A NOTE ON D-SPACES

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Abstract. We introduce notions of nearly good relations and N-sticky modulo a relation as tools for proving that spaces are D-spaces. As a corollary to general results about such relations, we show that $C_p(X)$ is hereditarily a D-space whenever X is a Lindelöf Σ-space. This answers a question of Matveev, and improves a result of Buzyakova, who proved the same result for X compact.

We also prove that if a space X is the union of finitely many D-spaces, and has countable extent, then X is linearly Lindelöf. It follows that if X is in addition countably compact, then X must be compact. We also show that Corson compact spaces are hereditarily D-spaces. These last two results answer recent questions of Arhangel'skii. Finally, we answer a question of van Douwen by showing that a perfectly normal collectionwise-normal non-paracompact space constructed by R. Pol is a D-space.

1. Introduction

The class of D-spaces, introduced by E. van Douwen in [vDP], is a very natural one. X is a *D-space* iff, given a "neighborhood assigment" $\{N(x) : x \in X\}$ (i.e, $x \in \mathrm{Int}N(x)$ for each $x \in X$), there is a closed discrete subset D of X such that $X = \bigcup\{N(x) : x \in D\}$.

There has been some interesting recent work on D-spaces due especially to Arhangel'skii and Buzyakova [AB], Buzyakova[Bu], and Fleisner and Stanely [FS]. In particular, Arhangel'skii and Buzyakova show that spaces having a point-countable base are D. Fleissner and Stanely introduce the notion of N-sticky for a neighborhood assignment N, a tool which simplifies many D-space arguments. Buzyakova obtained an interesting result in C_p-theory which illustrates how D-spaces can be useful: she proved that $C_p(X)$ is hereditarily D for compact X. This can be viewed as an "explanation" for the important, now classical, result of Baturov[Ba], that Lindelöf degree equals extent for subspaces of these $C_p(X)$'s.

In the first part of this note, we introduce the notion of a nearly good relation, and generalize the Fleissner-Stanley N-sticky notion. We observe that the point-countable base result and the $C_p(X)$ result mentioned above follow easily from general results about these notions. Baturov's result holds more generally for Lindelöf Σ-spaces X, and Matveev asked if $C_p(X)$ is hereditarily D for such X. We exploit our general results to obtain a positive answer to Matveev's question.

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I would like to dedicate this paper to my colleague and friend A.V. Arhangel'skii on the occasion of his 65^{th} birthday.

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Another corollary of Buzyakova's result that $C_p(X)$ for compact X is hereditarily D is that Eberlein compacta, which are embeddable in such function spaces, are hereditarily D. This led Arhangel'skii to ask if Corson compacts are hereditarily D. We show that the answer is positive. We answer another question of Arhangel'skii on D-spaces by showing that a countably compact space which is a finite union of D-spaces must be compact. Finally, we solve a problem of van Douwen by showing that a perfectly normal collectionwise-normal space of R. Pol is a D-space.

The following notation will be used throughout: If N is a neighborhood assignment on X, and $D \subset X$, we let $N(D) = \bigcup_{d \in D} N(d)$.

2. Nearly good sticky relations

Let X be a space. We say that a relation R on X (resp., from X to $[X]^{<\omega}$) is *nearly good* if $x \in \overline{A}$ implies xRy for some $y \in A$ (resp., $xR\tilde{y}$ for some $\tilde{y} \in [A]^{<\omega}$).

Further, if N is a neighborhood assignment on X, $X' \subset X$, and $D \subset X$, we say D is *N-sticky mod R on X'* if whenever $x \in X'$ and xRy for some $y \in D$ (resp., $xR\tilde{y}$ for some $\tilde{y} \in [D]^{<\omega}$), then $x \in N(D)$. (In other words, it means that $N(D)$ contains all the "relatives" of members (resp, finite subsets) of D that are in X'.) We say more briefly that *D is N-sticky mod R* if D is N-sticky mod R on X.

For example, if N is a neighborhood assignment and we define $xRy \iff y \in N(x)$, then "N-sticky mod R" means "$x \in N(D)$ whenever $N(x) \cap D \neq \emptyset$ and is what Fleissner and Stanley[FS] called simply "N-sticky" . Obviously this R is nearly good.

We begin with the following lemma, which is an immediate consequence of the definitions.

Lemma 2.0. *Let N be a neighborhood assignment on X, and R a nearly good relation (on X, or from X to $[X]^{<\omega}$). If D is N-sticky mod R on X', then $\overline{D} \cap X' \subset N(D)$.*

The next lemma will help us build closed discrete sets D with $X = N(D)$.

Lemma 2.1.

