7560 THEOREMS WITH PROOFS

Definition A *compactification* of a completely regular space X is a compact T_2-space αX and a homeomorphic embedding $\alpha : X \to \alpha X$ such that $\alpha(X)$ is dense in αX.

Remark. To explain the separation assumptions in the above definition: we will only be concerned with compactifications which are Hausdorff. Note that only completely regular spaces can have a T_2-compactification.

Remark. A few of the concepts and results below are from first year topology, and are included here for completeness.

Definition. Let X be a locally compact non-compact Hausdorff space. The *one-point compactification* ωX of X is the space $X \cup \{\infty\}$, where points of X have their usual neighborhoods, and a neighborhood of ∞ has the form

$$\{\infty\} \cup (X \setminus K)$$

where K is a compact subset of X. In the framework of the previous definition, the map $\omega : X \to \omega X$ is the identity map on X.

Of course, X is open in its one-point compactification. We'll see that this is the case for any compactification of a locally compact X.

Lemma 1. *If X is a dense subset of a T_2 space Z, and X is locally compact, then X is open in Z.*

Proof. If $x \in X$, let N_x be a compact neighborhood x in X and $U_x = \text{int}(N_x)$. Then $U_x = X \cap V_x$ for some V_x open in Z. Thus U_x is dense in V_x and so

$$V_x \subset \overline{V}_x = \overline{U}_x \subset \overline{N}_x = N_x$$

where closures are taken in Z. Note that $\overline{N}_x = N_x$ since N_x is compact and Z is T_2. Thus for any $x \in X, \exists\ V_x$ open in Z such that $x \in V_x \subset X$ and so X is open. □

Corollary 2. *If αX is a compactification of a locally compact space X, then $\alpha(X)$ is open in αX.*

Proof. Immediate from Lemma 1 and the definition of a compactification. □

Definition. If αX and γX are compactifications of X, we say $\alpha X \geq \gamma X$ iff there exists a continuous function $f : \alpha X \to \gamma X$ such that $f \circ \alpha = \gamma$. If such f exists which is a homeomorphism, we say that αX and γX are *equivalent* compactifications, and write $\alpha X \approx \gamma X$.

Theorem 3. *If X is locally compact, ωX is the one-point compactification, and αX is any compactification, then $\alpha X \geq \omega X$.*

Proof. Define $f : \alpha X \to \omega X$ by

$$f(x) = \begin{cases} y \text{ if } x \in \alpha(X) \text{ and } x = \alpha(y) \\ \infty \text{ if } x \in \alpha X \setminus \alpha(X) \end{cases}$$

Clearly $f \circ \alpha = \omega$. We show f is continuous:

If $U \subset X$ is open, $f^{-1}(U) = \alpha(U)$ is open since α is an embedding and $\alpha(X)$ is open.

Otherwise, if $\infty \in U$, then $\omega X \setminus U$ is compact and thus closed and

$$f^{-1}(\omega X \setminus U) = \alpha(\omega X \setminus U) \cup \alpha X \setminus \alpha(X)$$

is closed since $\alpha(\omega X \setminus U)$ is compact and $\alpha X \setminus \alpha(X)$ is closed by Lemma 1. $\square$

We often think of obtaining a compactification αX of X by adding some points to X to make it compact, i.e., X is a dense subset of αX (instead of merely homeomorphic to one) and the map $\alpha : X \to \alpha X$ is the identity on X. The next result says we don't lose anything by thinking of compactifications in this way.

Theorem 4. *If αX is any compactification of X, then there is an equivalent compactification γX such that $X \subset \gamma X$ and the map $\gamma : X \to \gamma X$ is the identity map.*

Proof. As a set, let $\gamma X = X \cup (\alpha X \setminus \alpha(X))$. Let $h : \gamma X \to \alpha X$ by

$$h(x) = \begin{cases} \alpha(x) \text{ if } x \in \alpha(X) \\ x \text{ if } x \in \alpha X \setminus \alpha(X) \end{cases}$$

Then h is a bijection. Topologize γX by forcing h to be a homeomorphism. Then h is an equivalence $\gamma X \approx \alpha X$. $\square$

Theorem 5. *Let f and g be continuous mappings from a space X to a Hausdorff space Y. If f and g agree on a dense subset of X, then $f = g$.*

Proof. Let $h : X \to Y^2$ by $h(x) = (f(x), g(x))$. Since Y is Hausdorff $\Delta \subset Y^2$ is closed and so $h^{-1}(\Delta) = \{x \in X | f(x) = g(x)\}$ is closed and contains a dense set. Thus $h^{-1}(\Delta) = X$ and $f(x) = g(x)$ for all $x \in X$. $\square$

Theorem 6. *$\alpha X \approx \gamma X$ iff $\alpha X \geq \gamma X$ and $\gamma X \geq \alpha X$.*

Proof.
$\Rightarrow$:

Since $\alpha X \approx \gamma X$, then there exists an $f : \alpha X \to \gamma X$, f a homeomorphism, such that $f \circ \alpha = \gamma$. Hence $\alpha X \geq \gamma X$. Now, since f is a homeomorphism, f^{-1} exists and is continuous, so $f^{-1} \circ \gamma = \alpha$, and $\gamma X \geq \alpha X$.

$\Leftarrow$: Since $\alpha X \geq \gamma X$, then there exists $f : \alpha X \to \gamma X$, f continuous, such that $f \circ \alpha = \gamma$. Similarly, since $\gamma X \geq \alpha X$, then there exists a $g : \gamma X \to \alpha X$, g continuous, with $g \circ \gamma = \alpha$.

Now $g \circ f = id_{\gamma X} |_{\alpha(X)}$, and since $\gamma(X)$ is dense in γX, then by Lemma 5, $g \circ f = id_{\gamma X}$. By a similar argument, $f \circ g = id_{\alpha X}$. Hence $\alpha X \approx \gamma X$. $\square$

Definition. Let $\mathcal{F}$ be a family of continuous functions from a space X to the unit interval $I = [0, 1]$. We say $\mathcal{F}$ *separates points* if, given $x_1 \neq x_2 \in X$, there exists $f \in \mathcal{F}$ such that $f(x_1) \neq f(x_2)$. We say f *separates points from closed sets* if, given $x \in X$ and any closed set H with $x \notin H$, there exists $f \in \mathcal{F}$ with $f(x) \notin \overline{f(H)}$. The *evaluation function determined by* $\mathcal{F}$ is the function $e_{\mathcal{F}} : X \to I^{\mathcal{F}}$ defined by $e_{\mathcal{F}}(x) = < f(x) >_{f \in \mathcal{F}}$.

