st(x, G_\epsilon) = \cup\{G \in G_\epsilon : x \in G\}$ and is called the “star” of $G_\epsilon$ at $x$. (It is easy to check that the collection of stars of a locally finite collection is closure-preserving.) Since a locally finite collection in a compact space must be finite, it follows then that the Hilbert cube admits a metric $d$ such that, for each $\epsilon > 0$, $B_d(\epsilon)$ is finite. Hence, so does any separable metrizable space, for the restriction of such $d$ to any subspace of the Hilbert cube has the same property.

In [N2], Nagata asks if every metrizable space admits a metric such that each $B(\epsilon)$ is locally finite. (He uses the term “hereditarily closure-preserving” in place of “locally finite”, but these notions are equivalent in the class of first-countable, in particular metrizable, spaces.) In this note, we characterize the class $\mathcal{N}$ of metrizable spaces which admit such a metric as precisely the class of strongly metrizable spaces, where a metrizable space $X$ is strongly metrizable if $X$ has a base which is the union of countably many star-finite open covers, or equivalently (see [P], Proposition 3.27), $X$ is embeddable in $\kappa^\omega \times I^\omega$ for some cardinal $\kappa$, where $I = [0, 1]$ and $\kappa$ carries the discrete topology. This gives a negative answer to Nagata’s question;

1991 Mathematics Subject Classification. 54E35.

Key words and phrases. metric, locally finite, star-finite.

Research of the first author was partially supported by NSF grant DMS 997-7099.

Research of the second author was partially supported by NSF grant DMS 970-4849.
in particular, any space with a non-separable component, such as a hedgehog with uncountably many spines, is not embeddable in $\kappa^\omega \times I^\omega$ and hence does not admit a metric such that each $B(\epsilon)$ is locally finite.

2. Main results

We first show that it matters not if one changes the question by replacing “locally finite” with “point-finite” or “star-finite”. (Recall that a collection $\mathcal{U}$ of subsets of $X$ is star-finite if each member of $\mathcal{U}$ meets only finitely other members.)

**Lemma 1.1.** Let $(X,d)$ be a metric space, and $\epsilon > 0$. Then the following are equivalent:

(a) $B_d(\epsilon)$ is locally finite;
(b) $B_d(\epsilon)$ is point-finite;
(c) $B_d(\epsilon)$ is star-finite;
(d) There is a star-finite open cover $\mathcal{U}$ such that $B_d(\epsilon) = \{st(x,\mathcal{U}) : x \in X\}$.

**Proof.** That $(c) \Rightarrow (a) \Rightarrow (b)$ is trivial, and $(d) \Rightarrow (b)$ is easy to check.

We now prove $(b) \Rightarrow (c)$, thereby establishing equivalence of $(a)$-$(c)$. To this end, suppose $B_d(\epsilon)$ is point-finite. For each $B \in B_d(\epsilon)$, let $C(B) = \{x \in B : B = B_d(x, \epsilon)\}$; i.e., $C(B)$ is the set of centers of the ball $B$. Note that the $C(B)$’s partition $X$. That $(b) \Rightarrow (c)$ is then immediate from the following two claims.

Claim 1. Each $B \in B_d(\epsilon)$ meets at most finitely many $C(B)$’s. Suppose by way of contradiction that $x_n \in C(B_n) \cap B$, $n \in \omega$. Let $x \in C(B)$. Then $d(x, x_n) < \epsilon$ for all $n$, so since $x_n \in C(B_n)$ we have $x \in B_n$ for all $n$, contradicting $(b)$.

Claim 2. Each $C(B)$ meets only finitely many $B$’s in $B_d(\epsilon)$. To see this, suppose $x_n \in C(B) \cap B_n$ for all $n$. Let $y \in C(B)$, and pick $z_n \in C(B_n)$. Then $d(x_n, z_n) < \epsilon$, so $z_n \in B_d(x_n, \epsilon) = B$. This implies $d(y, z_n) < \epsilon$ for all $n$, whence $y \in B_n$ for all $n$, again contradicting $(b)$.