(a) *Let R be a nearly good relation on X. If for each $\alpha < \lambda$, D_α is a closed discrete subset of $X \setminus N(\bigcup_{\beta<\alpha} D_\beta)$ and N-sticky mod R on $X \setminus N(\bigcup_{\beta<\alpha} D_\beta)$, then $\bigcup_{\alpha<\lambda} D_\alpha$ is closed discrete.*

(b) *Let R be a nearly good relation from X to $[X]^{<\omega}$. If for each $\alpha < \lambda$, D_α is a closed discrete subset of $X \setminus N(\bigcup_{\beta<\alpha} D_\beta)$ and $\bigcup_{\beta \le \alpha} D_\beta$ is N-sticky mod R on $X \setminus N(\bigcup_{\beta<\alpha} D_\beta)$, then $\bigcup_{\alpha<\lambda} D_\alpha$ is closed discrete.*

Proof. We prove (b) first. Suppose x is a limit point of $\bigcup_{\alpha<\lambda} D_\alpha$. Since R is nearly good, there are some $\alpha' < \lambda$ and $\tilde{y} \in [\bigcup_{\beta \le \alpha'} D_\beta]^{<\omega}$ with $xR\tilde{y}$. By the N-stickiness of $\bigcup_{\beta \le \alpha'} D_\beta$, we must have $x \in N(\bigcup_{\beta \le \alpha'} D_\beta)$. Let α be least such that $x \in N(D_\alpha)$. Then by the same argument, x is not a limit point of $\bigcup_{\beta<\alpha} D_\beta$. Since $N(D_\alpha) \cap \bigcup_{\gamma>\alpha} D_\gamma = \emptyset$, x is not a limit point of D, contradiction.

Part (a) is similar, noting that for relations on X we only need to apply N-stickiness to individual D_α's, instead of unions of the type $\bigcup_{\beta \le \alpha} D_\beta$. $\square$

Proposition 2.2. *Let N be a neighborhood assignment on X.*

(a) *Suppose R is a nearly good relation on X such that every non-empty closed subset F of X contains a non-empty closed discrete subset D which is N-sticky mod R on F. Then there is a closed discrete D^* in X with $N(D^*) = X$.*

(b) *Let R be a nearly good relation from X to $[X]^{<\omega}$. Suppose that given any closed discrete D and non-empty closed $F \subset X \setminus N(D)$ such that D is N-sticky mod R on F, there is a non-empty closed discrete $E \subset F$ such that $D \cup E$ is N-sticky mod R on F. Then there is a closed discrete D^* in X with $N(D^*) = X$.*

Proof. The proofs of (a) and (b) are essentially the same. Inductively define closed discrete $D_\alpha \subset X \setminus N(\bigcup_{\beta<\alpha} D_\beta)$ satisfying the stickiness property given by (a) or (b), until a stage λ is reached such that $X = N(\bigcup_{\alpha<\lambda} D_\alpha)$. Then apply Lemma 2.1 to see that $D^* = \bigcup_{\alpha<\lambda} D_\alpha$ is closed discrete. □

By part (a) of this proposition, if we wish to prove that a certain closed-hereditary property implies D, we just need to prove that any neighborhood assignment N on a space with the property contains some non-empty N-sticky mod R closed discrete subset for some nearly good R (as long as R is defined only in terms of N and the property).

For example, suppose X is left-separated and N is a neighborhood assignment on X. W.l.o.g., $N(x) \subset [x, \rightarrow)$, where the implied order is the order that left-separates X. Then every non-empty subset F of X has a non-empty closed discrete N-sticky subset, namely the least element of F. So by Proposition 2.2(a), left-separated spaces are D. van Douwen and Pfeffer[vDP] show this for the so-called "generalized left-separated" spaces, and this also follows from Proposition 2.2(a) by a similarly easy argument.

In certain more complicated applications of Proposition 2.2(a), it is natural to build a countable closed discrete N-sticky set. Here the use of countable elementary submodels can significantly simplify arguments. At first glance, countable elementary submodels do not seem as relevant to Proposition 2.2(b). However, they are relevant because it turns out that if 2.2(b) is true for all countable D, it is true for all D.

Proposition 2.3. *Let N be a neighborhood assignment on X , and let R be a nearly good relation from X to $[X]^{<\omega}$. Suppose that given any countable closed discrete D and non-empty closed $F \subset X \setminus N(D)$ such that D is N-sticky mod R on F, there is a countable non-empty closed discrete $E \subset F$ such that $D \cup E$ is N-sticky mod R on F. Then there is a closed discrete D^* in X with $N(D^*) = X$.*

Proof. Suppose D is closed discrete, $\emptyset \neq F \subset X \setminus N(D)$ is closed, and D is N-sticky mod R on F. We need to show that there is a non-empty closed discrete $E \subset F$ such that $D \cup E$ is N-sticky mod R on F. Then the required D^* exists by 2.2(b).

Let $D = \{d_\alpha < \alpha < \kappa\}$, where $\kappa = |D|$, and suppose any D of cardinality smaller than κ can be extended as required to an E such that $|E| \leq |D| + \omega$. By our assumption, $\kappa > \omega$. Inductively define non-empty closed discrete sets $E_\alpha \subset F \setminus N(\bigcup_{\beta<\alpha} E_\beta)$ of cardinality $\leq |\alpha| + \omega$ such that $\{d_\beta : \beta < \alpha\} \cup \bigcup_{\beta\leq\alpha} E_\beta$ is N-sticky mod R on F. Stop the induction either at $\lambda = \kappa$, or at any $\lambda < \kappa$ for which $F \setminus N(\bigcup_{\alpha<\lambda} E_\alpha) = \emptyset$. It is easy to see that $D \cup E$ is N-sticky mod R on F, and, using Lemma 2.1(b), that E is closed discrete. $\square$

We now describe a general situation in which spaces can be shown to be D-spaces because the hypothesis of Proposition 2.3 holds.