Theorem 7. *Let $\mathcal{F}$ be a collection of continuous functions from X into the unit interval. Then:*

(a) $e_{\mathcal{F}}$ is continuous;

(b) If $\mathcal{F}$ separates points, then $e_{\mathcal{F}}$ is one-to-one;

(c) If X is a T_1-space and $\mathcal{F}$ separates points from closed sets, then $e_{\mathcal{F}} : X \rightarrow e_{\mathcal{F}}(X)$ is a homeomorphic embedding of X into $I^{\mathcal{F}}$.

Proof.

(a) Well, since a product of continuous functions is continuous if and only if the individual projections are continuous, and each $f \in \mathcal{F}$ is continuous, then $e_{\mathcal{F}}$ is continuous.

(b) Let $e_{\mathcal{F}}(x) = e_{\mathcal{F}}(y)$. Then $< f(x) >_{f\in\mathcal{F}} = < f(y) >_{f\in\mathcal{F}}$, and hence $x = y$. Therefore $e_{\mathcal{F}}$ is one-to-one.

(c) From previous, $e_{\mathcal{F}}$ is injective and continuous. So sufficient to show $e_{\mathcal{F}}^{-1} : e_{\mathcal{F}}(X) \rightarrow X$ is continuous. So, let $H \subseteq X$ be closed. Need to show that $e_{\mathcal{F}}(H) = (e_{\mathcal{F}}^{-1})^{-1}(H)$ is closed in $e_{\mathcal{F}}(X)$. Let $y \in e_{\mathcal{F}}(X) \setminus e_{\mathcal{F}}(H)$. Then $y = < f(x) >_{f\in\mathcal{F}}$ for some $x \in H$. By hypothesis, there exists a $g \in \mathcal{F}$ such that $g(x)$ is not in $\overline{g(H)}$. So, there exists a $U \subset I$, U open, such that $g(x) \in U$, and $U \cap g(H) = \emptyset$. Let $O_f = U$ for $f = g$, and I otherwise. Then $y \in \prod_{f\in\mathcal{F}} O_f$, and $e_{\mathcal{F}}(H) \cap O_f = \emptyset$. Now, for $z \in H$, $y(z) = < f(z) >_{f\in\mathcal{F}}, soone_{\mathcal{F}}(z)(g) = g(z)$ is not in U. Hence $e_{\mathcal{F}}(z)$ is not in $\prod_{f\in\mathcal{F}} O_f$. □

Corollary 8.*A space X is completely regular if and only if X is homeomorphic to a subspace of I^{κ} for some cardinal κ.*

Proof. If X is completely regular, then the collection $\mathcal{F}$ of all continuous functions from X to $[0, 1]$ separates points from closed sets. □

Corollary 9. *Any completely regular space X has a Hausdorff compactification.*

Proof. Embed X in I^{κ}; then it's closure there is a compactification of X. □

Definition. Let $\mathcal{C}(X, I)$ be the collection of all continuous functions from a completely regular space X into the unit interval I, and let $\mathcal{F}$ be any subfamily of $\mathcal{C}(X, I)$ which separates points from closed sets. Let $e_{\mathcal{F}}$ be the embedding of X into $I^{\mathcal{F}}$ defined above, and let $e_{\mathcal{F}}X = \overline{e_{\mathcal{F}}(X)}$, where the closure is taken in $I^{\mathcal{F}}$. It follows from Theorem 6(c) that $e_{\mathcal{F}}X$ is a compactification of X.

Theorem 10. *If αX is any compactification of X, then there exists a subfamily $\mathcal{F}$ of $\mathcal{C}(X, I)$ for which $e_{\mathcal{F}}X \approx \alpha X$.*

Proof. Since αX is a compactification of X, we then have:

$$X \rightarrow^{\alpha} \alpha X \rightarrow^{f} I$$

So $f \circ \alpha : X \rightarrow I$.

Consider the collection $\mathcal{F} = \{f \circ \alpha |\ f : \alpha X \rightarrow I, \textit{and f is continuous}\}$. We now need to show that $e_{\mathcal{F}}(X) \approx \alpha X$.

So, $e_{\mathcal{F}} : X \rightarrow I^{\mathcal{F}}$ by $e_{\mathcal{F}}(x) = < f \circ \alpha(x) >_{f\in\mathcal{C}(\alpha X,I)}$.

For $p \in \alpha X$, let $h(p) = < f(p) >_{f\in\mathcal{C}(\alpha X,I)}$. Now, since $p \in \alpha X$, then $p = \alpha(x)$ for some unique $x \in X$. Then $h(p) = < f(\alpha(x)) >_{f\in\mathcal{C}(\alpha X,I)} = e_{\mathcal{F}}(x)$.

Claim: h is a homeomorphism.

h is continuous since each projection is continuous and between compact T_2 spaces. Also, $\mathcal{C}(\alpha X, I)$ separates points from closed sets. So, h is the evaluation function from αX into I. So, since $h = e_{\mathcal{C}(\alpha X,I)} : \alpha X \to I^{\mathcal{C}(\alpha X,I)}$, and $\mathcal{C}(\alpha X, I)$ separates points from closed sets, then h is a homeomorphic embedding.

