BASE-PARACOMPACTNESS AND BASE-NORMALITY OF GO-SPACES

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Abstract. We answer a question of J.E. Porter by proving that every paracompact GO-space is base-paracompact, and we answer questions of K. Yamazaki by providing an example of a countably compact LOTS which is not base-normal. This example is also the first known ZFC example of a normal space which is not base-normal.

1. Introduction

J.E. (Ted) Porter[1] calls a space $X$ base-paracompact if there is a base $B$ for $X$ such that $|B| = w(X)$, and every open cover of $X$ has a locally finite refinement by members of $B$. K. Yamazaki[1, 2] defined $X$ to be base-normal if there is a base $B$ for $X$ such that $|B| = w(X)$, and any two-element open cover of $X$ has a locally finite refinement by members of $B$. Yamazaki also observed that a paracompact space is base-paracompact iff it is base-normal.

As noted by Yamazaki, it is easy to see that base-normal spaces are normal, and that for normal spaces, $X$ is base-normal iff there is a base $B$ for $X$ such that $|B| = w(X)$, and any two-element open cover of $X$ has a locally finite refinement by members of $B$. Yamazaki also observed that a paracompact space is base-paracompact iff it is base-normal.

Recall that a space $X$ is a linearly ordered topological space (LOTS) if there is a linear order on $X$ such that the collection of open intervals with endpoints in $X$ (or $\pm \infty$) is a base for $X$. $X$ is a generalize ordered (GO) space if $X$ is homeomorphic to a subspace of a LOTS.

Porter showed that every paracompact GO-space of weight $\aleph_1$ is base-paracompact, and asked if the weight restriction is necessary. Yamazaki[2] asked if GO-spaces are base-normal, and she asked if countably compact normal spaces are base-normal. In this note, we answer these questions by (1) proving that every paracompact GO-space is base-paracompact, and (2) providing an example (in ZFC) of a countably compact LOTS which is not base-normal.

It remains an unsolved problem of Porter whether or not every paracompact space is base-paracompact. In [1], it is noted that some well-known consistent examples of separable normal non-metrizable Moore spaces fail to be base-normal. Our LOTS example provides a ZFC example of a normal space which is not base-normal, answering another question of Yamazaki[1, 2].

2. A non-base-normal LOTS

For each ordinal number $\alpha$, the cardinal $\beth_\alpha$ is defined as follows (see, e.g., [K]):

(i) $\beth_0 = \aleph_0$;
(ii) \( \bigwedge_{\alpha+1} = 2^{\bigwedge_\alpha} \);  
(iii) For a limit ordinal \( \alpha \), \( \bigwedge_\alpha = \sup \{ \bigwedge_\beta : \beta < \alpha \} \).

**Example 2.1.** Let \( \kappa = \bigwedge_{\omega_1} \) and give \( 2^\kappa \) the lexicographic order topology. Let \( X \) be the subspace of \( 2^\kappa \) consisting of all “eventually constant” points (i.e., points \( \bar{x} = (x_\alpha)_{\alpha < \kappa} \in 2^\kappa \) such that, for some \( \delta < \kappa \) and \( j = 0, 1 \), \( x_\beta = j \) for all \( \beta > \delta \)). Then \( X \) is a countably compact LOTS which is not base-normal.

**Proof.** Since \( X \) is dense in \( 2^\kappa \) with the lexicographic ordering, it is easy to see that \( X \) with the restricted ordering is a LOTS.

To see that \( X \) is countably compact, suppose \( A = \{ a_\alpha \}_{\alpha \in \omega} \) is a countably infinite subset of \( X \). Since \( 2^\kappa \) is compact, there is a limit point \( p \) of \( A \) in \( 2^\kappa \). We aim to show that \( p \in X \). We may assume that either \( a_\alpha > p \) for every \( \alpha \in \omega \), or \( a_\alpha < p \) for every \( \alpha \in \omega \). We assume the former, the latter case being analogous. Let \( \alpha_n \) be the least coordinate where \( a_\alpha \) and \( p \) differ, and let \( \sup \{ a_\alpha \}_{\alpha < \kappa} < \alpha < \kappa \). We claim that \( p \) is constant 1 at \( \alpha \) and beyond, hence in \( X \). Suppose on the other hand that \( p(\delta) = 0 \) for some \( \delta \geq \alpha \). Then if \( q \) agrees with \( p \) below \( \delta \), and \( q(\delta) = 1 \), no point of \( A \) is in the interval between \( p \) and \( q \), contradiction. Hence \( X \) is countably compact.