It remains to prove that $(c) \Rightarrow (d)$. Assume $B_d(\epsilon)$ is star-finite. For each $p \neq q \in X$ with $d(p, q) < \epsilon$, let

$$U(p, q) = \bigcap \{B \in B_d(\epsilon) : \{p, q\} \subset B\}.$$ 

We first show that $d(x, y) < \epsilon$ for any two points $x, y \in U(p, q)$. Clearly $U(p, q) \subset B_d(p, \epsilon) \cap B_d(q, \epsilon)$. It follows that $p, q \in B_d(x, \epsilon)$, whence $y \in U(p, q) \subset B_d(x, \epsilon)$. So $d(x, y) < \epsilon$ as claimed.

Now let $\mathcal{U}' = \{U(p, q) : p \neq q \text{ and } d(p, q) < \epsilon\}$. Since each member of $\mathcal{U}'$ is a finite intersection of members of a star-finite collection, $\mathcal{U}'$ is also star-finite. It covers all non-isolated points, but possibly not all isolated points, so we let $\mathcal{U} = \mathcal{U}' \cup (\{\{x\} : x \in X \setminus \mathcal{U}'\})$, which of course is also star-finite. Consider any $x \in X$. By the previous paragraph, we have $st(x, \mathcal{U}) \subset B_d(x, \epsilon)$. Suppose $y \in B_d(x, \epsilon) \setminus \{x\}$. Then $x, y \in U(x, y) \in \mathcal{U}$, so $y \in st(x, \mathcal{U})$. Thus $st(x, \mathcal{U}) = B_d(x, \epsilon)$, which completes the proof. $\square$

The proof of the next lemma is similar to the proof of the equivalence of 1.1(b) and 1.1(c) and hence is omitted.
Lemma 1.2. Let \((X,d)\) be a metric space, and \(\varepsilon > 0\). Then the following are equivalent:

(i) \(B_d(\varepsilon)\) is point-countable;
(ii) \(B_d(\varepsilon)\) is star-countable.

Let \(\mathcal{N}\) be as defined in the introduction. We note the following easy lemma.

Lemma 1.3. \(\mathcal{N}\) is closed under subspaces and countable products.

Proof. If the metric \(d\) witnesses that \(X \in \mathcal{N}\), clearly the restriction of \(d\) to any subspace \(X'\) witnesses that \(X' \in \mathcal{N}\). To prove closure under countable products, suppose \(X_0, X_1, \ldots\) are in \(\mathcal{N}\), witnessed by \(d_0, d_1, \ldots\). Let \(d_i(x,y)\) be the minimum of \(1/2^i\) and \(d_i(x,y)\). It is easy to check that \(d_i\) also witnesses that \(X_i\) is in \(\mathcal{N}\), and that \(d(\bar{x}, \bar{y}) = \max_{i \in \omega} d_i(x_i, y_i)\) witnesses that \(\Pi_{i \in \omega} X_i \in \mathcal{N}\). 

Since discrete spaces are obviously in \(\mathcal{N}\), and the Hilbert cube is in \(\mathcal{N}\) by Nagata’s result mentioned in the Introduction, it follows from Lemma 1.3 that, for any cardinal \(\kappa\), any subspace of \(\kappa^{\omega} \times [0, 1]^{\omega}\) is in \(\mathcal{N}\) (where \(\kappa\) has the discrete topology). The next result, our main one, shows that this characterizes \(\mathcal{N}\).

Theorem 1.4. The following are equivalent for a metrizable space \(X\):

(i) \(X \in \mathcal{N}\), i.e., \(X\) admits a metric such that, for all \(\varepsilon > 0\), \(B(\varepsilon)\) is locally finite;
(ii) \(X\) admits a metric such that, for all \(\varepsilon > 0\), \(B(\varepsilon)\) is point-finite;
(iii) \(X\) admits a metric such that, for all \(\varepsilon > 0\), \(B(\varepsilon)\) is star-finite;
(iv) \(X\) admits a metric such that, for all \(\varepsilon > 0\), \(B(\varepsilon)\) is point-countable;
(v) \(X\) admits a metric such that, for all \(\varepsilon > 0\), \(B(\varepsilon)\) is star-countable;
(vi) There is a sequence of star-finite open covers \(\mathcal{G}_n, n < \omega\), of \(X\) such that, for each \(n\), \(\mathcal{G}_{n+1}\) refines \(\mathcal{G}_n\) and, for each \(x \in X\), \(\{st(x, \mathcal{G}_n) : n \in \omega\}\) is a base at \(x\).
(vii) There is a sequence of star-countable open covers \(\mathcal{G}_n, n < \omega\), of \(X\) such that, for each \(n\), \(\mathcal{G}_{n+1}\) refines \(\mathcal{G}_n\) and, for each \(x \in X\), \(\{st(x, \mathcal{G}_n) : n \in \omega\}\) is a base at \(x\).
(viii) \(X\) is homeomorphic to a subspace of \(\kappa^{\omega} \times [0, 1]^{\omega}\) for some cardinal \(\kappa\).