Now, $\alpha(X)$ is dense in αX, so $h(\alpha(X))$ is dense in $h(\alpha X)$. Since $h(\alpha(X)) = e_{\mathcal{F}}(X) \subset e_{\mathcal{F}}X$, then $e_{\mathcal{F}}(X)$ is dense in $h(\alpha X)$. And, since $\overline{e_{\mathcal{F}}(X)} = e_{\mathcal{F}}X = h(\alpha X)$ then $h(\alpha X) = e_{\mathcal{F}}X$. $\square$

Definition. The *Stone-Cech compactification* of a completely regular space X is defined to be $e_{\mathcal{F}}X$, where $\mathcal{F} = \mathcal{C}(X, I)$. The Stone-Cech compactification of X is denoted by βX, and the embedding $e_{\mathcal{C}(X,I)}$ is denoted by β_X (or just β if X is understood).

Theorem 11. *Suppose $h : X \to Y$ is a continuous map from X into Y (both $T_{3.5}$) Then there exists a continuous $H : I^{\mathcal{C}(X,I)} \to I^{\mathcal{C}(Y,I)}$ such that $H \circ \beta_X = \beta_Y \circ h$. In particular, H restricted to β_X maps β_X into β_Y.*

Proof. Consider $I^{\mathcal{C}(X,I)}$ as the set $\{F : \mathcal{C}(X, I) \to I\}$. In regards to notation, we can think of $e_{\mathcal{C}(X,I)}(x)$ as $F_x \in I^{\mathcal{C}(X,I)}$ where $F_x(f) = f(x)$. Define a function $H : I^{\mathcal{C}(X,I)} \to I^{\mathcal{C}(Y,I)}$ as follows. If $F \in I^{\mathcal{C}(X,I)}$ and $g \in \mathcal{C}(X, I)$ then define $H(F)(g) = F(g \circ h)$. Then $H(F_x)(g) = F_x(g \circ h) = g \circ h(x) = F_{h(x)}(g)$; so the diagram commutes.

We will show that H is continuous by showing that the projection maps are continuous. Note: $\pi_g(F) = F(g)$, in other words: $\pi_g(\langle a_f \rangle_{f\in\mathcal{C}(X,I)}) = a_g$. So $\pi_g \circ H\left(\langle a_f \rangle_{f\in\mathcal{C}(X,I)}\right) = a_{g\circ h}$. Hence $\pi_g \circ H = \pi_{g\circ h}$ is continuous. It follows that H is continuous. $\square$

Theorem 12. *For every compact T_2-space Y and each continuous map $f : X \to Y$, there exists a continuous map $F : \beta_X \to Y$ such that $f = F \circ \beta_X$.*

Proof. Suppose Y is a compact T_2-space, and $f : X \to Y$ is continuous. Since Y is compact T_2, it is completely regular, and hence has a Stone-Čech compactification β_Y. Let $H : I^{\mathcal{C}(X,I)} \to I^{\mathcal{C}(Y,I)}$ be a continuous function as in Theorem 11. Define $F = e^{-1}_{\mathcal{C}(Y,I)} \circ H \restriction_{\beta X}$. Then $F : \beta X \to Y$ is continuous, and $F \circ \beta_X = f$. $\square$.

Remark. If, as often done, we think of X as a subset of βX, and dispense with the map $\beta_X : X \to \beta X$, then Theorem 12 says that any continuous map from X to a compact space can be extended to βX.

Corollary 13. *If αX is any compactification of X, then $\beta X \geq \alpha X$. Also any compactification of X which satisfies the condition in Theorem 12 is equivalent to βX.*

Proof. Note: $\alpha_X : X \to \alpha X$ is continuous and αX is compact T_2. By theorem 12, let $F : \beta X \to \alpha X$ be continuous such that $\alpha = F \circ \beta_X$. It follows that $\beta_X \geq \alpha X$. For the second part of the Corollary, suppose γX is a compactification of X such for any continuous $f : X \to Y$, where Y is compact T_2, there is an extension of f to a continuous function from γX into Y. Note: $\beta_X : X \to \beta X$ is continuous and βX is compact T_2. Let $G : \gamma X \to \beta X$ be a continuous extension of β_X. It follows that $\gamma X \geq \beta X$. By theorem 6 and the first part of the corollary. $\gamma X \approx \beta X$. $\square$

Theorem 14. *$[0, \omega_1]$ is the only (up to equivalence) Hausdorff compactification of the space $[0, \omega_1)$ of countable ordinals. Hence $\beta[0, \omega_1) = [0, \omega_1]$.*

Proof. Suppose $p \neq q$ where $p, q \in \beta[0, \omega_1) \setminus [0, \omega_1)$. We will show that p and q can't be separated by open sets (and hence contradict the fact that $\beta[0, \omega_1)$ is

Hausdorff). Towards this goal we will show that every neighborhood of p intersects every neighborhood of q. Let N_p be a closed neighborhood of p and consider $N_p \cap [0, \omega_1)$.

Claim: $N_p \cap [0, \omega_1)$ is unbounded. Suppose, towards a contradiction, that it is bounded by $\alpha \in \omega_1$. Then $N_p \cap [0, \omega_1)$ is a closed subset of compact $[0, \alpha]$. It follows that $N_p \cap [0, \omega_1)$ is a compact subset of Hausdorff $\beta[0, \omega_1)$, thus it's closed in $\beta[0, \omega_1)$. However, this would mean that $N_p \setminus (N_p \cap [0, \omega_1))$ is an open neighborhood of p missing $[0, \omega_1)$, which is a contradiction since $[0, \omega_1)$ is dense in $\beta[0, \omega_1)$. Therefore $N_p \cap [0, \omega_1)$ is closed and unbounded.

Similarly if N_q is a closed neighborhood of q then $N_q \cap [0, \omega_1)$ is a club. Since any two clubs necessarily intersect, it follows that $N_q \cap N_p \neq \emptyset$. Hence p and q can't be separated by open sets. This contradicts the fact that $\beta[0, \omega_1)$ is Hausdorff, and we must conclude that there aren't two distinct points in $\beta[0, \omega_1) \setminus [0, \omega_1)$. So the one-point compactification $[0, \omega_1]$ is the only compactification of $[0, \omega_1)$.□

Theorem 15. *Let ω be the discrete space of natural numbers. Then $\beta\omega$ maps continuously onto any separable compact Hausdorff space.*

Proof. Suppose Y is a separable compact Hausdorff space, with countable dense subset D. Let $f : \omega \to D$ be any bijection from ω onto D. Then f is necessarily continuous since X is ω is discrete. Let $F : \beta\omega \to Y$ be a continuous extension of f. Note: $F(\beta\omega) \supseteq F(\omega) = f(\omega) = D$, hence $F(\beta\omega)$ is dense in Y. Also, since $\beta\omega$ is compact, consequently $F(\beta\omega)$ is a compact subspace of Hausdorff Y; hence $F(\beta\omega)$ is closed in Y. It follows that $F(\beta\omega) = Y$. And therefore F maps $\beta\omega$ continuously onto Y. □.

