CONNECTIFICATIONS OF METRIZABLE SPACES

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Abstract. We answer a question of Alas, Tkacenko, Tkachuk, and Wilson by constructing a metrizable space with no compact open subsets which cannot be densely embedded in a connected metrizable (or even perfectly normal) space. We also obtain a result that implies that every nowhere locally compact metrizable space can be densely embedded in a connected metrizable space.

0. Introduction

A space Y is called a *connectification* of a space X if X is dense in Y and Y is connected. It is easy to see that if X has a compact (or H-closed) open subset, then X has no Hausdorff connectification. There seem to be no other obvious general conditions which preclude spaces from having connectifications which are Hausdorff or Tychonoff or even more if the space in question satisfies some stronger property. E. K. van Douwen [vD] briefly discusses the question of when a Tychonoff space has a connected compactification, which is equivalent to having a Tychonoff connectification, and gives an example of a nowhere locally compact Tychonoff space with no Tychonoff connectification. In the same paper he conjectures that the familiar Sorgenfrey line, while of course very nice in terms of separation and covering properties, has no Tychonoff connectification; this was later shown by Emeryk and Kulpa [EK] to be the case.

Watson and Wilson [WW] give the first systematic study of when spaces have at least a Hausdorff connectification. They give an example of a Tychonoff space with no Hausdorff connectification, as well as some sufficient conditions for a Hausdorff connectification to exist. In particular, they show that every metrizable nowhere locally compact space has a Hausdorff connectification, and ask if this is true for any metrizable space with no compact open sets. This was recently answered positively by Porter and Woods [PW]; their technique it should be noted does not necessarily produce a connectification with higher separation properties (e.g., Tychonoff). Alas, Tkachuk, Tkacenko, and Wilson [ATTW] then showed that every separable metrizable space with no compact open sets has a metrizable connectifaction, and asked if this is true in the non- separable case as well.

In this paper we answer the question of Alas et al in the negative by constructing a metrizable space with no compact open sets which does not have a metrizable, or even perfectly normal, connectification. We also obtain a partial positive result that implies that nowhere locally compact metrizable spaces do have metrizable connectifications. This improves the result in [WW] mentioned above, and by our example

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it is in some sense a best possible positive result. Our example does have a Tychonoff connectification, however, so the question, also stated in [ATTW], whether every metrizable space with no compact open sets has a Tychonoff connectification remains open.

1. COUNTEREXAMPLE

We first prove a lemma which says that a metrizable space satisfying certain conditions will not have a metrizable, or even perfectly normal, connectification. Then we construct a metrizable space (with no compact open sets) satisfying those conditions.

Lemma 1.1. *Suppose X contains a closed subspace G and a collection $\mathcal{R}$ of disjoint (from each other and from G) clopen locally compact non-compact spaces (e.g., copies of the real line) such that:*

- (i) *every point in G has a neighborhood N such that $N \cap R$ is compact for every $R \in \mathcal{R}$;*
- (ii) *whenever $\mathcal{U}$ is a countable collection of opens sets covering G, some finite subcollection of $\mathcal{U}$ covers some member of R.*

Then X has no perfectly normal connectification.

Proof. Suppose Y is connected and contains X as a dense subspace. For each $R \in \mathcal{R}$, choose a point $y(R) \in \overline{R} \cap (Y \setminus X)$. Let $Y_0 = \{y(R) : R \in \mathcal{R}\}$. Note that by (i), $\overline{Y_0} \cap G = \emptyset$. If Y were perfectly normal, then G would be contained in a countable collection $\mathcal{U}$ of open sets (in Y) whose closures miss Y_0. By (ii), there would be a finite subcollection $\mathcal{V}$ of $\mathcal{U}$ and an $R \in \mathcal{R}$ such that $R \subset \cup\mathcal{V}$. But then $y(R)$ would be in the closure of some member of $\mathcal{V}$, contradiction. □

Example 1.2. *There is a metrizable space with no compact open sets which has no perfectly normal connectification.*

Proof. Let G be a complete graph on c^+ vertices, where c denotes the cardinality of the continuum, and where we think of a point of G as being either a vertex of G or a point in the interior of one of the edges of G. Define the distance between any two vertices of G to be 1, and extend this distance function in a natural way to all of G, thinking of the edges as having unit length.