Given a neighborhood assignment N on X, let us call a subset Z of X *N-close* if $x, x' \in Z \Rightarrow x \in N(x')$ (equivalently, $Z \subset N(x)$ for every $x \in Z$).

Proposition 2.4. *Let N be a neighborhood assignment on X. Suppose there is a nearly good R on X (resp., from X to $[X]^{<\omega}$) such that for any $y \in X$ (resp., $\tilde{y} \in [X]^{<\omega}$), $R^{-1}(y) \setminus N(y)$ (resp., $R^{-1}(\tilde{y}) \setminus N(\tilde{y})$) is the countable union of N-close sets. Then there is a closed discrete D such that $N(D) = X$.*

Remark. Note that if N and R satisfy the hypotheses of the proposition, then so does their restriction to any subspace. So, if for any N on X we can produce such an R, then X is hereditarily D.

Proof of the proposition. We prove this in case R is a relation from X to $[X]^{<\omega}$. By Proposition 2.3, we need only show that if D is countable, closed discrete, and N-sticky mod R on some non-empty closed $X' \subset X \setminus N(D)$, then there is a non-empty countable closed discrete $E \subset X'$ such that $D \cup E$ is N-sticky mod R on X'.

For $\tilde{y} \in [X]^{<\omega}$, let $R^{-1}(\tilde{y}) \setminus N(\tilde{y}) = \bigcup_{n\in\omega} G_n(\tilde{y})$, where each $G_n(\tilde{y})$ is N-close. Put all relevant objects in a countable elementary submodel $\mathcal{M}$. Let $<_{\mathcal{M}}$ well-order $\mathcal{M}$ in type ω. Choose $e_0 \in X' \cap \mathcal{M}$. If $e_i \in X' \cap \mathcal{M}$ has been defined for all $i < n$, look at

$$X'_n = \{x \in X' \setminus N(\{e_i : i < n\}) : xR\tilde{y} \text{ for some } \tilde{y} \in [D \cup \{e_i : i < n\}]^{<\omega}\}.$$

If $x \in X'_n$, then $x \in G_n(\tilde{y}) \subset N(x)$ for some $\tilde{y} \in [D \cup \{e_i : i < n\}]^{<\omega}\}$. Note any such $G_n(\tilde{y})$ is in $\mathcal{M}$. Choose $e_n \in X'_n \cap \mathcal{M}$ such that the corresponding $G_n(\tilde{y})$ is $<_{\mathcal{M}}$ least possible.

If $X'_n = \emptyset$ for any $n > 0$, then $D \cup \{e_i : i < n\}$ is closed discrete and N-sticky mod R relative to X' and we are done. If $X'_n \neq \emptyset$ for all $n > 0$, let us show that if $E = \{e_i : i < \omega\}$, then $D \cup E$ is N-sticky mod R on X' and closed discrete. Clearly E is relatively discrete in $N(E)$, so by Lemma 2.0, it suffices to prove $D \cup E$ is N-sticky mod R on X'. To this end, suppose $x \in X' \setminus N(D \cup E)$ and $xR\tilde{y_0}$ for some $\tilde{y_0} \in [D \cup E]^{<\omega}$. Then for all sufficiently large n, we have $x \in X'_n$. Let n_0 be such that $x \in G_{n_0}(\tilde{y_0})$, and note that $G_{n_0}(\tilde{y_0}) \in \mathcal{M}$. Since $N(e_n)$ always contains the $<_{\mathcal{M}}$-least $G_n(\tilde{y})$ corresponding to some $x \in X'_n$, eventually we chose e_n with $N(e_n) \supset G_{n_0}(\tilde{y_0})$, which puts $x \in N(e_n)$, contradiction. $\square$

Recall that X satisfies *open (G)* if for each $x \in X$ we have a countable open neighborhood base $\mathcal{B}_x$ of x such that whenever $x \in \overline{A}$ and $N(x)$ is a neighborhood of x, then for some $a \in A$ we have $x \in B \subset N(x)$ for some $B \in \mathcal{B}_a$. Spaces with a point-countable base satisfy open (G), but whether or not the reverse holds is an unsolved problem [CR]. We illustrate the use of Proposition 2.4 by proving the following generalization of the Arhangel'skii-Buzyakova result about point-countable bases (which can similarly be derived from 2.4).

Proposition 2.5. *Any space satisfying open (G) is a D-space.*

Proof. Let X satisfy open (G), and let N be a neighborhood assignment. Define

$$xRy \iff \exists B \in \mathcal{B}_y \text{ with } x \in B \subset N(x).$$

It is clear from the definition of open (G) that this R is nearly good.

