BASE-PARACOMPACTNESS AND BASE-NORMALITY OF GO-SPACES

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Abstract. We answer a question of J.E. Porter by proving that every paracompact GO-space is base-paracompact, and we answer questions of of K. Yamazaki by providing an example of a countably compact LOTS which is not base-normal. This example is also the first known ZFC example of a normal space which is not base-normal.

1. Introduction

J.E. (Ted) Porter[P] calls a space X *base-paracompact* if there is a base $\mathcal{B}$ for X such that $|\mathcal{B}| = w(X)$, and every open cover of X has a locally finite refinement by members of $\mathcal{B}$. K. Yamazaki[Y_1] defined X to be *base-normal* if there is a base $\mathcal{B}$ for X such that $|\mathcal{B}| = w(X)$, and any two-element open cover of X has a locally finite refinement by members of $\overline{\mathcal{B}} = \{\overline{B} : B \in \mathcal{B}\}$.

As noted by Yamazaki, it is easy to see that base-normal spaces are normal, and that for normal spaces, X is base-normal iff there is a base $\mathcal{B}$ for X such that $|\mathcal{B}| = w(X)$, and any two-element open cover of X has a locally finite refinement by members of $\mathcal{B}$. Yamazaki also observed that a paracompact space is base-paracompact iff it is base-normal.

Recall that a space X is a *linearly ordered topological space (LOTS)* if there is a linear order on X such that the collection of open intervals with endpoints in X (or $\pm\infty$) is a base for X. X is a *generalize ordered(GO)-space* if X is homeomorphic to a subspace of a LOTS.

Porter showed that every paracompact GO-space of weight $\aleph_1$ is base-paracompact, and asked if the weight restriction is necessary. Yamazaki[Y_2] asked if GO-spaces are base-normal, and she asked if countably compact normal spaces are base-normal. In this note, we answer these questions by (1) proving that every paracompact GO-space is base-paracompact, and (2) providing an example (in ZFC) of a countably compact LOTS which is not base-normal.

It remains an unsolved problem of Porter whether or not every paracompact space is base-paracompact. In [Y_1], it is noted that some well-known consistent examples of separable normal non-metrizable Moore spaces fail to be base-normal. Our LOTS example provides a ZFC example of a normal space which is not base-normal, answering another question of Yamazaki[Y_1][Y_2].

2. A non-base-normal LOTS

For each ordinal number α, the cardinal $\beth_\alpha$ is defined as follows (see, e.g., [K]):

(i) $\beth_0 = \aleph_0$;

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(ii) $\beth_{\alpha+1} = 2^{\beth_\alpha}$;
(iii) For a limit ordinal α, $\beth_\alpha = sup\{\beth_\beta : \beta < \alpha\}$.

Example 2.1. *Let $\kappa = \beth_{\omega_1}$, and give 2^κ the lexicographic order topology. Let X be the subspace of 2^κ consisting of all "eventually constant" points (i.e., points $\vec{x} = (x_\alpha)_{\alpha<\kappa} \in 2^\kappa$ such that, for some $\delta < \kappa$ and $j = 0, 1$, $x_\beta = j$ for all $\beta > \delta$). Then X is a countably compact LOTS which is not base-normal.*

Proof. Since X is dense in 2^κ with the lexicographic ordering, it is easy to see that X with the restricted ordering is a LOTS.

To see that X is countably compact, suppose $A = \{a_n\}_{n\in\omega}$ is a countably infinite subset of X. Since 2^κ is compact, there is a limit point p of A in 2^κ. We aim to show that $p \in X$. We may assume that either $a_n > p$ for every $n \in \omega$, or $a_n < p$ for every $n \in \omega$. We assume the former, the latter case being analogous. Let α_n be the least coordinate where a_n and p differ, and let $sup\{\alpha_n\}_{n\in\omega} < \alpha < \kappa$. We claim that p is constant 1 at α and beyond, hence in X. Suppose on the other hand that $p(\delta) = 0$ for some $\delta \geq \alpha$. Then if q agrees with p below δ, and $q(\delta) = 1$, no point of A is in the interval between p and q, contradiction. Hence X is countably compact.

From the definition of $\beth_{\omega_1}$, it is easy to see that for any cardinal $\lambda < \kappa$, we have $2^\lambda < \kappa$. Thus $|X| = sup\{2^\lambda : \lambda < \kappa\} = \kappa$, and it is easy to see that X also has weight κ. Suppose $\mathcal{B}$ is a base of X of cardinality κ. We will prove that $\mathcal{B}$ cannot witness base-normality of X.

