IF X IS σ-COMPACT POLISH, THEN $C_k(X)$ HAS A σ-CLOSURE-PRESERVING BASE

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Abstract. We prove that if X is a σ-compact Polish space, then the space $C_k(X)$ of all continuous real-valued functions on X with the compact-open topology is a μ-space, and hence is M_1, i.e., it has a σ-closure-preserving base. We also construct an explicit σ-closure-preserving base for $C_k(X)$.

0. Introduction

Recall that a space X is *stratifiable* if it has a σ-closure-preserving quasi-base $\mathcal{B}$, i.e., for each point x in an open set U, there is some $B \in \mathcal{B}$ with $x \in B^o \subset B \subset U$, and $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ where each $\mathcal{B}_n$ is closure-preserving. X is an M_1*-space* if X has a σ-closure-preserving base. It is an old problem of J. Ceder [C] whether or not every stratifiable space is an M_1-space.

P. Gartside and E. Reznichenko [GR] have shown that the function space $C_k(X)$ is stratifiable whenever X is a complete separable metrizable space, i.e., Polish. Interestingly, Gartside and Reznichenko's proof of their theorem gives no clue as to whether or not these function spaces are M_1.

A space is a μ*-space* if it is embeddable in some $\Pi_{n\in\omega} X_n$, where each X_n is paracompact and a countable union of closed metrizable subspaces. It is known [M] that every stratifiable μ-space is M_1. Recently, Gartside and A. Glyn[GG] proved that $C_k(X)$ is a μ-space if X is the "metric fan", i.e., the space that may be described as the hedgehog with countably many "spines", where the spines are convergent sequences.

In his talk at the 2002 Summer Topology Conference in Auckland, New Zealand, Gartside asked if the conclusion of his theorem with Glyn was true for all σ-compact Polish spaces. In this note, we give a positive answer to this question. We also give a direct construction of a σ-closure-preserving base for such spaces, in hopes that this may be of some use in more general cases. The main related question which remains open is whether or not $C_k(\mathbb{P})$ is M_1, where $\mathbb{P}$ is the space of irrationals. A negative answer would settle Ceder's question.

Recently, T. Mizokami, N. Shimane, and Y. Kitamura [MSK] proved that stratifiable "WAP-spaces" (they used the term "δ-space" instead of WAP-space) are M_1. So another possible way to prove that the function spaces we are considering are M_1 would be to show that they are WAP. But we haven't been able to decide whether or not $C_k(X)$ is WAP for Polish spaces X, even if X is σ-compact. See the end of Section 2 for more details about this, including a simple example of a stratifiable space which is not WAP.

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1. Background and definitions

Let $\mathcal{U}$ be a collection of subsets of X. For each $\mathcal{C} \subset \mathcal{U}$, let $E_{\mathcal{C}} = \cap\mathcal{C} \setminus \cup(\mathcal{U} \setminus \mathcal{C})$. Then $\mathcal{P}_{\mathcal{U}} = \{E_{\mathcal{C}} : \mathcal{C} \subset \mathcal{U}\}$ is a partition of X called the *partition induced by* $\mathcal{U}$. $\mathcal{U}$ is said to be *mosaical* if $\mathcal{P}_{\mathcal{U}}$ has a σ-discrete closed refinement $\mathcal{F}$; note that it is equivalent to say that each member of $\mathcal{F}$ is contained in every member of $\mathcal{U}$ that it meets.

K. Tamano[Ta] noted that a stratifiable space X is a μ-space iff X has a σ-mosaical base. Note that for a Lindelöf space, a family $\mathcal{U}$ of Y is mosaical if and only if $\mathcal{P}_{\mathcal{U}}$ is countable and each member of $\mathcal{P}_{\mathcal{U}}$ is an F_σ-set; equivalently, there is a countable closed cover $\mathcal{F}$ of X such that each member of $\mathcal{F}$ is contained in every member of $\mathcal{U}$ that it meets.