From the definition of \( \bigwedge_{\omega_1} \), it is easy to see that for any cardinal \( \lambda < \kappa \), we have \( 2^\lambda < \kappa \). Thus \( |X| = \sup \{ 2^\lambda : \lambda < \kappa \} = \kappa \), and it is easy to see that \( X \) also has weight \( \kappa \). Suppose \( B \) is a base of \( X \) of cardinality \( \kappa \). We will prove that \( B \) cannot witness base-normality of \( X \).

Let \( S = \{ \sup(B) : B \in B \} \) (where \( \sup(B) \) is taken in \( 2^\kappa \)). Since \( 2^\kappa > \kappa \), there is a point \( p \in 2^\kappa \setminus (S \cup X) \).

We aim to show that there is no locally finite refinement by members of \( B \) of the two-element clopen partition \( \{ (-\infty, p) \cap X, (p, \infty) \cap X \} \) of \( X \).

To this end, for each \( \alpha < \omega_1 \), let \( p_\alpha \) be the point (in \( X \)) whose coordinates agree with \( p \) below \( \bigwedge_\alpha \), and then are constant 0. Let \( P = \{ p_\alpha : \alpha < \omega_1 \} \). Note that for any \( B \in B \) with \( B \subset (-\infty, p) \cap X \), we have \( |B \cap P| \leq \omega \) (for otherwise, \( p = \sup(B) \in S \)). A standard application of the pressing down lemma (see, e.g., Lemma II.6.15 of [K]) shows that there is no locally finite cover of the space \( \omega_1 \) of countable ordinals by sets bounded in \( \omega_1 \). Thus we are done once we prove the following claim.

**Claim.** \( P \) is homeomorphic to \( \omega_1 \).

To see this, define \( \theta : \omega_1 \to P \) by \( \theta(\alpha) = p_\alpha \). It is easy to check that \( \theta \) is continuous and non-decreasing. Hence the fibers of \( \theta \) are countable closed intervals. Then if we let \( C = \{ \min \{ \theta^{-1}(p_\alpha) \} : \alpha < \omega_1 \} \), \( C \) is a closed unbounded subset of \( \omega_1 \), hence homeomorphic to \( \omega_1 \), and \( \theta \mid C \) is a homeomorphism from \( C \) to \( P \). This completes the proof. \( \square \)

**Remark.** No cover of the space of countable ordinals by bounded open sets is point-countable. In particular, if in the definition of base-normal, “locally finite” were weakened to “\( \sigma \)-locally finite”, the above example would still not satisfy the property. Yamazaki has asked if these two reasonable notions of base-normality are equivalent; this is still unsettled.

3. **Paracompact GO-spaces**

The following theorem answers a question of Porter [P], who proved it for spaces of weight \( \leq \aleph_1 \).
Theorem 3.1. Every paracompact GO-space is base-paracompact.

It will be convenient to first show:

Proposition 3.2. Let $X$ be a paracompact GO-space which is a subspace of a compact LOTS $\hat{X}$. Then $X$ base-paracompact iff $X$ is base-paracompact with respect to all two-element open covers of the form $\{(-\infty, g) \cap \hat{X}, (g, \infty) \cap \hat{X}\}$, where $g \in \hat{X} \setminus X$.

Proof. Suppose $X$ is as stated, and has a base $\mathcal{B}$ with $|\mathcal{B}| = w(X)$ such that, for any $g \in \hat{X} \setminus X$, there is a locally finite subcollection of $\mathcal{B}$ covering $X$ and refining $\{(-\infty, g), (g, \infty)\}$.

Let $D$ be a dense subset of $X$ which includes all points $x$ such that either $(-\infty, x]$ or $[x, \infty)$ is open. Note that $|D| \leq w(X)$. Let $C$ be the collection of all convex open subsets of $X$ with endpoints in $D$, and let $\mathcal{B}^*$ be smallest collection containing $\mathcal{B} \cup C$ which is closed under finite intersections. Then $|\mathcal{B}^*| = w(X)$. We will prove that any open cover of $X$ has a locally finite refinement by members of $\mathcal{B}^*$.

To this end, let $U$ be an open cover of $X$; since $X$ is paracompact, we may assume $U$ is locally finite. We may also assume each $U \in U$ is convex. To see this, consider a closed shrinking $\{F(U) : U \in U\}$ of $U$. Let $W(U)$ be the collection of convex components of $U$ that $F(U)$ meets. It is easy to check that $W(U)$ is locally finite. Then $\bigcup \{W(U) : U \in U\}$ is a locally finite cover of $X$.