Let $S = \{sup(B) : B \in \mathcal{B}\}$ (where $sup(B)$ is taken in 2^κ). Since $2^\kappa > \kappa$, there is a point

$$p \in 2^\kappa \setminus (S \cup X).$$

We aim to show that there is no locally finite refinement by members of $\mathcal{B}$ of the two-element clopen partition $\{(-\infty, p) \cap X, (p, \infty) \cap X\}$ of X.

To this end, for each $\alpha < \omega_1$, let p_α be the point (in X) whose coordinates agree with p below $\beth_\alpha$, and then are constant 0. Let $P = \{p_\alpha : \alpha < \omega_1\}$. Note that for any $B \in \mathcal{B}$ with $B \subset (-\infty, p) \cap X$, we have $|B \cap P| \leq \omega$ (for otherwise, $p = sup(B) \in S$). A standard application of the pressing down lemma (see, e.g., Lemma II.6.15 of [K]) shows that there is no locally finite cover of the space ω_1 of countable ordinals by sets bounded in ω_1. Thus we are done once we prove the following claim.

Claim. P is homeomorphic to ω_1.

To see this, define $\theta : \omega_1 \to P$ by $\theta(\alpha) = p_\alpha$. It is easy to check that θ is continuous and non-decreasing. Hence the fibers of θ are countable closed intervals. Then if we let $C = \{min\{\theta^{-1}(p_\alpha)\} : \alpha < \omega_1\}$, C is a closed unbounded subset of ω_1, hence homeomorphic to ω_1, and $\theta \restriction C$ is a homeomorphism from C to P. This completes the proof. $\square$

Remark. No cover of the space of countable ordinals by bounded open sets is point-countable. In particular, if in the definition of base-normal, "locally finite" were weakened to "σ-locally finite", the above example would still not satisfy the property. Yamazaki has asked if these two reasonable notions of base-normality are equivalent; this is still unsettled.

3. Paracompact GO-spaces

The following theorem answers a question of Porter[P], who proved it for spaces of weight $\leq \aleph_1$.

Theorem 3.1. *Every paracompact GO-space is base-paracompact.*

It will be convenient to first show:

Proposition 3.2. *Let X be a paracompact GO-space which is a subspace of a compact LOTS $\hat{X}$. Then X base-paracompact iff X is base-paracompact with respect to all two-element open covers of the form $\{(-\infty, g) \cap X, (g, \infty) \cap X\}$, where $g \in \hat{X} \setminus X$.*

Proof. Suppose X is as stated, and has a base $\mathcal{B}$ with $|\mathcal{B}| = w(X)$ such that, for any $g \in \hat{X} \setminus X$, there is a locally finite subcollection of $\mathcal{B}$ covering X and refining $\{(-\infty, g), (g, \infty)\}$.

Let D be a dense subset of X which includes all points x such that either $(-\infty, x]$ or $[x, \infty)$ is open. Note that $|D| \leq w(X)$. Let $\mathcal{C}$ be the collection of all convex open subsets of X with endpoints in D, and let $\mathcal{B}^*$ be smallest collection containing $\mathcal{B} \cup \mathcal{C}$ which is closed under finite intersections. Then $|\mathcal{B}^*| = w(X)$. We will prove that any open cover of X has a locally finite refinement by members of $\mathcal{B}^*$.

To this end, let $\mathcal{U}$ be an open cover of X; since X is paracompact, we may assume $\mathcal{U}$ is locally finite. We may also assume each $U \in \mathcal{U}$ is convex. To see this, consider a closed shrinking $\{F(U) : U \in \mathcal{U}\}$ of $\mathcal{U}$. Let $\mathcal{W}(U)$ be the collection of convex components of U that $F(U)$ meets. It is easy to check that $\mathcal{W}(U)$ is locally finite. Then $\bigcup\{\mathcal{W}(U) : U \in \mathcal{U}\}$ is a locally finite cover of X by convex components of members of $\mathcal{U}$.