For a space X, let $LC(X) = \{x \in X : x \text{ has a compact neighborhood}\}$; of course, $LC(X)$ is open. Let $X^{(0)} = X$. Suppose $X^{(\beta)}$ has been defined for all $\beta < \alpha$. If $\alpha = \gamma + 1$, let $X^{(\alpha)} = X^{(\gamma)} \setminus LC(X^{(\gamma)})$, and if α is a limit ordinal, let $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$. Then X is said to be *C-scattered* [Te] if $X^{(\alpha)} = \emptyset$ for some α; in this case, the least such α is called the *C-scattered rank*, $r(X)$, of X.

Using the Baire property, it is easy to check that every σ-compact Polish space is C-scattered of countable rank.

Some notation for $C_k(X)$. For any $\varphi : X \to \mathbb{R}$, compact set K, and $\varepsilon > 0$, we let

$$B(\varphi, K, \varepsilon) = \{f \in C_k(X) : \forall x \in K(|f(x) - \varphi(x)| < \varepsilon)\}.$$

Of course, $B(\varphi, K, \varepsilon)$ is a typical basic open neighborhood of φ in $C_k(X)$. We also denote by $\mathbf{0}$ the constant zero function in $C(X)$, and use $[K, \varepsilon]$ to denote $B(\mathbf{0}, K, \varepsilon)$.

2. $C_k(X)$ is μ

In this section we prove:

Theorem 2.1. *$C_k(X)$ is a μ-space for any σ-compact Polish space X.*

The following facts on mosaical collections are used:

Fact 1. *If $\mathcal{A}$ and $\mathcal{B}$ be mosaical families of Y, then $\mathcal{A} \cup \mathcal{B}$ and $\mathcal{A} \wedge \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$ are also mosaical.*

Fact 2. *Let $\mathcal{A}$ be a family of subsets of a space X. If there is a countable closed cover $\mathcal{F}$ of X such that $\mathcal{A}|F = \{A \cap F : A \in \mathcal{A}\}$ is mosaical for each $F \in \mathcal{F}$, then $\mathcal{A}$ is mosaical.*

Fact 3. *Suppose that $\{\mathcal{A}_n : n \in \omega\}$ is a sequence of mosaical families of Y. Let K be a closed set of Y and $\{U_n : n \in \omega\}$ a decreasing sequence of open F_σ-sets of Y with $K \subset A$ for each $A \in \bigcup_{n\in\omega} \mathcal{A}_n$ and $K = \bigcap_{n\in\omega} U_n$. Then $\mathcal{A} = \{A \cap U_n : A \in \mathcal{A}_n, n \in \omega\}$ is mosaical.*

Facts 1 and 2 are very easy to verify from the definitions. For Fact 3, note that each $U_n \setminus U_{n+1}$ is the union of a countable collection $\mathcal{H}_n$ of closed sets. Let $\mathcal{F} = \{K\} \cup \bigcup_{n\in\omega} \mathcal{H}_n$, and let $\mathcal{A}'_n = \{A \cap U_n : A \in \mathcal{A}_n\}$. Note that each $\mathcal{A}'_n$ is mosaical and $\mathcal{A} = \bigcup_{n\in\omega} A'_n$. For each $F \in \mathcal{F}$, there is some $n \in \omega$ such that $\mathcal{A}|F - \{\emptyset\} = (\bigcup_{i\leq n} \mathcal{A}'_i)|F - \{\emptyset\}$, and hence is mosaical by Fact 1. So by Fact 2, $\mathcal{A}$ is mosaical.

Let (X, q, r) be a triple such that X is a space and q, r are real numbers with $0 < q < r$. Consider the following property $(*)$ for (X, q, r):

($*$): There are a dominating family $\mathcal{K}$ of compact sets of X and a mosaical open family $\mathcal{B} = \{B(K) : K \in \mathcal{K}\}$ such that $\{\varphi \in C_k(X) : |\varphi(x)| \leq q$ for any $x \in K\} \subset B(K) \subset [K, r]$ for any $K \in \mathcal{K}$.