Corollary 16. $|\beta\omega| \geq 2^{\mathfrak{c}}$.

Proof. It was shown that I^I is compact, separable and Hausdorff with cardinality $2^{\mathfrak{c}}$. By theorem 15, there is a surjection from $\beta\omega$ onto I^I. Consequently $|\beta\omega| \geq |I^I| = 2^{\mathfrak{c}}$. □

Lemma 17. *If X is separable T_2-space then $|X| \leq 2^{\mathfrak{c}}$.*

Proof. Let X be a separable T_2 space and $D \subset X$ be countable and dense. Define $f : X \to 2^{2^D}$ by $f(x) = \{U \cap D : x \in U \text{ and } U \text{ is open}\}$. Let $x, y \in X$ be distinct, then there exist disjoint open $U, V \subset X$ with $x \in U$ and $y \in V$. Clearly, $U \cap D \in f(x)$, and if $y \in V'$ open, then there is some $p \in V \cap V' \cap D \subset V' \cap D$ but $p \notin U \cap D$ and so $U \cap D \notin f(y)$, in particular $f(x) \neq f(y)$ and f is injective. Thus $|X| \leq |2^{2^D}| = 2^{\mathfrak{c}}$.

Corollary 18. $|\beta\omega| = 2^{\mathfrak{c}}$.

Proof. Follows from Corollary 16 and Lemma 17.

Theorem 19. *Let Z be a compact Hausdorff space and X be a dense subspace of Z. If every continuous $f : X \to I$ extends to some continuous $f : Z \to I$ then $Z \approx \beta X$.*

Proof. By Corollary 13, $\beta X \geq Z$. Now, for every $f \in C(X, I)$, $f = \pi_f \circ \beta_X : X \to I$ is continuous and extends to a continuous $\pi_f^* : Z \to I$. Define $F : Z \to I^{C(X,I)}$ by $F =< \pi_f^* >_{f \in C(X,I)}$. F is clearly continuous and for each $x \in X$ and $f \in C(X, I)$,

$$\pi_f^*(x) = \pi_f \circ \beta_X(x) = f(x)$$

and so

$$F(x) =< \pi_f^*(x) >_{f \in C(X,I)} =< f(x) >_{f \in C(X,I)} = \beta_X(x)$$

i.e.,

$$F \circ 1_X = F \restriction X = \beta_X$$

. Since X is dense in Z, $F(Z) \subset \overline{F(X)} = \overline{\beta_X(X)} = \beta X$, and $F : Z \to \beta X$, hence $Z \geq \beta X$. Thus $Z \approx \beta X$ by Theorem 6. $\square$

In the next 4 results, assume, as we may, that $X \subset \beta X$ and that the mapping $\beta_X : X \to \beta X$ is the identity on X.

Theorem 20. *If X is normal, then any two disjoint closed subsets of X have disjoint closures in βX.*

Proof. Let $H, K \subset X$ be closed and disjoint. Since X is normal, there exists $f : X \to I$ such that $H \subset f^{-1}(0)$ and $K \subset f^{-1}(1)$. f extends to a continuous $f^* : \beta X \to I$, and so $\overline{H}^{\beta X} \subset f^{-1}(0)$ and $\overline{K}^{\beta X} \subset f^{-1}(1)$, thus the closures of the two are clearly disjoint. $\square$

Theorem 21. *Let X be a normal space, and let H be a closed subset of X. Then $\overline{X}^{\beta X} \approx \beta X$.*

Proof. Let $f : H \to I$ be continuous, then by Tietze Extension theorem, f extends to a continuous $\overline{F} : X \to I$. By theorem 19, $\overline{f}$ extends to some continuous $g : \beta X \to I$. So, $g \restriction \overline{H}^{\beta X}$ extends f. Since H is dense in $\overline{H}^{\beta X}$, applying Theorem 19 we get our result.

Corollary 22. *$\beta\omega$ is homeomorphic to a subspace of $\beta[0,1)$ also of $\beta\mathbb{R}$.*

Proof. Follows from Theorem 21.

Theorem 23. *$\beta[0,1)\backslash[0,1)$ is connected, and $\beta\mathbb{R}\backslash\mathbb{R}$ has exactly two components.*

Proof. (a) $\beta[0,1) \setminus [0,1)$ is connected.

Proof. Claim: If $a \in (0,1)$, then $cl_{\beta[0,1)}([0,a)) = [0,a]$. If $a \in [0,1)$, then $[0,a]$ is compact and so $[0,a]$ is closed in $\beta[0,1) \Rightarrow cl_{\beta[0,1)}([0,a)) \subset [0,a]$. Because $[0,a)$ is not compact (if $a > 0$), it is not closed in $\beta[0,1)$, and so $cl_{\beta[0,1)}([0,a))$ must contain a point of $[0,a] \setminus [0,a) = \{a\}$. Thus, $cl_{\beta[0,1)}([0,a)) = [0,a]$.

If $a \in (0,1)$, then $[a,1)$ is closed and $cl_{\beta[0,1)}([a,1)) = \beta[a,1)$; meaning,

$$\beta[0,1) = cl_{\beta[0,1)}([0,a) \cup [a,1)) = [0,a] \cup \beta[a,1),$$

and so

$$\beta[0,1) \setminus [0,1) = ([0,a] \setminus [0,1]) \cup ([a,1) \setminus [0,1)) \subset \beta[a,1) \setminus [a,1). \tag{1}$$

If $i \in \mathbb{N}$, let $a_i = 1 - \frac{1}{i}$ and let $C_i = cl_{\beta[0,1)}([0,a_i))$. $[a_i,1)$ is connected $\Rightarrow C_i = cl_{\beta[0,1)}([a_i,1))$ is connected, compact, and $\beta[0,1) \setminus [0,1) \subset C(a_i)$, by (1).

$$[0,a_i) \subset [0,a_{i+1}) \Rightarrow [a_i,1) \supset [a_{i+1},1) \Rightarrow C(a_i) \supset C(a_{i+1}),$$

therefore, $\mathbf{C} = \cap_{i=1}^{\infty} C(a_i)$ is connected, and $\beta[0,1) \setminus [0,1) \subset \mathbf{C}$.