For each $B \in \mathcal{B}_y$, let $C(B) = \{x : x \in B \subset N(x)\}$. Then $C(B)$ is N-close, and $R^{-1}(y) = \bigcup_{B \in \mathcal{B}_y} C(B)$. By Proposition 2.4, X is D. $\square$

The framework encompassed by our Propositions 2.2(b),2.3, and 2.4 is implicit in Buzyakova's proof of the following, which we give here as another illustration of the use of our Proposition 2.4.

Proposition 2.6 [Bu]**.** *If X is compact, then $C_p(X)$ is hereditarily D.*

Proof. Let $\mathcal{B}$ be a countable base for the real line $\mathbb{R}$. For $S \subset X$ and $B \in \mathcal{B}$, let $[S, B] = \{f \in C(X) : f(S) \subset B\}$. For $A \subset C(X)$, let $\mathcal{G}_A$ be the set of all $G = \bigcap_{i<n}[S_i, B_i]$ where $B_i \in \mathcal{B}$ and S_i can be written in the form $X \setminus \bigcup_{a \in A'} a^{-1}(B_a)$ for some finite $A' \subset A$.

Let N be a neighborhood assignment on $C_p(X)$. For $f \in C_p(X)$ and $\tilde{g} \in [C_p(X)]^{<\omega}$, define

$$fR\tilde{g} \iff \exists G \in \mathcal{G}_{\tilde{g}}(f \in G \subset N(f)).$$

Without using the terminology, Lemma 2.3 of [Bu] says exactly that this R is nearly good. For each $G \in \mathcal{G}_{\tilde{g}}$, let $C(G) = \{f \in R^{-1}(\tilde{g}) : f \in G \subset N(f)\}$. Note that $C(G)$ is N-close. Since $\mathcal{G}_{\tilde{g}}$ is countable, we have that $R^{-1}(\tilde{g})$ is a countable union of N-close sets. So $C_p(X)$ is hereditarily D by Proposition 2.4. $\square$

A similar use of Proposition 2.4 answers a question of Matveev [M].

Proposition 2.7. *Let X be a Lindelöf Σ-space. Then $C_p(X)$ is hereditarily D.*

Proof. Since X is Lindelöf Σ, there are a cover $\mathcal{K}$ by compact sets and a countable collection $\mathcal{F}$ such that, whenever $K \in \mathcal{K}$ and $K \subset U$, where U is open, then $K \subset F \subset U$ for some $F \in \mathcal{F}$.

For $A \subset C(X)$, define $\mathcal{G}_A$ just like in the proof of Proposition 2.6, except that the S_i's may have the form $F \setminus \bigcup_{a \in A'} a^{-1}(B_a)$ for some finite $A' \subset A$, where $F \in \mathcal{F}$. Then define the relation R just like before. Since $\mathcal{F}$ is countable, $\mathcal{G}_A$ for countable A is too, so by the same argument each $R^{-1}(\tilde{g})$ is a countable union of N-close sets.

Thus it remains to prove that R is nearly good. To this end, suppose $f \in \overline{A}$ for some $A \subset C_p(X)$. We need to show that $f \in G \subset N(f)$ for some $G \in \mathcal{G}_A$. (Note that any $G \in \mathcal{G}_A$ is in $\mathcal{G}_{A'}$ for some finite $A' \subset A$.) Since $\mathcal{G}_A$ is closed under finite intersections, we may assume $N(f)$ is a subbasic open set $[\{p\}, B]$, where $p \in X$ and $B \in \mathcal{B}$.

Let $p \in K$, where $K \in \mathcal{K}$. Let B' be open in $\mathbb{R}$ with $f(p) \subset B' \subset \overline{B'} \subset B$. For each $y \in K$ with $f(y) \notin B$, choose $B_y \in \mathcal{B}$ containing $f(y)$ with $B_y \cap B' = \emptyset$. Since $f \in \overline{A}$, we can choose some $a_y \in A$ with $a_y(y) \in B_y$ and $a_y(p) \in B'$. By compactness, there are y_i, $i < n$, such that the sets $a_{y_i}^{-1}(B_{y_i})$ cover $K \setminus f^{-1}(B)$. Let $F \in \mathcal{F}$ such that $K \subset F \subset f^{-1}(B) \cup \bigcup_{i<n} a_{y_i}^{-1}(B_{y_i})$. Let $S = F \setminus \bigcup_{i<n} a_{y_i}^{-1}(B_{y_i})$. Then $[S, B] \in \mathcal{G}_A$ and $f \in [S, B] \subset [\{p\}, B] = N(f)$. $\square$

3. Corson compacts

A corollary of Bouziakova's result that $C_p(X)$ is hereditarily a D-space whenever X is compact is that Eberlein compact spaces are hereditarily D. This prompted the natural question, due to Arhangel'skii, whether Corson compact spaces are hereditarily D. We will show that the answer is positive.