Lemma 2.2. *Let $0 < q < r$. Suppose that X is a σ-compact space with $X = \bigoplus_{n\in\omega} X_n$, where (X_n, q, r) satisfies ($*$) for each $n \in \omega$. Then (X, q, r) also satisfies ($*$).*

Proof. Suppose that for each $n \in \omega$, $\mathcal{K}_n$ and $\mathcal{B}_n = \{B_n(K) : K \in \mathcal{K}_n\}$ are families satisfying ($*$) for (X_n, q, r). Let $X = \bigcup_{n\in\omega} C_n$, where $\{C_n : n \in \omega\}$ is an increasing sequence of compact sets of X. For each $n \in \omega$, let $U_n = \{\varphi \in C_k(X) : |\varphi(x)| < q + 2^{-n}$ for any $x \in C_n\}$. Let $F = \{\varphi \in C_k(X) : |\varphi(x)| \leq q$ for any $x \in X\}$. Then F is a closed set of $C_k(X)$ and $F = \bigcap_{n\in\omega} U_n$. For each $n \in \omega$, let $\mathcal{H}_n = \{(\bigcup_{i\leq n} K_i) \cup C_n : K_i \in \mathcal{K}_i$ for each $i \leq n\}$. For each $K \in \mathcal{H}_n$, take some $\{K_i : i \leq n\}$ with $K = (\bigcup_{i\leq n} K_i) \cup C_n$ (note that such $\{K_i : i \leq n\}$ is not unique), and define $B(K) = (\bigcap_{i\leq n} B(K_i)) \cap U_n$. Let $\mathcal{K} = \bigcup_{n\in\omega} \mathcal{H}_n$ and $\mathcal{B} = \{B(K) : K \in \mathcal{K}\}$. Then by using Fact 1 and Fact 2, one can easily show that $\mathcal{K}$ and $\mathcal{B}$ satisfies ($*$) for (X, q, r). □

Lemma 2.3. *Let X be a zero-dimensional σ-compact Polish space. Then for any positive real numbers q, r with $0 < q < r$, (X, q, r) satisfies ($*$).*

Proof. We prove this by induction on the C-scattered rank $\alpha = r(X)$.

Case 1. $\alpha = 1$. Then X is a topological sum of countably many compact sets. If X is a compact space, then X trivially satisfies ($*$). Now apply Lemma 2.2.

Case 2. α is a limit ordinal. Then $X = \bigoplus_{n\in\omega} X_n$, where $r(X_n) < \alpha$ for each $n \in \omega$, and we can apply Lemma 2.2.

Case 3. $\alpha = \gamma + 1$ ($\gamma > 0$). By Lemma 2.2, we may assume that $X = (\bigoplus_{n\in\omega} X_n) \cup C$ such that $(\bigoplus_{n\in\omega} X_n) \cap C = \emptyset$, each X_n is a clopen set with $r(X_n) < \alpha$, and $\{(\bigcup_{i\geq n} X_i) \cup C : n \in \omega\}$ is an open neighborhood base of a compact set C.

Take s with $q < s < r$. By our assumption, each X_n has $\mathcal{K}_n$ and $\mathcal{B}_n = \{B_n(K) : K \in \mathcal{K}_n\}$ satisfying ($*$) for (X_n, s, r).

Let $\mathcal{K} = \{(\bigcup_{n\in\omega} K_n) \cup C : K_n \in \mathcal{K}_n$ for each $n \in \omega\}$. For each $K \in \mathcal{K}$, define $K_n = K \cap X_n$, and $B(K) = \bigcap_{n\in\omega} B(K_n) \cap [C, s]$. We show that $\mathcal{K}$ and $\mathcal{B} = \{B(K) : K \in \mathcal{K}\}$ has the desired property ($*$).

Claim.

(1) *Suppose $\varphi_0 \in C_k(X)$, $m \in \omega$, and $\varepsilon > 0$ satisfies $|\varphi_0(x)| < s - \varepsilon$ for any $x \in (\bigcup_{n>m} X_n) \cup C$. Then*

$$B(\varphi_0, K, \varepsilon) \cap B(K) = B(\varphi_0, K, \varepsilon) \cap (\bigcap_{n\leq m} B(K_n)) \cap [C, s])$$

for each $K \in \mathcal{K}$.