Lastly, because $[0,1)$ is locally compact, $[0,a_i)$ is open in $\beta[0,1)$. $[0,a_i)$ open and $[0,a_i) \cap [a_i,1) = \emptyset \Rightarrow [0,a_i) \cap cl_{\beta[0,1)}([a_i,1)) = \emptyset \Rightarrow [0,a_i) \cap C_i = \emptyset$.

$$\Rightarrow [0,1) \cap \mathbf{C} = (\cup_{i=1}^{\infty}[0,a_i)) \cap \mathbf{C} \subset \cup_{i=1}^{\infty}([0,a_i) \cap C_i) = \emptyset,$$

$\Rightarrow \mathbf{C} \subset \beta[0,1) \setminus [0,1) \Rightarrow \mathbf{C} = \beta[0,1) \setminus [0,1)$

$\therefore \beta[0,1) \setminus [0,1)$ is connected. $\square$

(b) $\beta\mathbb{R} \setminus \mathbb{R}$ has exactly two components.

Proof. First note that $[0,\infty) \cong [0,1) \cong (-\infty,0]$ and so $\beta[0,1) \cong \beta[0,\infty) \cong \beta(-\infty,0] \Rightarrow \beta[0,\infty) \setminus [0,\infty)$ and $\beta(-\infty,0] \setminus (-\infty,0]$ are each connected. Because $\mathbb{R} = (-\infty,0] \cup [0,\infty)$, $\beta\mathbb{R} = cl_{\beta\mathbb{R}}((-\infty,0]) \cup cl_{\beta\mathbb{R}}([0,\infty))$. Both intervals are closed in $\mathbb{R}$, which means their closures are equivalent to there compactifications $\Rightarrow \beta\mathbb{R} = \beta(-\infty,0] \cup \beta[0,\infty)$; meaning $\beta\mathbb{R} \setminus \mathbb{R}$ has at most two components.

Let $D = [1,\infty)$ and $E = (-\infty,-1]$, by the same argument used to show (1) in (a), $\beta[1,\infty)\setminus[1,\infty) \subset \beta[1,\infty)\setminus[1,\infty)$ and $\beta(-\infty,0]\setminus(-\infty,0] = \beta(-\infty,1]\setminus(-\infty,1]$.

$[1,\infty)$ and $(-\infty,-1]$ are disjoint closed sets (in $\mathbb{R}$) and $\mathbb{R}$ is normal; thus, $\beta[1,\infty)$ and $\beta(-\infty,-1]$ are disjoint $\Rightarrow \beta[1,\infty) \setminus [1,\infty)$ and $\beta(-\infty,-1] \setminus (-\infty,-1]$ are disjoint $\Rightarrow \mathbb{R}$ has exactly two components. □

Now we will investigate the space $\beta\omega$. We'll see that this space can be conveniently described as the space of all ultrafilters on ω with a certain topology. First, let's recall the definition and a few facts about filters and ultrafilters.

Definition. A collection $\mathcal{F}$ of subsets of a set X is called a *filter on X* if:

(i) Whenever $F_1, F_2, \ldots, F_n \in \mathcal{F}$, then $\bigcap_{i=1}^n F_i \in \mathcal{F}$;
(ii) If $F \in \mathcal{F}$ and $F \subset G \subset X$, then $G \in \mathcal{F}$;
(iii) $\emptyset \notin \mathcal{F}$.

In other words, a filter on X is a collection of nonempty subsets of X which is closed under supersets and finite intersections.

Also, a filter $\mathcal{F}$ on X is called an *ultrafilter* if $\mathcal{F}$ is not properly contained in any other filter on X. A trivial example of an ultrafilter on X is the collection of all subsets of X containing a fixed element x_0 in X. Ultrafilters like this are called *fixed* ultrafilters. A filter $\mathcal{F}$ is said to be *free* if $\cap\mathcal{F} = \emptyset$. The Axiom of Choice is needed to show the existence of free ultrafilters. A standard Zorn's Lemma argument shows that any filter on X is contained in an ultrafilter, so applying this to any free filter, e.g., to the filter of co-finite subsets of an infinite set X, gets you a free ultrafilter on X.

Theorem 24. *Let $\mathcal{F}$ be a filter on X. Then the following are equivalent:*

(a) $\mathcal{F}$ is an ultrafilter.
(b) If $G \subset X$ and $G \cap F \neq \emptyset$ for every $F \in \mathcal{F}$, then $G \in \mathcal{F}$.
(c) For ever $G \subset X, either G \in \mathcal{F}$ or $X \setminus G \in \mathcal{F}$.

Proof.

(i) $(a) \Rightarrow (b)$.

Proof. Suppose F is an ultrafilter and $G \subset X$ such that $G \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. Let $\mathcal{F}' = \{F' : F' \supset (G \cap F)$ for some $F \in \mathcal{F}\}$.

Note:$G \subset \mathcal{F}'$ because $X \in \mathcal{F}$ and $G \supset X \cap G$, and $\mathcal{F} \subset \mathcal{F}'$ because $F \supset (F \cap G)\ \forall\ F \in \mathcal{F}$

$\emptyset$ is not in $\mathcal{F}'$, for if $F \in \mathcal{F}$, $F \cap G \neq \emptyset \Rightarrow$ if $F' \in \mathcal{F}'$, then F' contains a nonempty set.