Denote the set of vertices of G by V, and let Z be the integers. For every injection $f : Z \to V$, and every $k \in \omega$, let $R_{k,f}$ be a copy of the real line. We think of f as coding a bi-infinite path in G, naturally homeomorphic to the real line, and we are taking countably many copies of each such path. Indeed, let P_f denote the path in G corresponding to f, and fix the natural homeomorphisms $h_{k,f} : R_{k,f} \to P_f$ such that $h_{k,f}(i) = f(i)$. Let $\mathcal{R}$ be the collection of all of the $R_{k,f}$'s.

Let $X = G \cup (\cup\mathcal{R})$. Define a neighborhood of a point in

any $R \in \mathcal{R}$ to be a usual real line neighborhood. If $p \in G$, let $B(p, n) = \{x \in G : d(p, x) < 1/2^n\}$ be the $1/2^n$-ball about p in G, and define the n^{th} open neighborhood of p in X to be

$$B(p, n)^* = B(p, n) \cup (\bigcup\{h_{k,f}^{-1}(B(p, n) \cap P_f) : k \geq n, p \in P_f\}.$$

(In other words, for every bi-infinite path P_f containing p and for every $k \geq n$, we add to $B(p,n)$ the open interval in $R_{k,f}$ corresponding to the trace of $B(p,n)$ on P_f.)

It is easy to check that X is regular. Note that the collection $\{B(p,n)^* : p \in A\}$ is discrete if either $A = V$ and $n > 1$, or, for some $\epsilon > 0$, A contains one point from each edge and this point is at least ϵ away from the nearest vertex and $1/2^n < \epsilon$. It easily follows that X has a σ-discrete base, hence is metrizable.

Suppose that $\{U_n\}_{n\in\omega}$ is an open cover of G. Given an edge e, there is $n(e) \in \omega$ such that $e \subset \bigcup_{i<n(e)} U_i$. By compactness again there is a finite subset F of e and integers $\{n_x : x \in F\}$ such that each $B(x, n_x)^*$ is contained in $\bigcup_{i<n(e)} U_i$ and $\{B(x, n_x) : x \in F\}$ covers e. Now using the Lebesgue covering theorem we see that there is a single integer $m(e)$ such that for every $x \in e$ we have $B(x, m(e))^* \subset \bigcup_{i<n(e)} U_i$.

By the Erdos-Rado theorem [ER], there is an infinite (even uncountable, but this is not needed) subset W of V and integers m_0, n_0 such that $m(e) = m_0$ and $n(e) = n_0$ for each edge e with endpoints in W. Let f be any injection of Z into W. Then if $k \geq m_0$ it follows that $R_{k,f} \subset \bigcup_{i<n_0} U_i$. That completes the proof. □

Remark. It is easy to see that the space X just constructed does have a Tychonoff connectification, the Stone- Cech compactification of which would be a normal connectification. Let $\mathcal{B}$ be any base for X such that

(i) $cl(B \cap R)$ is compact for every $R \in \mathcal{R}$;
(ii) for each $B \in \mathcal{B}$ there is $B' \in \mathcal{B}$ with $cl(B) \subset B'$.

(E.g., the collection of all open sets satisfying (i) would satisfy (ii) as well.) Add a point ∞ to X, declaring the compelement of the closure of any finite union of members of $\mathcal{B}$ to be a neighborhood of ∞. It is easy to check that $X \cup \infty$ is connected and Tychonoff.

2. A positive result.

In this section, we prove a theorem (Theorem 2.2) from which it easily follows that every nowhere locally compact metrizable space has a metrizable connectification. The result of Alas et al [ATTW] that every separable metrizable space has a metrizable connectification is another easy corollary of our theorem.

We first establish the following lemma.

Lemma 2.1. *Let Z be a metrizable space with metric d, and let $D = \bigcup_{n\in\omega} D_n$ satisfy:*

(1) *Each D_n is closed discrete;*
(2) $d(D_n, D_m) \geq 1/2^n$ *if* $n > m$;
(3) $D_n = \bigcup\{D_n^\alpha : \alpha < \kappa_n\}$, *where* $diam(D_n^\alpha) < 1/2^{n-3}$;
(4) $d(D_n^\alpha, D_n^\beta) \geq 1/2^n$ *for all* $\alpha \neq \beta \in \kappa_n$.