Proof. By the previous lemmas, (i)-(iii) are equivalent, as are (iv) and (v). As noted following the proof of Lemma 1.3, we also have (viii) \(\Rightarrow\) (i). Furthermore, it is clear that (iii) implies (iv), (v), and (vi), and (v) and (vi) both imply (vii). We now prove (vii) implies (viii); the theorem then follows from this and the aforementioned implications.

Let \(\{\mathcal{G}_n\}_{n \in \omega}\) satisfy condition (vii). For \(U, V \in \mathcal{G}_n\), define \(U \sim_n V\) iff there is a finite sequence \(U_0, U_1, \ldots, U_k\) of elements of \(\mathcal{G}_n\) with \(U = U_0, V = U_k, U_i \cap U_{i+1} \neq \emptyset\) for all \(i < k\). Then each equivalence class \(\bar{U}^n\) is countable, and the collection \(\mathcal{P}_n = \{\bigcup \bar{U}^n : U \in \mathcal{G}_n\}\) is a clopen partition of \(X\). Since \(\mathcal{G}_{n+1}\) refines \(\mathcal{G}_n\), \(\mathcal{P}_{n+1}\) refines \(\mathcal{P}_n\).

Let \(\mathcal{P}_0 = \mathcal{P}_0 = \{P(\alpha) : \alpha < \kappa_0\}\) for some cardinal \(\kappa_0\). Then for each \(\alpha < \kappa_0\), let \(\mathcal{P}_{\langle \alpha \rangle} = \{P \in \mathcal{P}_1 : P \subset P_{\langle \alpha \rangle}\} = \{P_{\alpha, \beta} : \beta < \kappa_{\langle \alpha \rangle}\}\) for some cardinal \(\kappa_{\langle \alpha \rangle}\). If \(\mathcal{P}_{\langle \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \rangle} = \{P_{\langle \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \rangle, \alpha_n} : \alpha_n < \kappa_{\langle \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \rangle}\}\) is defined, then let \(\mathcal{P}_{\langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle} = \{P \in \mathcal{P}_{n+1} : P \subset P_{\langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle}\} = \{P_{\langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha_{n+1} \rangle} : \alpha_{n+1} < \kappa_{\langle \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha_{n+1} \rangle}\}\).
\( \kappa_{\langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle} \). For each finite sequence \( \sigma \) for which \( P_\sigma \) has been defined, select \( \mathcal{G}_\sigma \in \mathcal{G}_k \), where \( k = |\sigma| - 1 \), such that \( P_\sigma = \bigcup [G_\sigma]_k \). Let \( \mathcal{G}_\sigma \) denote the equivalence class \([G_\sigma]_k\); since equivalence classes are countable, we may index \( \mathcal{G}_\sigma \) by \( \{G_{\sigma,j} : j \in \omega \} \).

Let \( \kappa \) be the supremum of all the defined \( \kappa_\sigma \)'s. For any \( \sigma \in \kappa^{<\omega} \) for which \( P_\sigma \) has not been defined, let \( P_\sigma = \emptyset = \mathcal{G}_\sigma \). Note that for each \( n \),

\[
\mathcal{G}_n = \bigcup \{ \mathcal{G}_\sigma : \sigma \in \kappa^{n+1} \}
\]

Since \( X \) is metrizable, hence paracompact, there is a locally finite closed shrinking \( \{ H(G) : G \in \mathcal{G}_n \} \) of \( \mathcal{G}_n \). For \( G \in \mathcal{G}_n \), let \( f_G : X \to [0, 1/2^n] \) be continuous such that \( f_G(H(G)) = \{1/2^n\} \) and \( f_G(X \setminus G) = \{0\} \). Let \( \mathcal{F}_n = \{ f_G : G \in \mathcal{G}_n \} \).