If $F_1', F_2', \ldots F_n' \in \mathcal{F}'$ choose $F_1, F_2, \ldots F_n \in \mathcal{F}$ such that $F_i' \supset F_i \cap G$. It follows that

$$\cap_{i=1}^n F_1' \supset \cap_{i=1}^n (F_i \cap G) = (\cap_{i=1}^n F) \cap G.$$

Because $\mathcal{F}$ is a filter, $\cap_{i=1}^n F_i \in \mathcal{F} \Rightarrow \cap_{i=1}^n F_i' \in \mathcal{F}'$.

If $H \subset X$ and $F' \in \mathcal{F}'$ such that $H \supset F'$, then $H \supset F' \supset F \cap G$ for some $F \in \mathcal{F}, \Rightarrow H \in \mathcal{F}'$.

$\Rightarrow \mathcal{F}'$ is a filter and $\mathcal{F} \subset \mathcal{F}' \Rightarrow \mathcal{F} = \mathcal{F}' \Rightarrow G \in \mathcal{F}$.

$\square$

(ii) $(b) \Rightarrow (c)$

Proof. Suppose $G \subset X$. If G and $X \setminus G$ are each not in $\mathcal{F}$, then there is $E \in \mathcal{F}$ and $F \in \mathcal{F}$ such that $E \cap G = F \cap (X \setminus G) = \emptyset = \Rightarrow (E \cap F) \cap G = (E \cap F) \cap (X \setminus G) = \emptyset \Rightarrow (E \cap F) \cap X = \emptyset \Rightarrow \mathcal{F}$ is not a filter. $\square$

(iii) $(c) \Rightarrow (b)$.

Proof. By contrapositive, if $\mathcal{F}$ is not an ultrafilter, there is $G \subset X$ and an ultrafilter $\mathcal{F}' \supset \mathcal{F}$ such that $G \in \mathcal{F}'$. This means $X \setminus G \notin \mathcal{F}' \Rightarrow X \setminus G \notin \mathcal{F}$.

$\square$

Theorem 25. *Let $p \in \beta\omega \setminus \omega$, and let*

$$\mathcal{F}_p = \{U \cap \omega : U \text{ is a nbhd of } p\}.$$

Then:

(i) $\mathcal{F}_p$ is a free ultrafilter on ω
(ii) $p \neq q \Rightarrow \mathcal{F}_p \neq \mathcal{F}_q$
(iii) If $\mathcal{F}$ is a free ultrafilter on ω, then there is a unique $p \in \beta\omega \setminus \beta$ with $\mathcal{F}_p = \mathcal{F}$.

Proof.

(i)$\mathcal{F}_p$ is a filter on ω:

- Since ω is dense in $\beta\omega$ by corollary 2, $U \cap \omega \neq \emptyset$ for any neighborhood U of p. Thus $\emptyset \notin \mathcal{F}_p$

- If $F_1, F_2 \in \mathcal{F}_p$, let U_i be neighborhoods of p such that $F_i = U_i \cap \omega$ for $i = 1, 2$. Then $U_1 \cap U_2$ is also a neighborhood of p and thus $F_1 \cap F_2 = (U_1 \cap U_2) \cap \omega \in \mathcal{F}_p$

- If $F \in \mathcal{F}_p$ and $F \subset G \subset \omega$, let U be a neighborhood of p such that $F = U \cap \omega$. Then $U \cup G$ is a neighborhood of p and $G = (U \cup G) \cap \omega \in \mathcal{F}_p$

$\mathcal{F}_p$ is an ultrafilter:

- If $G \subset \omega$ and $G \cap F \neq \emptyset$ for all $F \in \mathcal{F}_p$ and $G \notin \mathcal{F}_p$, then for every $F \in \mathcal{F}_p$, $F \not\subset G$ and so $F \cap (\omega \setminus G) \neq \emptyset$. Thus $p \in \overline{G}^{\beta\omega} \cap \overline{\omega \setminus G}^{\beta\omega}$, but by theorem 20, this set is empty. Thus, by theorem 24, $\mathcal{F}_p$ is an ultrafilter

$\mathcal{F}_p$ is free:

- If $x \in \omega$ then since $\beta\omega$ is T_2, there exists an open U containing p such that $x \notin \overline{U}$. Then $p \notin \overline{U} \cap \omega \in \mathcal{F}_p$ and thus $\cap \mathcal{F} = \emptyset$

(ii) If $p \neq q$ then since $\beta\omega$ is T_2, there exist disjoint open sets U and V with $p \in U$ and $q \in V$. Then $U \cap \omega \in \mathcal{F}_p$ and $V \cap \omega \in \mathcal{F}_p$ would imply $(U \cap \omega) \cap (V \cap \omega) = \emptyset \in \mathcal{F}_p$. Similarly, $V \cap \omega \in \mathcal{F}_q$ and $U \cap \omega \notin \mathcal{F}_q$ so $\mathcal{F}_p \neq \mathcal{F}_q$

(iii) If $\mathcal{F}$ is a free ultrafilter on ω, let $\mathcal{F}' = \{\overline{F} \subset \beta\omega : F \in \mathcal{F}\}$. Then $\mathcal{F}'$ has the f.i.p. since $\mathcal{F}$ is a filter and so by compactness of $\beta\omega$, $\cap \mathcal{F}' \neq \emptyset$. Choose some $p \in \cap \mathcal{F}'$. Then for every open U containing p and $F \in \mathcal{F}$, $U \cap F \neq \emptyset$ since $p \in \overline{F}$. Thus $(U \cap \omega) \cap F = U \cap F \neq \emptyset$ for every $F \in \mathcal{F}$ and so by theorem 24, $U \cap \omega \in \mathcal{F}$. Thus $\mathcal{F}_p \subset \mathcal{F}$ and so since $\mathcal{F}_p$ is an ultrafilter, $\mathcal{F}_p = \mathcal{F}$. $\square$

Theorem 26.

(i) If $A \subset \omega$, then $\overline{A}^{\beta\omega} = A \cup \{p \in \beta\omega \setminus \omega : A \in \mathcal{F}_p\}$
(ii) For each $A \subset \omega$, $\overline{A}^{\beta\omega}$ is clopen in $\beta\omega$

(iii) The collection $\{\overline{A}^{\beta\omega} : A \subset \omega\}$ *forms a base for* $\beta\omega$*.*

Proof.