So it remains to show that a locally finite cover $\mathcal{U}$ by convex open sets has a locally finite refinement by members of $\mathcal{B}^*$. Let $\{H(U) : U \in U\}$ be a closed shrinking of $\mathcal{U}$. For each $U \in U$, let $l_U = \inf(H(U))$ and $r_U = \inf(U)$ (where the inf's are taken in $\hat{X}$); obviously, $l_U \leq l_U$. Define $r_U \leq r_U$ analogously.

Claim. There is a locally finite subcollection $\mathcal{B}^*(U)$ of $\mathcal{B}^*$ such that $H(U) \subset \bigcup \mathcal{B}^*(U) \subset U$.

First note that if we prove the claim, we are done, for then $\bigcup \{\mathcal{B}^*(U) : U \in U\}$ is a locally finite refinement of $\mathcal{U}$ by members of $\mathcal{B}^*$.

We proceed to prove the claim. If $(l_U, l_U) \cap X \neq \emptyset$, then there is some $d_U \in D$ with $l_U < d_U < l_U$. In this case, let $\mathcal{B}^+_U(U) = \{(d_U, \infty)\}$. If $(l_U, l_U) \cap X = \emptyset$, and $l_U \in X$, then let $r_U \in D$; in this case, let $\mathcal{B}^+_U(U) = \{l_U, \infty\}$. Finally, if $(l_U, l_U) \cap X = \emptyset$, and $l_U \not\in X$, then let $\mathcal{B}^+_U(U)$ be a locally finite cover of $(l_U, \infty) \cap X$ by members of $\mathcal{B}$. In all cases, $\mathcal{B}^+_U(U)$ is a locally finite subcollection of $\mathcal{B}^*$ covering $H(U)$ and contained in $(l_U, \infty)$. Define $\mathcal{B}^+_U(U)$ analogously, and let

$$\mathcal{B}^*(U) = \{B_t \cap B_r : B_t \in \mathcal{B}^+_U(U), B_r \in \mathcal{B}^+_U(U)\}.$$  

It is easy to see that $\mathcal{B}^*(U)$ satisfies the desired conditions. □

Proof of Theorem 3.1. Let $X$ be a paracompact GO-space. Then $X$ may be viewed as a dense subspace of a compact LOTS $\hat{X}$ [L]. We may also assume that if $a, b \in \hat{X}$ and $b$ is the immediate successor (or predecessor) of $a$, then either $a$ or $b$ is in $X$ (since we can modify $\hat{X}$ if necessary by collapsing to a point any such pair where neither $a$ nor $b$ is in $X$). Then it is easy to see that $w(\hat{X}) = w(X)$ for such $\hat{X}$.

Let $\kappa = w(X)$. Then there is a subset $D$ of $\hat{X}$ of cardinality $\kappa$ such that the collection of all open intervals with endpoints in $D \cup \pm\infty$ is a base for $\hat{X}$. Note that this implies that if $a, b \in \hat{X}$ and $b$ is the immediate successor of $a$, then $a, b \in D$. Let $D = \{d_\alpha : \alpha < \kappa\}$. 
For $\alpha < \kappa$, inductively construct collections $\mathcal{I}_\alpha$ of closed intervals of $\hat{X}$ with disjoint interiors satisfying the following conditions:

(i) $\mathcal{I}_0 = \{\hat{X}\}$.

(ii) Given $\mathcal{I}_\alpha$, for each non-degenerate $I \in \mathcal{I}_\alpha$, let $I_0, I_1$ be two closed subintervals of $I$ with disjoint interiors whose union is $I$, with the following stipulation: (*) If $d_\alpha$ is in $I$, then $d_\alpha$ is an endpoint of $I_0$ or $I_1$.

Let $I_{\alpha+1} = \{I_j : I \in \mathcal{I}_\alpha, j \in \{0,1\}\}$.

(iii) Suppose $\mathcal{I}_\beta$ has been defined for all $\beta < \alpha$, $\alpha$ a limit ordinal. For each $x \in \hat{X} \cap \bigcap_{\beta < \alpha} (\cup \mathcal{I}_\beta)$ such that $x$ is not an endpoint of any member of $\bigcup_{\beta < \alpha} \mathcal{I}_\beta$, choose the unique member $I^x_\beta = [l^x_\beta, r^x_\beta] \in \mathcal{I}_\beta$ with $x \in I^x_\beta$. Let $r^x_\alpha = \inf \{r^x_\beta : \beta < \alpha\}$ and define $l^x_\alpha$ analogously. Then let $\mathcal{I}_\alpha$ be the collection of all intervals of the form $[l^x_\alpha, r^x_\alpha]$ such that $l^x_\alpha \neq r^x_\alpha$.