Now we can define our desired embedding \( \theta \), which will map \( X \) into \( \kappa^\omega \times I^{\omega \times \omega} \). Pick \( x \in X \). Then for each \( n \), there is a unique \( \sigma^x_n \in \kappa^{n+1} \) with \( x \in P_{\sigma^x_n} \). Note that \( \sigma^x_{n+1} \) extends \( \sigma^x_n \), so there is a unique \( \tau^x \in \kappa^\omega \) such that \( \tau^x \upharpoonright (n + 1) = \sigma^x_n \) for all \( n \). Then let

\[
\theta(x) = \{ \tau^x \} \times \{ (f_{G_{\tau^x \upharpoonright (k+1),j}}(x))_{(k,j)} \in \omega \times \omega \}.
\]

Let us check that \( \theta \) is one-to-one. Suppose \( x \neq y \), yet \( \theta(x) = \theta(y) \). Then \( \tau^x = \tau^y = \tau \in \kappa^\omega \), i.e., \( x \) and \( y \) are always in the same member of the partitions \( P_n \). Choose \( k \) sufficiently large so that \( y \notin st(x, \mathcal{G}_k) \), and choose \( G \in \mathcal{G}_k \) with \( x \in H(G) \). Then for some \( j \), \( G = G_{\tau^x \upharpoonright (k+1),j} \), and \( f_{G_{\tau^x \upharpoonright (k+1),j}}(x) = 1/2^k \) while \( f_{G_{\tau^y \upharpoonright (k+1),j}}(y) = 0 \). Thus \( \theta(x) \) and \( \theta(y) \) differ on coordinate \((k,j)\), contradiction.

Now we show that \( \theta \) is continuous. Let \( \theta(x) = (\theta_1(x), \theta_2(x)) \), where \( \theta_1(x) \in \kappa^\omega \) and \( \theta_2(x) \in I^{\omega \times \omega} \). Suppose a sequence of points \( x_n, n < \omega \), converges to \( x \in X \). Fix \( k \in \omega \). Then for sufficiently large \( n \), \( x_n \in P_{\tau^x \upharpoonright (k+1)} \) and hence also \( \tau^x_n \upharpoonright (k+1) = \tau^x \upharpoonright (k+1) \). It easily follows from this that as \( n \) gets large, \( \theta_1(x_n)(k) = \tau^x_n(k) \) converges to (in fact, is equal to) \( \theta_1(x)(k) = \tau^x(k) \) and that, if we also fix \( j \), \( \theta_2(x_n)(k,j) \) converges to \( \theta_2(x)(k,j) \). Thus \( \theta \) is continuous.

It remains to prove that \( \theta \) is a homeomorphism onto its range. To this end, let \( O \) be open in \( X \); it will suffice to show that \( \theta(O) \) is relatively open in \( \theta(X) \). Take a point \( \theta(x) \in \theta(O) \). We need to find a neighborhood \( V \) of \( \theta(x) \) in \( \kappa^\omega \times I^{\omega \times \omega} \) such that \( V \cap \theta(X) \subset \theta(O) \).

Choose \( k \) sufficiently large so that \( st(x, \mathcal{G}_k) \subset O \). Let \( J = \{ j \in \omega : x \in H(G_{\tau^x \upharpoonright (k+1),j}) \} \); obviously, \( J \) is finite. Let \( \epsilon = 1/2^k \), and let \( k' = k + 1 \). Then the set

\[
V = \{ (z_1, z_2) \in \kappa^\omega \times I^{\omega \times \omega} : z_1 \upharpoonright k' = \tau^x \upharpoonright k' \& \forall j \in J \left( |z_2(k,j) - \theta_2(x)(k,j)| < \epsilon \right) \}
\]

is an open neighborhood of \( \theta(x) \).

It remains to show \( V \cap \theta(X) \subset \theta(O) \). Suppose \( \theta(y) \in V \cap \theta(X) \setminus \theta(O) \). Then \( y \notin O \). Since \( \theta(y) \in V \), \( \tau^y \upharpoonright (k+1) = \tau^x \upharpoonright (k+1) \). Let \( G \in \mathcal{G}_k \) with \( x \in H(G) \). Then \( G \in [G_{\tau^x \upharpoonright (k+1)}]_k \) and so \( G = G_{\tau^y \upharpoonright (k+1),j} \) for some \( j \); of course, \( j \in J \). Since \( y \notin O \), \( y \notin G \). Thus \( \theta_2(y)(k,j) = f_{G_{\tau^y \upharpoonright (k+1),j}}(y) = f_{G_{\tau^x \upharpoonright (k+1),j}}(y) = f_G(y) = 0 \), while \( \theta_2(x)(k,j) = f_G(x) = 1/2^k \). But this contradicts \( \theta(y) \in V \), and thus completes the proof. \( \square \)
Corollary 1.5. \( \mathcal{N} \) is precisely the class of strongly metrizable spaces.