Let $\{z_n^\alpha : \alpha < \kappa_n\}$ be a collection of distinct points not in Z. Let

$$Z^* = (Z \setminus D) \cup \{z_n^\alpha : \alpha < \kappa_n\}$$

and define $f : Z \to Z^*$ by:

(5) $f \restriction (Z \setminus D) = id$;
(6) $D_n^\alpha = f^{-1}(z_n^\alpha)$.

Then there is a metrizable topology on Z^* *such that* $f \restriction (Z \backslash D)$ *is a homeomorphic embedding.*

Proof.

We need to define the topology on Z^* so that everything works. Let $N_k(D_n^\alpha) = \bigcup\{B(d, 1/2^{n+k} : d \in D_n^\alpha\}$, where $B(d,\epsilon) = \{z \in Z : d(d,z) < \epsilon\}$. Note that by (2) and (4),

(5) $N_k(D_n^\alpha) \setminus D_n^\alpha$ misses D_j for all $j \leq n+k$.

Also, since $diam(D_n^\alpha < 1/2^{n-3}$, we see

(6) $diam(N_0(D_n^\alpha)) < 1/2^{n-3} + 2(1/2^n) < 1/2^{n-4}$.

Call an open subset U of Z *full* if whenever $U \cap D_n^\alpha \neq \emptyset$, then there exists $k \in \omega$ with $N_k(D_n^\alpha) \subset U$, and furthermore $k = 0$ for all but finitely many such D_n^α. Define V to be basic open in Z^* iff $f^{-1}(V)$ is open and full. The collection of open full sets in Z is clearly closed under finite intersections, so the collection of such V's in Z^* is a base for a topology on Z^*, and f is a continuous mapping.

Given an open set $U \subset Z$, define an open and full set $U^* \supset U$ as follows. Let $U_0^* = U$, and let

$$U_{n+1}^* = U_n^* \cup \bigcup\{N_0(D_m^\alpha) : D_m^\alpha \cap U_n^* \neq \emptyset\}$$

. Then $U^* = \bigcup_{n\in\omega} U_n^*$.

Since $N_0(D_m^\alpha) \cap D_n = \emptyset$ whenever $n < m$, and $N_0(D_m^\alpha) \cap D_m = D_m^\alpha$, and $diam(N_0(D_m^\alpha)) < 1/2^{m-4}$, it follows that if $U \cap (\bigcup_{i\leq n} D_i = \emptyset$, then any point in U_1^* is within $1/2^{n-3}$ of $U_0^* = U$, any point in U_2^*, is within $1/2^{n-2}$ of U_1^*, and so on. Thus we have:

$$(*) \qquad \text{If } U \cap (\bigcup_{i\leq n} D_i = \emptyset, \text{ then any point in } U^* \text{ is within } 1/2^{n-4} \text{ of } U.$$

From (*) we see that:

(7) If O is open in Z and $y \in O \setminus D$, then there is an open full set containing y and contained in O.

It follows that $f \restriction (Z \setminus D)$ is a homeomorphic embedding of $Z \setminus D$ into Z^*.

Let $N_k^*(D_m^\alpha) = D_m^\alpha \cup (N_k(D_m^\alpha) \setminus D_m^\alpha)^*$. Then $\{N_k^*(D_n\alpha) : k \in \omega\}$ is a collection of open full sets. By (5)and (*), for each $n \in \omega$ there exists $k > n$ such that $N_k^*(D_m^\alpha) \subset (N_n(D_m^\alpha)$. It follows that the images under f of the $N_k^*(D_n^\alpha)$'s for $k \in \omega$ form a base at the point z_n^α in Z^*.

We use the Moore metrization theorem [see, e.g., [E], Ch. 5] to prove that Z^* is metrizable. Let

$$\mathcal{B}_n = \{W^* : W \text{ is open in } Z \text{ and } diam(W^*) < 1/2^n\}$$

$$\cup\{N_k^*(D_m^\alpha) : m \in \omega, \alpha < \kappa_m, \text{ and } N_k^*(D_m^\alpha) \subset N_n(D_m^\alpha)\}.$$

Let $f(\mathcal{B}_n) = \{f(B) : B \in \mathcal{B}_n\}$. Let $p \in V \subset Z^*$, where V is open. Then $U = f^{-1}(V)$ is open and full in Z.