(2) *Let $m \in \omega$. Define $T = \{\varphi \in C_k(X) : |\varphi(x)| < s$ for any $x \in (\bigcup_{n>m} X_n) \cup C$. Then $T \cap B(K) = T \cap (\bigcap_{n\leq m} B(K_n)) \cap [C, s]$ for each $K \in \mathcal{K}$.*

Proof of Claim. This follows from the fact that for each $K = (\bigcup_{n\in\omega} K_n) \cup C \in \mathcal{K}$ and for each $n > m$, we have $B(\varphi_0, K, \varepsilon) \cup T \subset [K_n, s] \subset B(K_n)$. □

Now we show that $B(K)$ is an open set of $C_k(X)$ for each $K = (\bigcup_{n\in\omega} K_n) \cup C$, where $K_n \in \mathcal{K}_n$ for each $n \in \omega$. Let $\varphi \in B(K)$. Then $|\varphi(x)| < s$ for any $x \in C$. Hence there are $m \in \omega$ and $\varepsilon > 0$ such that $|\varphi(x)| < s-\varepsilon$ for any $x \in (\bigcup_{n>m} X_n)\cup C$. Then by Claim (1), $B(\varphi, K, \varepsilon) \cap B(K)$ is the intersection of finitely many open sets of $C_k(X)$, hence is an open set of $C_k(X)$.

Next we show that $\mathcal{B}$ is mosaical. Let $S = C_k(X) - [C, s]$, and $S_{k,m} = \{\varphi \in C_k(X) : \varphi(x) \leq s \cdot (1 - \dfrac{1}{2^{k+1}})$ for any $x \in (\bigcup_{i>m} X_i) \cup C\}$. Then S and $S_{k,m}$, $k, m \in \omega$ are closed sets of $C_k(X)$ and $C_k(X) = S \cup (\bigcup_{k,m\in\omega} S_{k,m})$. Note that $S \cap B = \emptyset$ for any $B \in \mathcal{B}$. So by Fact 2, it suffices to show that $\mathcal{B}|S_{k,m}$ is mosaical for each $k, m \in \omega$. Since $\{B(K_n) : K_n \in \mathcal{K}_n\}$ is mosaical for each $n \in \omega$, by Fact 1, we have that $\{\bigcap_{n\leq m} B(K_n) : K_n \in \mathcal{K}_n$ for each $n \leq m\}$ is mosaical. Let T be the set in Claim (2). Then $S_{k,m} \subset T$. Hence $S_{k,m} \cap B(K) = S_{k,m} \cap (\bigcap_{n\leq m} B(K_n)) \cap [C, s])$ for each $K \in \mathcal{K}$. Hence by Fact 1, $\mathcal{B}|S_{k,m}$ is mosaical. □

Proof of Theorem 2.1. Let X be a σ-compact Polish space. Note that X is the perfect image of a zero-dimensional σ-compact Polish space Y, hence $C_k(X) \subset C_k(Y)$. Since it is hereditary to be a μ-space, we may assume that X is zero-dimensional.

By Lemma 2.3, for each pair of rationals q, r with $0 < q < r$, there is a collection $\mathcal{B}_{qr}$ satisfying condition $(*)$. Then

$$\mathcal{B} = \bigcup\{\mathcal{B}_{qr} : 0 < q < r \text{ and } q, r \in \mathbb{Q}\}$$

is σ-mosaical and a neighborhood base for the constant zero function in $C_k(X)$. Since $C_k(X)$ is a separable topological group, there are countably many translations of $\mathcal{B}$ which, taken together, form a σ-mosaical base for $C_k(X)$. Hence $C_k(X)$ is a μ-space. □

As we remarked in the introduction, another possible way to prove spaces are M_1 is to show that they are WAP-spaces. Recall that a space X is said to have the property of *Weak Approximation by Points (WAP)* if for every non-closed set A, there is some $x \in \overline{A} \setminus A$ and a subset B of A such that x is the unique point of $\overline{B}$ not in A. The original motivation for these spaces was in categorical topology where they had another name, and they were introduced into general topology under this name by P. Simon[S]. Sequential spaces are easily seen to be WAP. See [BY] and [TY] for more information about WAP spaces.