Recall that X is Corson compact iff X is compact and can be embedded into a Σ-product of real lines. Using the fact that any closed interval in the real line containing 0 is a ≤ 2-to-one continuous image of the Cantor set under a map f with $f^{-1}(0) = 0$, it is easy to see (and well-known) that any Corson compact space is the continuous image of a 0-dimensional Corson compact space. Also, the D-space property is preserved by closed mappings [BW]. It follows that it suffices to prove that 0-dimensional Corson compact spaces are hereditarily D.

Lemma 3.1. *Let X be Corson compact and 0-dimensional. Then X has a point-countable T_0-separating cover $\mathcal{B}$ by compact open sets which is closed under finite intersections.*

Proof. From 0-dimensionality and the fact that a compact space X is Corson compact iff X has a point-countable T_0-separating cover by open F_σ-sets [MR], it easily follows that there is a point-countable T_0-separating collection $\mathcal{B}$ of compact open sets. Take any such $\mathcal{B}$ and close it under finite intersections. □

Lemma 3.2. *Let $\mathcal{B}$ a point-countable T_0-separating cover by compact open sets of a compact space X which is closed under finite intersections. Then every $x \in X$ has a neighborhood base of sets of the form $B \setminus \cup\mathcal{C}$, where $B \in \mathcal{B}$ and and $\mathcal{C}$ is a finite subcollection of $\mathcal{B}$.*

Proof. Let $x \in U$, U open. For each $y \in X \setminus U$, either there is $B_y \in \mathcal{B}$ with $x \in B_y$ and $y \notin B_y$, or there is $C_y \in \mathcal{B}$ with $y \in C_y$ and $x \notin C_y$. By compactness, there is a finite subcollection of $\{C_y : y \in X \setminus U\} \cup \{X \setminus B_y : y \in X \setminus U\}$ which covers $X \setminus U$. Then take B to be the intersections of the B_y's from this finite subcover , and take $\mathcal{C}$ to be the C_y's. □

Given a collection $\mathcal{S}$ of finite sets, let $\mathcal{R}(\mathcal{S})$ denote the collection of all roots of uncountable Δ-systems from $\mathcal{S}$ (i.e., $R \in \mathcal{R}(\mathcal{S})$ iff there is an uncountable subcollection $\mathcal{S}'$ of $\mathcal{S}$ such that $S_0 \cap S_1 = R$ whenever S_0 and S_1 are distinct elements of $\mathcal{S}'$). Then let

$$\mathcal{M}(\mathcal{S}) = \{R \in \mathcal{R}(\mathcal{S}) : \nexists R' \in \mathcal{R}(\mathcal{S})(R' \subsetneq R)\} \cup \{S \in \mathcal{S} : \nexists R \in \mathcal{R}(R \subseteq S)\}.$$

Lemma 3.3. *For any collection $\mathcal{S}$ of finite sets, the collection $\mathcal{M}(\mathcal{S})$ is countable.*

Proof. Easy application of the Δ-system lemma (that any uncountable collection of finite sets contains an uncountable Δ-system). □

Theorem 3.4. *Every Corson compact space is hereditarily a D-space.*

Proof. Let X be Corson compact, and $Z \subset X$. By the remark preceding Lemma 3.1, we may assume X is 0-dimensional. Then by Lemma 3.1, there is a point-countable T_0-separating cover $\mathcal{B}$ of X consisting of compact open sets which is closed under finite intersections.

Let $N(z)$, $z \in Z$, be a neighborhood assignment. By Lemma 3.2, we may assume $N(z) = (B_z \setminus \cup\mathcal{C}_z) \cap Z$ for some $B_z \in \mathcal{B}$ and finite $\mathcal{C}_z \subset \mathcal{B}$. By our observation after Proposition 2.2, we need only show that there exists a non-empty closed discrete N-sticky subset D of Z. Recall that N-sticky means we need $D \cap N(z) \neq \emptyset$ to imply $z \in N(D)$.

To this end, put $X, \mathcal{B}, Z, N, \ldots$ in a countable elementary submodel M (of $H(\kappa)$ for some sufficiently large κ). Let $\{\mathcal{B}_i\}_{i\in\omega}$ enumerate $M \cap [\mathcal{B}]^{<\omega}$ in type ω such that each term is listed infinitely often.

At step k, $k \in \omega$, we are going to define a finite subset F_k of $Z \cap M$ to put in D. Look at $\mathcal{B}_k$. If $\mathcal{B}_k$ is not a singleton, let $F_k = \emptyset$. Otherwise, let $\mathcal{B}_k = \{B_k\}$, and consider the collection

$$\mathcal{S}_k = \{\mathcal{C} : \exists z \in Z \cap B_k \setminus N(\cup\{F_i : i < k\}) \text{ with } B_z = B_k \text{ and } \mathcal{C}_z = \mathcal{C}\}.$$

If $\mathcal{S}_k = \emptyset$, let $F_k = \emptyset$. Otherwise, continue as follows. Note that $\mathcal{S}_k$ is in M since all parameters in its definition are in M. Thus $\mathcal{M}(\mathcal{S}_k)$ is in M, and since by Lemma 2.3 it is countable, we have $\mathcal{M}(\mathcal{S}_k) \subset M$.