We need to show that there is an open set O' in Z^* containing p and $n \in \omega$ such that $st(O', f(\mathcal{B}_n)) \subset V$. Translated in Z, this means we need to find an open full set O in Z containing $q = f^{-1}(p)$ such that $st(O, \mathcal{B}_n) \subset U$.

Case 1. $p \notin f(D)$. Then $q = p$. Find $\epsilon > 0$ such that $B(p, \epsilon) \subset U$. By (6) we can find $k \in \omega$ such that $diam(N_0(D_j^\alpha)) < \epsilon/2$ if $j \geq k$; we may also assume $1/2^k < \epsilon/2$. Since $d(p, \bigcup_{i<k} D_i) > 0$, it follows from (5) that there is an open set W containing p and $n > k$ such that $N_n(D_m^\alpha) \cap W = \emptyset$ whenever $m < k$. Now let O be an open full set containing p and contained in $W \cap B(p, \epsilon/2)$. Then $st(O, \mathcal{B}_n) \subset B(p, \epsilon) \subset U$.

Case 2. $f^{-1}(p) = D_m^\alpha$. There is $j \in \omega$ with $N_j(D_m^\alpha) \subset U$. Now Case 2 follows in a similar way as Case 1, observing that by (4), $d(D_m^\alpha, (\bigcup_{i<k} D_i) \setminus D_m^\alpha) > 0$ for any $k \in \omega$. □

Theorem 2.2. *Suppose a metrizable space X can be densely embedded in a metrizable space Y in such a way that every clopen set in X has a limit point in $Y \setminus X$. Then X has a metrizable connectification.*

Proof.

Let d_0 be a metric on Y such that $diam(Y) < 1$. We shall first define a certain sequence $\{E_n\}_{n\in\omega}$ of closed discrete subsets of $Y \setminus X$, and "resolve" the points of $E = \bigcup_{n\in\omega} E_n$ into intervals.

Choose a point $e_0 \in Y \setminus X$ and let $E_0 = \{e_0\}$. Next, let E_1 be a maximal subset of $Y \setminus (X \cup E_0)$ such that $d_0(e, e') \geq 1/2$ for all $e \neq e' \in E_1$. Define a continuous function $f_0 : Y \setminus E_0 \to [0, 1/2]$ satisfying the following conditions:

(a) $f_0(U)$ is dense in $[0, 1/2]$ for every neighborhood U of e_0;
(b) $f_0(E_1) = 1/2$.

Define a new metric d_1 on $Y \setminus E_0$ by $d_1(y, y') = d_0(y, y') + |f_0(y) - f_0(y')|$; note that, by continuity of f_0, d_1 is equivalent to d_0 on $Y \setminus E_0$.

Now let E_2 be a maximal subset of $Y \setminus (X \cup E_0 \cup E_1)$ such that $d_1(e, e') \geq 1/4$ for each $e \neq e' \in E_2$. Let us notice that E_2 is closed discrete in Y. It clearly is closed discrete in $Y \setminus E_0$ (i.e., where d_1 is defined). We claim that $e_0 \in E_0$ cannot be a limit point of E_2. Suppose otherwise, and consider a sequence $y_0, y_1, ...$ in E_2 converging to e_0. Some subsequence of $f_0(y_0), f_0(y_1), ...$ converges to some point in the interval $[0, 1/2]$. But then the points y_n corresponding to this subsequence get arbitrarily near each other in metric d_1, contradiction.

Now define a continuous function $f_1 : Y \setminus E_1 \to [0, 1/4]$ satisfying the following conditions:

(a) $f_1(U)$ is dense in $[0, 1/4]$ for every neighborhood U of each point $e \in E_1$;
(b) $f_1(E_0) = 0$;
(c) $f_1(E_2) = 1/4$.