Stratifiable WAP spaces were shown to be M_1 in [MSK]. However, we do not not know the answer to the following:

Question. *Let X be a Polish space. Is $C_k(X)$ WAP? What if X is also σ-compact?*

One approach to a negative answer might be to find a non-WAP space which embeds in the function space. This is the idea used to prove that certain function spaces are not sequential; see, e.g., [P] where it is shown, in particular, that $C_k(X)$ is not sequential for any non-locally compact metric space X. To follow this approach for the above question, one would need an appropriate basic stratifiable non-WAP space. Stratifiable non-WAP spaces do exist, but we have been unable to find any mentioned in the literature. So we give here the following simple example.

Example. A countable stratifiable non-WAP space.

The example may be described as the "Alexandroff duplicate of the rationals modulo the nowhere dense sets". Let $Z = \mathbb{Q} \cup \mathbb{Q}'$, where $\mathbb{Q}$ is the set of rational numbers and $\mathbb{Q}'$ is a copy of $\mathbb{Q}$. Points of $\mathbb{Q}'$ are isolated. A neighborhood of a point z in $\mathbb{Q}$ is $U \cup (U' \setminus N')$, where U is a usual open set in the rationals, U' is its copy in $\mathbb{Q}'$, and N' is the copy in $\mathbb{Q}'$ of a nowhere dense subset N of $\mathbb{Q}$.

Suppose U is open in Z and $x \in U$. If x is in $\mathbb{Q}'$, i.e., isolated, let $U_x = \{x\}$. Otherwise choose n such that the usual $\frac{1}{2^n}$ neighborhood of x in $\mathbb{Q}$ is contained in U. Then let $U_x = U \cap (B \cup B')$, where B is the $\frac{1}{2^{n+1}}$ neighborhood of x. It is easy to check that $U_x \cap V_y \neq \emptyset \Rightarrow y \in U$ or $x \in V$. It follows (see, e.g., Theorem 5.19 of [Gr]) that Z is monotonically normal. Z, being countable, has a countable network, and monotonically normal spaces with a σ-discrete (in particular, countable) network are stratifiable (see, e.g., Theorems 5.9 and 5.16 of [Gr]).

But Z is not WAP, since the subset $\mathbb{Q}'$ is non-closed, yet there is no point $x \in \overline{\mathbb{Q}'} \setminus \mathbb{Q}'$ which satisfies the condition in the definition of WAP.

3. A σ-closure-preserving base

In this section, we construct an explicit σ-closure-preserving base for $C_k(X)$ for X σ-compact Polish. To do this, we first need to show that there is a dominating, closure-preserving collection of compact subsets of X that satisfies a certain convergence condition.

Lemma 3.1. *If X is σ-compact Polish, then there is a dominating, closure-preserving collection $\mathcal{K}$ of compact sets that satisfies the following condition:*

(∗∗) *Whenever $x_n \in K_n \in \mathcal{K}$, and $x_n \notin \bigcup_{j \neq n} K_j$, then the set $\{x_n\}_{n\in\omega}$ has a limit point.*

Proof. The proof is by induction on the C-scattered rank.

Case 1. X is locally compact. Write X as an increasing (not necessarily strictly) union of compact sets L_n, $n \in \omega$, where L_n is contained in the interior of L_{n+1}. Then $\{L_n\}_{n\in\omega}$ satisfies the desired conditions.

Case 2. X has a locally-finite cover $\mathcal{V}$ by closed sets satisfying Lemma 3.1. Let $\mathcal{V} = \{V_n : n \in \omega\}$. Let $\mathcal{H}_n$ be a collection of compact subsets of V_n satisfying the conclusion of Lemma 3.1 for V_n. Let $\{L_{nm}\}_{m\in\omega}$ be an increasing collection of compact subsets of V_n whose union is V_n. Then let

$$\mathcal{K}_n = \{\bigcup_{i \leq n} H_i \cup L_{in} : H_i \in \mathcal{H}_i\},$$

and let $\mathcal{K} = \bigcup_{n\in\omega} \mathcal{K}_n$.