(i) Clearly $\overline{A}^{\beta\omega} \cap \omega = A$. Let $B = \{p \in \beta\omega \setminus \omega : A \in \mathcal{F}_p\}$. If $p \in B$ then $A \in \mathcal{F}_p$ so for any neighborhood U of p, $U \cap A = (U \cap \omega) \cap A \in \mathcal{F}_p$. Thus $B \subset \overline{A}^{\beta\omega}$. Conversely, if $p \in (\beta\omega \setminus \omega) \setminus B$ then there exists a neighborhood U of p such that $(U \cap \omega) \cap A = \emptyset$ and thus $p \notin \overline{A}^{\beta\omega}$ and so $\overline{A}^{\beta\omega} \setminus A \subset B$. It follows that $\overline{A}^{\beta\omega} = A \cup B$

(ii) If $A \subset \omega$ then A and $\omega \setminus A$ are both closed in ω and so $\overline{A}^{\beta\omega}$ and $\overline{\omega \setminus A}^{\beta\omega}$ are disjoint by theorem 20. Given any $p \in \beta\omega \setminus \beta$, $\mathcal{F}_p$ is an ultrafilter by theorem 25 and so either $A \in \mathcal{F}_p$ or $\omega \setminus A \in \mathcal{F}_p$ by theorem 24. Thus $p \in \overline{A}^{\beta\omega}$ or $p \in \overline{\omega \setminus A}^{\beta\omega}$ so $\overline{A}^{\beta\omega} \cup \overline{\omega \setminus A}^{\beta\omega} = \beta\omega$

(iii) If $U \subset \beta\omega$ is open and $p \in U$, by regularity there exists an open set V such that $p \in V \subset \overline{V} \subset U$. Let $A = V \cap \omega$. Then $p \in \overline{A}^{\beta\omega} = \overline{V} \subset U$. □

Now define a space as follows. The set for the space is

$$\omega \cup \{\mathcal{F} : \mathcal{F} \text{ is a free ultrafilter on } \omega\}.$$

Then for each $A \subset \omega$, define

$$\overline{A} = A \cup \{\mathcal{F} : A \in \mathcal{F}\}.$$

It is easy to check that, for any $A, B \subset \omega$, we have $\overline{A} \cap \overline{B} = \overline{A \cap B}$. It follows that

$$\{\overline{A} : A \subset \omega\}$$

is a base for a topology on the space.

Theorem 27. *Define* $X = \omega \cup \{\mathcal{F} : \mathcal{F} \text{ is a free ultrafilter on } \omega\}$ *and define the basic open sets as* $\overline{A} = A \cup \{\mathcal{F} : A \in \mathcal{F}\}$*. Then this space* X *is homeomorphic to* $\beta\omega$*.*

Proof. Define $\varphi : X \rightarrow \beta\omega$ by $\varphi|_\omega = id_\omega$ and $\varphi(\mathcal{F}) = p$ where $p \in \beta\omega \smallsetminus \omega$ with $\mathcal{F} = \mathcal{F}_p$.

By Theorem 25, φ is well-defined and a bijection. By Theorem 26, $\{cl_{\beta\omega}(A) : A \subset \omega\} = \{A \cup \{p \in \beta\omega \smallsetminus \omega : A \in \mathcal{F}_p\}\}$ is a basis for $\beta\omega$.

Now, $\varphi^{-1}(cl_{\beta\omega}(A)) = A \cup \{\mathcal{F}_p : A \in \mathcal{F}_p\} = A \cup \{\mathcal{F} : A \in \mathcal{F}\} = \varphi(A \cup \{\mathcal{F} : A \in \mathcal{F}\}) = A \cup \{p : A \in \mathcal{F}_p\}$. So φ is a homeomorphism. □

We now explore some topological properties of $\beta\omega$. By compactness, every infinite subset of ω has a limit point in $\beta\omega$, and said limit point must of course be in $\beta\omega \setminus \omega$, but we'll see we do not have sequential convergence:

Theorem 28. *No non-trivial sequence in* ω *converges to a point of* $\beta\omega$*.*

Proof. Let $\{a_i : i \in \omega\}$ be a non-trivial sequence in ω.

i. If $A = \{a_i : i \in \omega\}$ is infinite, we then can choose a subsequence $\{a_i'\}_{i \in \omega}$ such that $a_i' = a_j'\ if\ i = j$.
Let $a_0', a_1', a_2', \ldots = c_0, d_0, c_1, d_1, \ldots$.
Let $C = \{c_0, c_1, c_2, \ldots\}$ and $D = \{d_0, d_1, d_2, \ldots\}$.
So $C \cap D = \emptyset$, which implies that $\overline{C} \cap \overline{D} = \emptyset$. Therefore $\{c_i\}_{i \in \omega}$ and $\{d_i\}_{i \in \omega}$ cannot coverge to the same point. So $\{a_i\}_{i \in \omega}$ cannot converge.

ii. If $A = \{a_i : i \in \omega\}$ is finite, then because $\{a_i\}_{i\in\omega}$ is non-trivial, there must be an $\alpha,\ \gamma \in \omega$, with $\alpha \neq \gamma$ such that $\{i : a_i = \alpha\}$ and $\{j : a_j = \gamma\}$ is infinite. Then $\{\alpha\}_{i\in\omega}$ and $\{\gamma\}_{j\in\omega}$ are two subsequences converging to different points, hence $\{a_i\}$ does not converge. $\square$

Of course, the "interesting" points of $\beta\omega$ are the points of the remainder $\beta\omega \setminus \omega$, which is often denoted by ω^*. Note that ω^*, being closed in $\beta\omega$, is also a compact space. For each infinite $A \subset \omega$, let $A^* = \overline{A}^{\beta\omega} \cap \omega^*$. Then by Theorem 26(ii),(iii), the collection

$$\{A^* : A \text{ is an infinite subset of } \omega\}$$

is a clopen base for ω^*.