Similarly define d_2 on $Y \setminus E_0 \cup E_1$ by

$$d_2(y, y') = d_1(y, y') + |f_1(y) - f_1(y')| = d_0(y, y') + \Sigma_{i=0}^1 |f_i(y) - f_i(y')|.$$

Suppose E_i and d_i have been defined for $i \leq n$, and f_i for $i < n$. Let E_{n+1} be a maximal subset of $Y \setminus (X \cup E_0 \cup E_1 \cup ...E_n)$ such that $d_n(y, y') \geq 1/2^{n+1}$ for all $y \neq y' \in E_{n+1}$. Let us check that E_{n+1} is closed discrete in Y. Since d_n is equivalent to d_0 on $Y \setminus (E_0 \cup E_1 \cup ...E_{n-1})$, we need only see that no point of

$E_i, i < n$, can be a limit point of E_{n+1}. Suppose otherwise, and consider a sequence $y_0, y_1, ...$ in E_{n+1} converging to $e \in E_i$, where $i < n$. For $j < n$, $j \neq i$, the sequence $f_j(y_0), f_j(y_1), ...$ converges to $f_j(e)$, since the function f_j is continuous on $Y \setminus E_j$. Also, some subsequence of $f_i(y_0), f_i(y_1), ...$ converges to some point in the interval $[0, 1/2^{i+1}]$. Then the points y_n corresponding to this subsequence get arbitrarily near each other in metric d_n, contradiction. Define a continuous function $f_n : Y \setminus E_n$ satisfying the following conditions:

(a) $f_n(U)$ is dense in $[0, 1/2^{n+1}]$ for every neighborhood U of each point $e \in E_n$;
(b) $f_n(E_i) = 0$ for $i < n$;
(c) $f_n(E_{n+1}) = \{1/2^{n+1}\}$.

Similarly define d_{n+1} on $Y \setminus (E_0 \cup E_1 \cup ...E_n)$ by

$$d_{n+1}(y, y') = d_n(y, y') + |f_n(y) - f_n(y')| = d_0(y, y') + \Sigma_{i=0}^{n} |f_i(y) - f_i(y')|.$$

Let $E = \bigcup_{n\in\omega} E_n$, and consider the homeomorphic embedding

$$\iota : Y \setminus E \to Y \times [0, 1/2] \times [0, 1/4] \times ...$$

defined by

$$y \to (y, f_0(y), f_1(y), ...).$$

For each $e \in E_n$,$n \in \omega$, let

$$I(e) = \{(e, f_0(e), .., f_{n-2}(e), f_{n-1}(e) = 1/2^n)\} \times [0, 1/2^{n+1}] \times \{(0, 0, ...)\},$$

and let

$$Z = \iota(Y \setminus E) \cup (\bigcup_{e\in E} I(e)).$$

We first check that $Z \subset \overline{\iota(Y \setminus E)}$ (in fact, $Z = \overline{\iota(Y \setminus E)}$, but we don't need this). To this end, suppose $e \in E_n$ and $\alpha \in [0, 1/2^{n+1}]$. Since X is dense in Y and $e \in Y \setminus X$, and since $f_n(U)$ is dense in $[0, 1/2^{n+1}]$ for every neighborhood U of e, we can choose $x_i \in X$, $i \in \omega$, such that $x_i \to e$ and $f_n(x_i) \to \alpha$. Then $\iota(x_i) \to (e, f_0(e), ...f_{n-1}(e), \alpha, 0, 0, ...)$.

Let the distance d on Z be the coordinate-wise sum. We proceed to define closed discrete sets $D_0, D_1, ...$ satsifying the conditions of Lemma 2.1. First, for $e \in E_n$, let

$$e^- = (e, f_0(e), ...f_{n-2}(e), 1/2^n, 0, 0, ...)$$

and

$$e^+ = (e, f_0(e), ...f_{n-2}(e), 1/2^n, 1/2^{n+1}, 0, 0, ...).$$

Let $D_0 = \{e_0^-\}$, and for $n > 0$ let

$$D_n = \{e^+ : e \in E_{n-1}\} \cup \{e^- : e \in E_n\}.$$

Note that each D_n is closed discrete in Z, since the projection π_0 onto the first coordinate maps D_n one-to-one and onto the closed discrete set $E_{n-1} \cup E_n$ in Y.

Note that each point of D_n has $1/2^n$ as its last non-zero coordinate, which is the n^{th} coordinate, hence $d(D_n, D_m) \geq 1/2^n$ whenever $n > m$.