Claim 1. For each $R \in \mathcal{M}(\mathcal{S}_k)$, there is a finite set $F(R) \subset M \cap Z \cap B_k \setminus N(\cup\{F_i : i < k\})$ such that $\bigcup_{z\in F(R)} N(z) = Z \cap B_k \setminus \cup R$.

To see this, consider $R \in \mathcal{M}(\mathcal{S}_k)$. Let

$$Z_k = \{z \in Z \cap B_k \setminus N(\cup\{F_i : i < k\}) : B_z = B_k\}.$$

Note that if $R \in \mathcal{S}_k$, then $\mathcal{C}_z = R$ for some $z \in Z_k$. By elementarity, there is such a z, call it z_R, in M, and taking $F(R) = \{z_R\}$ works. Suppose on the other hand that $R \notin \mathcal{S}_k$. Then R is the root of some uncountable Δ-system $\mathcal{S}' \subset \mathcal{S}_k$. Since $\mathcal{B}$ is point-countable, $\bigcap_{S\in\mathcal{S}'} \cup(S \setminus R)$ is empty. By compactness, some finite subcollection of these $S \setminus R$'s has empty intersection. It follows that there is some finite subset $F(R)$ of Z_k such that $R \subset \mathcal{C}_z$ for each $z \in F(R)$ and $\bigcap_{z\in F(R)} \cup(\mathcal{C}_z \setminus R) = \emptyset$. Again by elementarity, there is such an $F(R)$ in M, and it is clear that this $F(R)$ satisfies the desired condition.

Having established Claim 1, let R_k be the least member of $\mathcal{M}(\mathcal{S}_k)$ in our indexing of $M \cap [\mathcal{B}]^{<\omega}$, and let F_k be the set $F(R)$ guaranteed by Claim 1 with $R = R_k$.

Claim 2. If $j < k$ and $B_j = B_k$, then $R_j \neq R_k$.

Suppose that $R_j = R_k$. By the construction, at stage k there is some $z \in Z \cap B_k \setminus N(\cup\{F_i : i < k\})$ with $R_k \subseteq \mathcal{C}_z$. So z is in $B_k \setminus \cup R_k = B_j \setminus \cup R_j$ and is not in $N(\bigcup_{i\leq j} F_i)$, contradicting that $F(R_j)$ satisfies the conclusion of Claim 1.

We let $D = \bigcup_{i\in\omega} F_i$. Note that by the construction, if $z \in D$, then $N(z) \cap D$ is finite. Thus D is relatively discrete. It remains to prove that D is closed and N-sticky, which follows easily from:

Claim 3. If $p \in Z$ and $B_p \in M$, then $p \in N(D)$.

Suppose not. Then for each $k \in \omega$ such that $B_p = B_k$, the following holds at stage k of the inductive procedure for building D:

$$\exists z \in Z \cap B_p \setminus N(\bigcup_{i<k} F_i) \text{ with } B_z = B_p \text{ and } \mathcal{C}_z = \mathcal{C}_p.$$

Thus $\mathcal{C}_p \in \mathcal{S}_k$, and so there is also some $R'_k \in \mathcal{M}(\mathcal{S}_k)$ with $R'_k \subseteq \mathcal{C}_p$. Then there is $R^* \subset \mathcal{C}_p$ such that $R'_k = R^*$ for infinitely many k.

Recall that at a stage k like this, the least $R \in \mathcal{M}(\mathcal{S}_k)$ is selected and denoted by R_k. Since there are only finitely many possible R's less than R^*, and R^* has the possibility of being selected infinitely often, it follows from Claim 2 that $R^* = R_k$ for some k. So for this k we have $F_k = F(R^*)$. Then by Claim 1, $N(F_k) \supset Z \cap B_k \setminus \cup R^*$, which puts $p \in N(D)$, contradiction. $\square$

4. Finite unions of D-spaces and linearly Lindelöfness

In the problems section of the Zoltan Balogh Memorial Topology Conference booklet, and also in [A], Arhangel'skii asked whether the union of two D-spaces must be a D-space. He also asked if a countably compact space that is the union of two D-spaces must be compact. In this section, we give a positive answer to the second question. Our answer is a corollary to our more general result that any space of countable extent which can be written as the finite union of D-spaces must be linearly Lindelöf.

The first question, if it has a positive answer, would imply the second (since countably compact D-spaces are compact), but that one is still unsolved. Another related problem from [A] that also remains unsolved is whether or not a countably compact space that is a countable union of D-spaces must be compact.

Recall that a space X is *linearly Lindelöf* if every increasing open cover of X has a countable subcover. This is well-known to be equivalent to the statement that every subset of X of uncountable regular cardinality has a complete accumulation point. The following is another known characterization; for the benefit of the reader, we include its easy proof.