That $\mathcal{K}$ is dominating follows from the fact that each compact subset of X is contained in some finite union of members of $\mathcal{V}$. $\mathcal{K}$ is closure-preserving because its trace on each member of the locally finite closed cover $\mathcal{V}$ is closure-preserving. To see the convergence property (∗∗), suppose $x_n \in K_n \in \mathcal{K}$, and $x_n \notin \bigcup_{j \neq n} K_j$. Then $x_0 \in L_{km}$ for some k, m, and hence for $j > 0$, if $K_j \in \mathcal{K}_{n(j)}$, then $n(j) \leq max\{k, m\}$. It now easily follows that $\mathcal{K}$ has the convergence property because each $\mathcal{H}_n$ does.

Now suppose the C-scattered rank of X is α, and the Lemma holds for all X of smaller rank. If α is a limit ordinal, then the Lemma holds for X by Case 2. So it remains to prove:

Case 3. The C-scattered rank of X is a successor ordinal $\gamma + 1$ (where $\gamma \geq 1$) .

It follows from Case 2 that it is sufficient to prove this when the set C of points of rank γ is compact, and there is a strongly decreasing sequence U_n of neighborhoods of C (i.e., $\overline{U_{n+1}} \subset U_n$ for all n) forming a base in X for the set C. Assume $U_0 = X$, and let $V_n = \overline{U_n} \setminus U_{n+1}$. Let $\mathcal{K}_n$ be a collection of compact subsets of V_n satisfying the conclusion of Lemma 3.1 for V_n, and let

$$\mathcal{K} = \{C \cup (\cup \{K_i\}_{i\in\omega}) : K_i \in \mathcal{K}_i\}.$$

The verification that $\mathcal{K}$ satisfies the desired conditions is a little different from Case 2, but just as easy, so we leave it to the reader. □

Next, we need the following known fact which essentially follows from Nagata and Siwiec's [NS] analysis of σ-spaces. However, we only need it for spaces having a countable network, so we give here the simple proof of that case.

Lemma 3.2. *Suppose X is a regular space with a countable network, and $\mathcal{K}$ is a closure-preserving collection of closed sets. Then there is a countable cover $\mathcal{Q}$ of X by closed sets such that, for each $Q \in \mathcal{Q}$ and $K \in \mathcal{K}$, either $Q \subset K$ or $Q \cap K = \emptyset$. Furthermore, if X is σ-compact, then $\mathcal{Q}$ may consist of compact sets.*

Proof. Let $\mathcal{N}$ be a countable network for X. For each $x \in X$, let $\mathcal{K}_x = \{K \in \mathcal{K} : x \in K\}$. For $\mathcal{C} \subset \mathcal{K}$, let $E_\mathcal{C}$ be as defined in Section 1. Note that $x \in E_\mathcal{C} \iff \mathcal{C} = \mathcal{K}_x$. Given $x \in X$, pick $N_x \in \mathcal{N}$ such that $N_x \cap \bigcup(\mathcal{K} \setminus \mathcal{K}_x) = \emptyset$. It is easy to check that $N_x = N_y \Rightarrow \mathcal{K}_x = \mathcal{K}_y$. Thus the collection $\{\mathcal{K}_x : x \in X\}$ is countable, and hence $\{E_\mathcal{C} : \mathcal{C} \subset \mathcal{K}\}$ is also countable. Since each $E_\mathcal{C}$ is the difference of two closed sets in a perfect space, it can be written as the union of a countable collection $\{Q_{\mathcal{C},n} : n \in \omega\}$ of closed sets (compact sets if X is σ-compact). Then $\mathcal{Q} = \{Q_{\mathcal{C},n} : \mathcal{C} \subset \mathcal{K}, n \in \omega\}$ is the desired collection. □

Now we show how to use the collection $\mathcal{K}$ of Lemma 3.1 and the corresponding $\mathcal{Q} = \{Q_n : n \in \omega\}$ of Lemma 3.2 to construct a σ-closure-preserving base for the constant zero function $\mathbf{0}$ of $C_k(X)$. Translating this base at $\mathbf{0}$ by members of a countable dense subset of $C_k(X)$ gives a σ-closure-preserving base for the whole space [GR].