So it remains to show that a locally finite cover $\mathcal{U}$ by convex open sets has a locally finite refinement by members of $\mathcal{B}^*$. Let $\{H(U) : U \in \mathcal{U}\}$ be a closed shrinking of $\mathcal{U}$. For each $U \in \mathcal{U}$, let $l_U = inf(H(U))$ and $l'_U = inf(U)$ (where the inf's are taken in $\hat{X}$); obviously, $l'_U \leq l_U$. Define $r_U \leq r'_U$ analogously.

Claim. There is a locally finite subcollection $\mathcal{B}^*(U)$ of $\mathcal{B}^*$ such that $H(U) \subset \cup\mathcal{B}^*(U) \subset U$.

First note that if we prove the claim, we are done, for then $\bigcup\{\mathcal{B}^*(U) : U \in \mathcal{U}\}$ is a locally finite refinement of $\mathcal{U}$ by members of $\mathcal{B}^*$.

We proceed to prove the claim. If $(l'_U, l_U) \cap X \neq \emptyset$, then there is some $d_U \in D$ with $l'_U < d_U < l_U$. In this case, let $\mathcal{B}^*_l(U) = \{(d_U, \infty)\}$. If $(l'_U, l_U) \cap X = \emptyset$, and $l_U \in X$, then $l_U \in D$; in this case, let $\mathcal{B}^*_l(U) = \{[d_U, \infty)\}$. Finally, if $(l'_U, l_U) \cap X = \emptyset$, and $l_U \notin X$, then let $\mathcal{B}^*_l(U)$ be a locally finite cover of $(l_U, \infty) \cap X$ by members of $\mathcal{B}$. In all cases, $\mathcal{B}^*_l(U)$ is a locally finite subcollection of $\mathcal{B}^*$ covering $H(U)$ and contained in (l'_U, ∞). Define $\mathcal{B}^*_r(U)$ analogously, and let

$$\mathcal{B}^*(U) = \{B_l \cap B_r : B_l \in \mathcal{B}^*_l(U), B_r \in \mathcal{B}^*_r(U)\}.$$

It is easy to see that this $\mathcal{B}^*(U)$ satisfies the desired conditions. $\square$

Proof of Theorem 3.1. Let X be a paracompact GO-space. Then X may be viewed as a dense subspace of a compact LOTS $\hat{X}$ [L]. We may also assume that if $a, b \in \hat{X}$ and b is the immediate successor (or predecessor) of a, then either a or b is in X (since we can modify $\hat{X}$ if necessary by collapsing to a point any such pair where neither a nor b is in X). Then it is easy to see that $w(\hat{X}) = w(X)$ for such $\hat{X}$.

Let $\kappa = w(X)$. Then there is a subset D of $\hat{X}$ of cardinality κ such that the collection of all open intervals with endpoints in $D \cup \pm\infty$ is a base for $\hat{X}$. Note that this implies that if $a, b \in \hat{X}$ and b is the immediate successor of a, then $a, b \in D$. Let $D = \{d_\alpha : \alpha < \kappa\}$.

For $\alpha < \kappa$, inductively construct collections $\mathcal{I}_\alpha$ of closed intervals of $\hat{X}$ with disjoint interiors satisfying the following conditions:

(i) $\mathcal{I}_0 = \{\hat{X}\}$.
(ii) Given $\mathcal{I}_\alpha$, for each non-degenerate $I \in \mathcal{I}_\alpha$, let I_0, I_1 be two closed subintervals of I with disjoint interiors whose union is I, with the following stipulation: $(*)$ If d_α is in I, then d_α is an endpoint of I_0 or I_1.
Let $\mathcal{I}_{\alpha+1} = \{I_j : I \in \mathcal{I}_\alpha, j \in \{0,1\}\}$.
(iii) Suppose $\mathcal{I}_\beta$ has been defined for all $\beta < \alpha$, α a limit ordinal. For each $x \in \hat{X} \cap [\bigcap_{\beta<\alpha}(\cup\mathcal{I}_\beta)]$ such that x is not an endpoint of any member of $\bigcup_{\beta<\alpha}\mathcal{I}_\beta$, choose the unique member $I^x_\beta = [l^x_\beta, r^x_\beta] \in \mathcal{I}_\beta$ with $x \in I^x_\beta$. Let $r^x_\alpha = inf\{r^x_\beta : \beta < \alpha\}$ and define l^x_α analogously. Then let $\mathcal{I}_\alpha$ be the collection of all intervals of the form $[l^x_\alpha, r^x_\alpha]$ such that $l^x_\alpha \neq r^x_\alpha$.