Lemma 4.1. *A space X is linearly Lindelöf iff whenever $\mathcal{O}$ is an open cover of X of cardinality κ and $\mathcal{O}$ has no subcover of cardinality $< \kappa$, then $cf(\kappa) \leq \omega$.*

Proof. Suppose X is linearly Lindelöf and $\mathcal{O} = \{O_\alpha : \alpha < \kappa\}$ is an open cover of X of cardinality κ with no subcover of cardinality $< \kappa$. Let $U_\alpha = \bigcup_{\beta \leq \alpha} O_\beta$. Then $\{U_\alpha : \alpha < \kappa\}$ is an increasing open cover, which must therefore have a countable subcover $\{U_{\alpha_n}\}_{n\in\omega}$. Since $\mathcal{O}$ has no subcover of cardinality $< \kappa$, $\{\alpha_n\}_{n\in\omega}$ must be cofinal in κ.

For the other direction, suppose X is not linearly Lindelöf, i.e., there is an increasing open cover $\mathcal{U}$ with no countable subcover. There is a cofinal subcollection $\mathcal{O}$ of $\mathcal{U}$ of regular cardinality κ. Note that $\mathcal{O}$ has no subcover of cardinality $< \kappa$ (by regularity of κ). Since $\mathcal{O}$ has no countable subcover, we have $cf(\kappa) = \kappa > \omega$. $\square$

Theorem 4.2. *If X has countable extent and can be written as the union of finitely many D-spaces, then X is linearly Lindelöf.*

Proof. . Suppose X satisfies the hypotheses, where $X = \bigcup_{i \leq k} X_i$ with each X_i a D-space. Suppose also by way of contradiction that X is not linearly Lindelöf and that k is the least possible value for any counterexample to the theorem. Of course $k > 1$ since any D-space of countable extent is Lindelöf.

By Lemma 4.1, there is an open cover $\mathcal{O} = \{O_\alpha\}_{\alpha<\kappa}$ of some cardinality κ with $cf(\kappa) > \omega$ and such that $\mathcal{O}$ has no subcover of cardinality $< \kappa$. For each $x \in X$, let α_x be least such that $x \in O_{\alpha_x}$ and consider the neighborhood assignment defined by $N(x) = O_{\alpha_x}$.

For each $i \leq k$, there is a relative closed discrete subset D_i of X_i such that $\{N(d) : d \in D_i\}$ covers X_i. Since $\mathcal{O}$ has no subcover of smaller cardinality, there must be some $i_0 \leq k$ such that $|\{\alpha_d : d \in D_{i_0}\}| = \kappa$. Note that $Z = \overline{D_{i_0}} \setminus D_{i_0}$ is closed in X and is a subset of $\bigcup_{i \neq i_0} X_i$. By minimality of k, Z is linearly Lindelöf. Applying this to the increasing open cover $\{\cup_{\beta<\alpha} O_\beta : \alpha < \kappa\}$, there are $\alpha_n < \kappa$, $n \in \omega$, such that $\mathcal{U} = \{\cup_{\beta<\alpha_n} O_\beta : n \in \omega\}$ covers Z. Note that $D_{i_0} \setminus \cup\mathcal{U}$ is closed discrete in X, so by countable extent is countable. By $cf(\kappa) > \omega$, we have $\delta = sup\{\alpha_n : n \in \omega\} < \kappa$. Hence there is some $d \in D_{i_0}$ with $\alpha_d > \delta$ and $d \in \cup\mathcal{U}$. But $d \in \cup\mathcal{U}$ implies $d \in O_\beta$ for some $\beta < \delta$, whence $\alpha_d < \delta$, contradiction. □

Corollary 4.3. *Suppose X is countably compact and a finite union of D-spaces. Then X is compact.*

Proof. Countably compact linearly Lindelöf spaces are compact. □

5. Pol's space is D

In his talk at the International Conference in Topology in Matsue, Japan, 2002, P.J. Nyikos mentioned the following problem related to what he had called "Classic Problem II" in the first volume (1976) of Topology Proceedings: Is every (perfectly normal) collectionwise-normal space with a point-countable base paracompact? This problem remains unsolved, not even consistency results are known. Arhangel'skii, recalling his result with R. Buzyakova that spaces with a point-countable base are D-spaces, asked in a verbal communication if it may even be that every (perfectly normal) collectionwise-normal D-space is paracompact. It turns out this essentially was asked earlier by van Douwen [vD]. He asked for a non-paracompact collectionwise-normal space that is not "trivially so". He goes on to mention some properties the space should have, and then says "it would be even better if the space is a D-space". In this section we show that that a perfectly normal collectionwise-normal non-paracompact space constructed by R. Pol[P] is a D-space, so this is an example of the kind van Douwen asked for, and answers Arhangel'skii's question in the negative.

We use the following version of Pol's space X. For each $\alpha \in \omega_1$, choose a non-decreasing function $x_\alpha : \omega \to \alpha$ with $\alpha = sup\{x_\alpha(n) : n \in \omega\}$. The set for X is $\{x_\alpha : \alpha < \omega_1\}$. For each $n \in \omega$ and $\sigma \in \omega_1^n$, let $[\sigma] = \{x \in X : x \restriction n = \sigma\}$. Then for each $\alpha < \omega_1$ and $n \in \omega$, let $B(\alpha, n) = \{x_\beta : \beta \leq \alpha \text{ and } x_\beta \restriction n = x_\alpha \restriction n\}$. Note that $B(\alpha, n) = [x_\alpha \restriction n] \cap \{x_\beta : \beta \leq \alpha\}$. The $B(\alpha, n)$'s form a basis for Pol's topology on X, which is clearly finer than the metric topology generated by the $[\sigma]$'s, and is also finer than the "interval" topology generated by sets of the form $\{x_\gamma : \alpha < \gamma \leq \beta\}$, where $\alpha, \beta \in \omega_1$.