Let $\mathcal{I} = \bigcup_{\alpha < \kappa} \mathcal{I}_\alpha$, and let $E$ be the set of all endpoints of members of $\mathcal{I}$. Note that $|\mathcal{I}_\alpha| \leq \kappa$ for each $\alpha$. For limit $\alpha$, this follows immediately from $\omega(X) \leq \kappa$ and the fact that all $I \in \mathcal{I}_\alpha$ are non-degenerate. For successor $\alpha$, it follows from the limit case and that each $I \in \mathcal{I}_\beta$ is split into at most two pieces in the construction of $\mathcal{I}_{\beta+1}$. Hence $|\mathcal{I}| = \kappa = |E|$.

Let $\mathcal{B}$ be the collection of all convex open sets in $X$ formed by intersecting with $X$ an open interval in $\hat{X}$ with endpoints in $E \cup \{\pm \infty\}$. Then $|\mathcal{B}| = \kappa$. We will show that, for any $g \in \hat{X} \setminus X$, the two-element open cover $\{(-\infty, g) \cap X, (g, \infty) \cap X\}$ of $X$ has a locally finite refinement by members of $\mathcal{B}$. By Proposition 3.2, this will complete the proof.

If $g \in E$, then $(-\infty, g) \cap X$ and $(g, \infty) \cap X$ are already in $\mathcal{B}$, so we may assume $g \notin E$. Let $\delta = \sup \{\alpha < \kappa : \exists I \in \mathcal{I}_\alpha (g \in I)\}$. For $\alpha < \delta$, let $I_\alpha = [l_\alpha, r_\alpha]$ be the (unique) member of $\mathcal{I}_\alpha$ containing $g$. Then $l_\alpha < g < r_\alpha$, and it follows from the construction that $\delta$ is a limit ordinal, and that the map $\alpha \mapsto I_\alpha$ is continuous and non-decreasing, $\alpha \mapsto r_\alpha$ is continuous and non-increasing, and $\sup \{l_\alpha\}_{\alpha < \delta} = g = \inf \{r_\alpha\}_{\alpha < \delta}$. In the case $\delta = \kappa$, the last claim follows from the use of $D$ in the construction. If $\bigcap_{\alpha < \kappa} [l_\alpha, r_\alpha] = [l, r]$ were non-degenerate, then either $(l, r) \neq \emptyset$ and so there would be some $d_\alpha \in D$ with $l < d_\alpha < r$, or both $l$ and $r$ would be in $D$. In any case, (*) of condition (ii) of the construction would be violated.

Case 1. $\text{cof}(\delta) = \omega$. In this case, let $\{\gamma_\alpha\}_{\alpha < \omega}$ be a sequence cofinal in $\delta$ such that $l_{\gamma_0} < l_{\gamma_1} < \ldots$. Then $\{(-\infty, l_{\gamma_0}) \cap X, (l_{\gamma_0}, l_{\gamma_1}) \cap X, (l_{\gamma_1}, l_{\gamma_2}) \cap X, \ldots\}$ is a locally finite subcollection of $\mathcal{B}$ contained in and covering $(-\infty, g) \cap X$. Similarly there is another such subcollection contained in and covering $(g, \infty) \cap X$. This completes the proof in Case 1.

Case 2. $\text{cof}(\delta) > \omega$. Let $\nu = \text{cof}(\delta)$. For each $\alpha < \nu$, define $\gamma_\alpha$ to be the least $\gamma < \delta$ such that $\gamma \geq \sup \{\gamma_\beta : \beta < \alpha\}$ and $l_\gamma \notin \{l_\beta : \beta < \delta\}$. Then the map $\alpha \mapsto I_{\gamma_\alpha}$ is a continuous embedding of $\nu$ into $\{I_\beta : \beta < \delta\}$, and the image of this embedding is closed and cofinal in $(-\infty, g) \cap X$. Since $X$ is paracompact, $X$ does not contain a closed subset homeomorphic to a stationary subset of the regular uncountable cardinal $\nu$. Hence there is a club subset $C$ of $\nu$ such that $l_{\gamma_\alpha} \notin X$ for every $\alpha \in C$. For each $c \in C$, let $c^+$ be the successor of $c$ in $C$. Then $\{(-\infty, l_{\min(C)}) \cap X\} \cup \{(l_c, l_{c^+}) \cap X : c \in C\}$ is a clopen partition of $(-\infty, g) \cap X$ by members of $\mathcal{B}$. There is a similar clopen partition of $(g, \infty) \cap X$. It follows that $\mathcal{B}$ witnesses base-paracompactness of $X$ with respect to such two-element covers, and that completes our proof. \(\square\)
References


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