Remark. Recall that a space is strongly paracompact if every open cover has a star-finite open refinement. It is well-known and easy to see that every strongly paracompact metrizable space is strongly metrizable and hence, by our result, is in \( \mathcal{N} \). But not every member of \( \mathcal{N} \) is strongly paracompact; e.g., it is known [N3] that \( \omega_1^\omega \times (0,1) \), which of course embeds in \( \omega_1^\omega \times I^\omega \) and hence is strongly metrizable, is not strongly paracompact.

We also remark that Y. Hattori [H] obtained another characterization of strongly metrizable spaces in terms of a metric.

3. An Example

By Lemma 1.1, the only way the collection of \( \epsilon \)-balls (\( \epsilon \) fixed) can be locally finite is for this collection to be precisely the stars of some star-finite open cover. Recall that Nagata showed that every metrizable space admits a metric such that the collection of \( \epsilon \)-balls is closure-preserving by constructing a metric such that the collection of \( \epsilon \)-balls is precisely the collection of stars of some locally finite open cover. So it is natural to ask if this is the only way the collection of \( \epsilon \)-balls can be closure-preserving. The following example shows that the answer is “no”.

Example. There is a metric space \((X,d)\) such that, for every \( \epsilon > 0 \), \( \mathcal{B}_d(\epsilon) \) is closure-preserving but there is no locally finite open cover \( \mathcal{G}(\epsilon) \) with \( \mathcal{B}_d(\epsilon) = \{st(x,\mathcal{G}(\epsilon)) : x \in X\} \).

Proof. Let the set \( X \) be \( \omega \times \mathbb{R} \), viewed as a subset of the plane. For \( x, y \in X \), denote the usual Euclidean distance between \( x \) and \( y \) by \( |x - y| \). Then for \( x = (n_x,r_x) \) and \( y = (n_y,r_y) \) in \( X \), define \( d(x,y) \) to be \( |x - y| \) if \( x = y \), or if \( n_x \neq n_y \), or if \( n_x = n_y = n \) and \( |x - y| > 1/2^n \). Let \( d(x,y) = 1/2^n \) otherwise. It is easy to check that \( d \) is a metric on the set \( X \), and that \( d \) generates the discrete topology on \( X \). So any collection of subsets of \( X \) is closure-preserving, in particular \( \mathcal{B}_d(\epsilon) \).

Fix \( \epsilon > 0 \). We aim to show that \( \mathcal{B}_d(\epsilon) \) cannot be precisely the collection of stars of some locally finite open cover. To this end, choose \( n \) such that \( 1/2^n < \epsilon \), and note that the trace of \( \mathcal{B}_d(\epsilon) \) on \( \{n\} \times \mathbb{R} \) contains \( \{(n\} \times (x - \epsilon, x + \epsilon) : x \in \mathbb{R}\} \), i.e., it contains all open intervals on \( \{n\} \times \mathbb{R} \) of length \( 2\epsilon \). Thus establishing the following claim will complete the proof.

Claim. There is no point-finite cover \( \mathcal{G} \) of \( \mathbb{R} \) such that every open interval of length \( 2\epsilon \) is the union of some finite subcollection of \( \mathcal{G} \).

Proof of Claim. Suppose \( \mathcal{G} \) is such a point-finite cover of \( \mathbb{R} \). Let \( z \in \mathbb{R} \). There is a finite subcollection \( \mathcal{G}_z \) of \( \mathcal{G} \) such that \( \cup \mathcal{G}_z = (z - 2\epsilon, z) \). Since \( \mathcal{G} \) is point-finite, if we let

\[
z' = \sup(\cup\{G \in \mathcal{G}_z : \sup(G) < z\})
\]

then \( z' < z \). Choose \( q_z \in \mathbb{Q} \) between \( z' \) and \( z \). Pick \( G_z \in \mathcal{G}_z \) with \( q_z \in G_z \). Note that \( \sup(G_z) = z \), hence \( z \neq z' \) implies \( G_z \neq G_{z'} \). But there must be \( q \) such that \( q_z = q \) for uncountably many \( z \), contradicting point-finiteness of \( \mathcal{G} \) at \( q \). □

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