Theorem 5.1. *Pol's space X described above is a D-space.*

Proof. Recall that a non-stationary subset A of ω_1 is metrizable; similarly, $X_A = \{x_\alpha : \alpha \in A\}$ is metrizable whenever A is non-stationary.

Another fact about X we shall use is that every uncountable subset of X contains an uncountable closed discrete set. To see this, note that since the topology is finer than the interval topology, every uncountable subset has an uncountable relatively discrete subset; then apply perfect normality.

Now suppose we are given an open neighborhood assignment for X. W.l.o.g., this can be coded by $f : \omega_1 \to \omega$, where $B(\alpha, f(\alpha))$ is the assigned open neighborhood of x_α.

Let Σ denote all $\sigma \in \omega_1^{<\omega}$ satisfying:

(i) $\exists$ stationary S_σ such that $x_\alpha \upharpoonright f(\alpha) = \sigma$ for all $\alpha \in S_\sigma$;
(ii) σ is minimal w.r.t. (i) (i.e., no proper initial segment of σ satisfies (i)).

Let $A = \{\beta \in \omega_1 : x_\beta \notin \bigcup_{\sigma\in\Sigma}[\sigma]\}$, and let $X_A = \{x_\beta : \beta \in A\}$. Then X_A is closed in X. Also, an easy pressing-down argument shows that A is non-stationary and hence X_A is metrizable. Thus there is a closed discrete subset D_0 of X_A such that $\{B(\beta, f(\beta)) : x_\beta \in D_0\}$ covers X_A.

Let $U = \bigcup\{B(\beta, f(\beta)) : x_\beta \in D_0\}$.

Claim 1. If $\sigma \in \Sigma$ and $[\sigma] \not\subset U$, then for sufficiently large $\alpha \in S_\sigma$, $x_\alpha \notin U$.

To prove Claim 1, suppose that $x_\alpha \in U$ for unboundedly many $\alpha \in S_\sigma$. Let $x_\gamma \in [\sigma]$. Consider $\alpha \in S_\sigma$ with $\alpha > \gamma$ and $x_\alpha \in U$. Then $x_\alpha \in B(\beta, f(\beta))$ for some $x_\beta \in D_0$. Note that $\alpha \leq \beta$. Since $x_\beta \notin [\sigma] = [x_\alpha \upharpoonright f(\alpha)]$ and $x_\alpha \upharpoonright f(\beta) = x_\beta \upharpoonright f(\beta)$, it must be the case that $\alpha < \beta$ and $f(\alpha) > f(\beta)$. Then $x_\gamma \in [\sigma] \cap \{x_\delta : \delta \leq \alpha\} \subset [x_\beta \upharpoonright f(\beta)] \cap \{x_\delta : \delta \leq \beta\} = B(\beta, f(\beta)) \subset U$. Hence $[\sigma] \subset U$, which proves Claim 1.

For each $\sigma \in \Sigma$ with $[\sigma] \not\subset U$, by Claim 1 and the fact that every uncountable subset of X contains an uncountable closed discrete set, there exists an unbounded $T_\sigma \subset S_\sigma$ such that $x_\alpha \notin U$ for any $\alpha \in T_\sigma$ and $E_\sigma = \{x_\alpha : \alpha \in T_\sigma\}$ is closed discrete. Let $D = D_0 \cup \bigcup\{E_\sigma : \sigma \in \Sigma, [\sigma] \not\subset U\}$.

Claim 2. D is closed discrete. Let $x \in X$. If $x \in U$, then x is not a limit point of D since U misses all the E_σ's, and D_0 is closed discrete. On the other hand, if $x \notin U$, then there is a unique $\sigma \in \Sigma$ with $x \in [\sigma]$, and $[\sigma]$ misses D_0 and all E_τ's with $\tau \in \Sigma$ and $\tau \neq \sigma$.

The next claim completes the proof of the example.

Claim 3. $\{B(\alpha, f(\alpha)) : x_\alpha \in D\}$ covers X. Let $x \in X$. If $x \in U$, we are done, so suppose $x \notin U$. Then $x \notin X_A$, so there is a unique $\sigma \in \Sigma$ with $x \in [\sigma]$. Then $[\sigma] \not\subset U$, so T_σ and D_σ are defined. Say $x = x_\gamma$. Choose $\alpha \in T_\sigma$ with $\alpha > \gamma$. Then $x_\alpha \in D$ and $x = x_\gamma \in [\sigma] \cap \{x_\beta : \beta \leq \alpha\} = B(\alpha, f(\alpha))$, which completes the proof. □

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