We need to define partitions of the D_n's. For $n \leq 1$, take the trivial partition of D_n. We partition D_n for $n > 1$ as follows. For $y \in E_n$, by maximality of E_{n-1}, $d_{n-2}(y, E_{n-1}) < 1/2^{n-1}$. If there is $y' \in E_{n-1}$ such that $d_{n-2}(y, y') < 1/2^n$, choose one and call it $\chi(y)$; otherwise, let $\chi(y)$ be any member of E_{n-1} with $d_{n-2}(y, \chi(y)) < 1/2^{n-1}$. For each $e \in E_{n-1}$, let

$$D_n^e = \{e^+\} \cup \{y^- : \chi(y) = e\}.$$

Then $\{D_n^e : e \in E_{n-1}\}$ is a partition of D_n.

We show $diam(D_n^e) < 1/2^{n-3}$. If $y, z \in E_n$ and $\chi(y) = \chi(z)$, then, by the definition of y^- and the fact that $ran(f_{n-2}) \subset [0, 1/2^{n-1}]$, we see that

$$\begin{aligned} d(y^-, z^-) = d_0(y,z) + \Sigma_{i=0}^{n-2}|f_i(y) - f_i(z)| &= d_{n-2}(y,z) + |f_{n-2}(y) - f_{n-2}(z)| \\ &\leq 2(1/2^{n-1}) + 1/2^{n-1} < 1/2^{n-3}. \end{aligned}$$

Also note that $d(y^-, e^+) \leq d_{n-2}(y, e) + 1/2^{n-1} < 1/2^{n-2}$.

We show that if $e \neq f \in E_{n-1}$, $d(D_n^e, D_n^f) \geq 1/2^n$. Let $y, z \in E_n$ with $\chi(y) = e$ and $\chi(z) = f$. We see similar to above that $d(y^-, z^-) = d_{n-1}(y, z) \geq 1/2^n$ by definition of E_n. Similarly, $d(e^+, f^+) = d_{n-2}(e, f) \geq 1/2^{n-1}$. It remains to check $d(e^+, z^-)$. Suppose $d(e^+, z^-) < 1/2^n$. Then $d_{n-2}(e, z) < 1/2^n$, and so by definition of $\chi(z)$, $d_{n-2}(z, f) < 1/2^n$. It follows that $d_{n-2}(e, f) < 1/2^{n-1}$, contradiction.

Let Z^* and $f : Z \to Z^*$ be as in Lemma 2.1. Since X is dense in Y and contained in $Y \setminus E$, $\iota(X)$ is a dense copy of X in Z^*. So it remains to prove that Z^* is connected.

Suppose U, V is a clopen partition of Z^*. Note that $f(\bigcup_{e \in E} I(e))$ is connected by the identifications made, so we may assume $f(\bigcup_{e \in E} I(e)) \subset V$. By the assumption on Y, there is a point $y \in Y \setminus X$ such that y is in the closure of $(f \circ \iota)^{-1}(U) \cap X$.

Suppose $y \in E_n$. Choose a sequence of points $x_0, x_1, ...$ in $(f \circ \iota)^{-1}(U) \cap X$ converging to y. We may assume that $f_n(x_0), f_n(x_1), ...$ converges to a number $\alpha \in [0, 1/2^{n+1}]$. Then the point $(y, f_0(y), ... f_{n-1}(y), \alpha, 0, 0, ...)$ in $I(y)$ is a limit point of $\iota(x_0), \iota(x_1), ...$ But then $f(I(y)) \cap U \neq \emptyset$, contradiction.

Suppose on the other hand that $y \notin E$. Then by maximality of the E_n's, there are $e_n \in E_n$ with $d_{n-1}(y, e_n) < 1/2^n$. It follows that $e_0^-, e_1^-, ...$ converges to $\iota(y)$ in Z. Hence $f(\iota(y)) \in \overline{f(\bigcup_{e \in E} I(e))} \subset V$, contradiction. That completes the proof. □

Corollary 2.3 [ATTW]. *Every separable metric space X with no compact open subsets has a metric connectification.*

Proof. Let Y be any metric compactification of X; then X and Y satisfy the hypotheses of Theorem 2.2. □

Corollary 2.4. *Let X be any nowhere locally compact metric space. Then X has a metric connectification.*

Proof. It is not difficult to see that one can add a σ-discrete set of points to a nowhere locally compact metrizable space X, obtaining a metrizable Y such that $Y \setminus X$ is dense in Y. Alternatively, one can quote [FGO], Lemma 2, which says that there is a metrizable space Y containing X such that both X and $Y \setminus X$ are dense in Y. In any case, X and Y satisfy the hypotheses of Theorem 2.2. □

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