Define $P_n = \bigcup_{i \leq n} Q_i$. Then $\{P_n : n \in \omega\}$ is an increasing collection of compact sets whose union is X which satisfies the following property:

$(***)$ For any $n \in \omega$ and $K \in \mathcal{K}$, $P_n \setminus P_{n-1} \subset K$ or $(P_n \setminus P_{n-1}) \cap K = \emptyset$.

Let q be any positive rational, and let $q_n = (1 - 1/2^{n+1})q$. For each $K \in \mathcal{K}$, define

$$B_q(K) = \{f \in C(X) : \forall n \forall x \in K \cap P_n (|f(x)| < q_n)\}.$$

It is easy to check that $B_q(K)$ is open in $C_k(X)$, and as q varies over the rationals and K over $\mathcal{K}$, the $B_q(K)$'s form a local base at $\mathbf{0}$. So to finish our construction, it will suffice to show the following:

Proposition. *Fix $q > 0$. Let $\mathcal{K}$, $\{P_n\}_{n\in\omega}$, and $B_q(K)$ be as above. Then the collection $\{B_q(K) : K \in \mathcal{K}\}$ is closure-preserving.*

Proof. Suppose not. Then there is a subcollection $\mathcal{K}_0$ of $\mathcal{K}$ and a function $\theta \in C(X)$ such that $\theta \in \overline{\bigcup_{K\in\mathcal{K}_0} B_q(K)} \setminus \bigcup_{K\in\mathcal{K}_0} \overline{B_q(K)}$.

Call a point $x \in X$ a *bad* point if $x \in P_n$ but $\theta(x) > q_n$. It is easy to show:

(i) If θ has no bad points on K, then $\theta \in \overline{B_q(K)}$.
(ii) If x is a bad point of θ, then for sufficiently small $\epsilon > 0$, $B(\theta, \{x\}, \varepsilon) \cap B_q(K) = \emptyset$ whenever $x \in K$.

So θ has a bad point in every $K \in \mathcal{K}_0$. Let n_0 be least such that there is a bad point $x_0 \in P_{n_0}$ which is in some $K_0 \in \mathcal{K}_0$. By (ii), if ε_0 is sufficiently small and $K \in \mathcal{K}_0$, then $B(\theta, \{x_0\}, \varepsilon_0) \cap B_q(K) \neq \emptyset$ implies $x_0 \notin K$, which implies $K \cap (P_{n_0} \backslash P_{n_0 - 1}) = \emptyset$, hence K contains no bad points in P_{n_0}.

Thus $\mathcal{K}_1 = \{K \in \mathcal{K}_0 : x_0 \notin K\}$ is a *large* subcollection of $\mathcal{K}_0$ in the sense that the $B_q(K)$'s for $K \in \mathcal{K}_1$, together with θ, also witness non-closure-preserving. Let n_1 be least such that there is a bad point $x_1 \in P_{n_1}$ which is in some $K_1 \in \mathcal{K}_1$. Note that $n_1 > n_0$. $\mathcal{K}_2 = \{K \in \mathcal{K}_1 : x_1 \notin K\}$ is also large, and so on.

Thus we can inductively define $x_i \in K_i \in \mathcal{K}_i$, where x_i is in $P_{n_i} \setminus P_{n_i - 1}$ and is a bad point of θ, $n_0 < n_1 < ...$, and K_i contains no bad points of θ in $P_{n_i - 1}$. In particular this implies $x_i \notin K_j$ if $i < j$. We claim that the x_i's are discrete. Suppose they had a limit point y. Say $y \in P_k$. Then y is a bad point of θ (note $\theta(y) \geq q$). For sufficiently large j, $n_j > k$, which implies $y \notin K_j$. Then by closure-preserving, the set $\bigcup\{K_j : n_j > k\}$ is closed, contains all but finitely many x_i's, and misses y, contradiction.

Now, since $\{x_i\}_{i \in \omega}$ is discrete, we can pass to an infinite subset A of ω such that, for $i \neq j \in A$, we have $x_i \notin K_j$. Then by the convergence property $(**)$ of Lemma 3.1 the x_i's must have a limit point, contradiction. □

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