Let $\mathcal{I} = \bigcup_{\alpha<\kappa}\mathcal{I}_\alpha$, and let E be the set of all endpoints of members of $\mathcal{I}$. Note that $|\mathcal{I}_\alpha| \leq \kappa$ for each α. For limit α, this follows immediately from $w(X) \leq \kappa$ and the fact that all $I \in \mathcal{I}_\alpha$ are non-degenerate. For successor α, it follows from the limit case and that each $I \in \mathcal{I}_\beta$ is split into at most two pieces in the construction of $\mathcal{I}_{\beta+1}$. Hence $|\mathcal{I}| = \kappa = |E|$.

Let $\mathcal{B}$ be the collection of all convex open sets in X formed by intersecting with X an open interval in $\hat{X}$ with endpoints in $E \cup \{\pm\infty\}$. Then $|\mathcal{B}| = \kappa$. We will show that, for any $g \in \hat{X} \setminus X$, the two-element open cover $\{(-\infty, g) \cap X, (g, \infty) \cap X\}$ of X has a locally finite refinement by members of $\mathcal{B}$. By Proposition 3.2, this will complete the proof.

If $g \in E$, then $(-\infty, g) \cap X$ and $(g, \infty) \cap X$ are already in $\mathcal{B}$, so we may assume $g \notin E$. Let $\delta = sup\{\alpha < \kappa : \exists I \in \mathcal{I}_\alpha(g \in I)\}$. For $\alpha < \delta$, let $I_\alpha = [l_\alpha, r_\alpha]$ be the (unique) member of $\mathcal{I}_\alpha$ containing g. Then $l_\alpha < g < r_\alpha$, and it follows from the construction that δ is a limit ordinal, and that the map $\alpha \mapsto l_\alpha$ is continuous and non-decreasing, $\alpha \mapsto r_\alpha$ is continuous and non-increasing, and $sup\{l_\alpha\}_{\alpha<\delta} = g = inf\{r_\alpha\}_{\alpha<\delta}$. In the case $\delta = \kappa$, the last claim follows from the use of D in the construction. If $\bigcap_{\alpha<\kappa}[l_\alpha, r_\alpha] = [l, r]$ were non-degenerate, then either $(l, r) \neq \emptyset$ and so there would be some $d_\gamma \in D$ with $l < d_\gamma < r$, or both l and r would be in D. In any case, $(*)$ of condition (ii) of the construction would be violated.

Case 1. $cof(\delta) = \omega$. In this case, let $\{\gamma_n\}_{n\in\omega}$ be a sequence cofinal in δ such that $l_{\gamma_0} < l_{\gamma_1} < \dots$. Then $\{(-\infty, l_{\gamma_1}) \cap X, (l_{\gamma_0}, l_{\gamma_2}) \cap X, (l_{\gamma_1}, l_{\gamma_3}) \cap X, \dots\}$ is a locally finite subcollection of $\mathcal{B}$ contained in and covering $(-\infty, g) \cap X$. Similarly there is another such subcollection contained in and covering $(g, \infty) \cap X$. This completes the proof in Case 1.

Case 2. $cof(\delta) > \omega$. Let $\nu = cof(\delta)$. For each $\alpha < \nu$, define γ_α to be the least $\gamma < \delta$ such that $\gamma \geq sup\{\gamma_\beta : \beta < \alpha\}$ and $l_\gamma \notin \{l_{\gamma_\beta} : \beta < \alpha\}$. Then the map $\alpha \mapsto l_{\gamma_\alpha}$ is a continuous embedding of ν into $\{l_\beta : \beta < \delta\}$, and the image of this embedding is closed and cofinal in $(-\infty, g) \cap \hat{X}$. Since X is paracompact, X does not contain a closed subset homeomorphic to a stationary subset of the regular uncountable cardinal ν. Hence there is a club subset C of ν such that $l_{\gamma_\alpha} \notin X$ for every $\alpha \in C$. For each $c \in C$, let c^+ be the successor of c in C. Then $\{(-\infty, l_{min(C)}) \cap X\} \cup \{(l_c, l_{c^+}) \cap X : c \in C\}$ is a clopen partition of $(-\infty, g) \cap X$ by members of $\mathcal{B}$. There is a similar clopen partition of $(g, \infty) \cap X$. It follows that $\mathcal{B}$ witnesses base-paracompactness of X with respect to such two-element covers, and that completes our proof. $\square$

References

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