Lemma 29. *Let $A,\ B \subset \omega$. Then:*

(1) $A^* \cap B^* = (A \cap B)^*$;
(2) $A^* \cap B^* = \emptyset \iff A \cap B$ is finite;
(3) $A^* \subset B^*$ iff $A \smallsetminus B$ is finite.

Proof.

(1) Well, $\mathcal{F} \in A^* \cap B^* \iff A \in \mathcal{F}$ and $B \in \mathcal{F} \iff A \cap B = \mathcal{F} \iff \mathcal{F} \in (A \cap B)^*$.

(2) $\Longleftarrow$: Well, $A^* \cap B^*$ is finite implies that $\overline{A \cap B} \subset \omega$, so $(A \cap B)^* = \overline{A \cap B} \smallsetminus \omega = \emptyset$.
$\Longrightarrow$: (by contrapositive) If $A \cap B$ is not finite, then $A \cap B$ is infinite. So $\overline{(A \cap B)} \cap (\beta\omega \smallsetminus \omega) \neq \emptyset$ by compactness. So $(A \cap B)^* = (\overline{A \cap B}) \smallsetminus \omega \neq \emptyset$.

(3) $A^* \subset B^*$ iff $A \smallsetminus B$ is finite.

Claim: If $\beta \subset \omega$, then $\omega^* \smallsetminus \beta^* = (\omega \smallsetminus \beta)^*$.

$\subseteq$: If $\mathcal{F} \in \omega^*$, then $\beta \in \mathcal{F}$ or $\omega \smallsetminus \beta \in \mathcal{F}$. If $\mathcal{F} \in \omega^* \smallsetminus \beta^*$, then $\beta \notin \mathcal{F}$, and so $\omega \smallsetminus \beta \in \mathcal{F}$, implying $\mathcal{F} \in (\omega \smallsetminus \beta)^*$. So $\omega^* \smallsetminus \beta^* \subset (\omega \smallsetminus \beta)^*$.

$\supseteq$: If $\mathcal{F} \in (\omega \smallsetminus \beta)^*$, then $\omega \smallsetminus \beta \in \mathcal{F}$. So $\beta \notin \mathcal{F}$, and hence $\mathcal{F} \notin \beta^*$. Then $\mathcal{F} \in \omega^* \smallsetminus \beta^*$, since $\mathcal{F}$ has to be in ω^*. Then $(\omega \smallsetminus \beta)^* \subset \omega^* \smallsetminus \beta^*$. So $\omega^* \smallsetminus \beta^* = (\omega \smallsetminus \beta)^*$. $\square$

Now from the claim, $(A^* \smallsetminus B^*) = A^* \cap (\omega^* \smallsetminus B^*) = A^* \cap (\omega \smallsetminus B)^*$.
From (1), $A^* \cap (\omega \smallsetminus B)^* = (A \cap (\omega \smallsetminus B))^* = (A \smallsetminus B)^*$.
Now $A^* \subset B^* \iff A^* \smallsetminus B^* = \emptyset \iff (A \smallsetminus B)^* = \emptyset \iff A \smallsetminus B$ is finite. (from (2)). $\square$

Definition. For $A, B \subset \omega$, we say that:

(1) A and B are *almost disjoint* if $A \cap B$ is finite;
(2) A is *almost included* in B if $A \setminus B$ is finite.

Of course, any disjoint family of subsets of ω is countable, but:

Lemma 30. *There is a pairwise almost disjoint family $\mathcal{A}$ of subsets of ω of cardinality $\mathfrak{c}$.*

Proof. It suffices to find such a family of subsets of $\mathbb{Q}$. For each $x \in \mathbb{R}$, choose a sequence $\{q_{x,n}\}_{n\in\omega}$ of rationals converging to x. Let $A_x = \{q_{x,n}\}_{n\in\omega}$ and let $\mathcal{A} = \{A_x : x \in \mathbb{R}\}$. It is easy to check that $\mathcal{A}$ has the desired properties. $\square$

Corollary 31. *There is a family of* $\mathfrak{c}$*-many disjoint open subsets of* ω^**; in particular,* ω^* *is not separable.*

Proof. Let $\mathcal{A}$ be as in Lemma 30. Then $\{A^* : A \in \mathcal{A}\}$ is a $\mathfrak{c}$-sized pairwise disjoint collection of open subsets of ω^*. $\square$

Now we want to see a very strange property of $\beta\omega$: it contains no nontrivial convergent sequences at all. First, recall that a subset D of a space X is *relatively discrete* if the subspace topology on D is the discrete topology. (Equivalently, D being relatively discrete means that no point of D is a limit point of D; this should be compared with the concept of D being *closed discrete*, which means no point of the whole space X is a limit point of D.)

Lemma 32. *If* D *is a countable infinite relatively discrete subset of a regular space, then there is a family of disjoint open sets* $\{U_d : D \in D\}$ *with* $d \in U_d$ *for each* $d \in D$*.*

Proof. Let $K = \{k_i\}_{i\in\omega} \subseteq X$ be relatively discrete. For each $i \in \omega$ define let $U_i \subseteq X$ be an open set such that $U_i \cap K = \{k_i\}$. For each $i \in \omega$ let $V_i \subseteq X$ be open such that $k_i \in V_i \subseteq \overline{V_i} \subseteq U_i$; this is possible since X is regular. Let $W_0 = V_0$. For each $i > 0$ in ω, let $W_i = Vi \setminus \bigcup_{j<i} \overline{V_j}$. The family $\{W_i : i \in \omega\}$ satisfy the conclusion of the lemma. $\square$

Now our "no convergent sequences in $\beta\omega$" result follows quite easily from:

Theorem 33. *If* $D \subseteq \beta\omega$ *is countably infinite and relatively discrete, then* $\overline{D} \approx \beta D$*.*

Proof. Let $\{U_d : d \in D\}$ be a family as in Lemma 32. Let $f : D \to [0,1]$ be continuous. Note: since D is discrete, any function is continuous. Let $A = \bigcup_{d\in D} (U_d \cap \omega)$ and define a function $f^* : A \to [0,1]$ as follows: if $n \in U_d$ define $f^*(n) = f(d)$; this is well defined since the family $\{U_d : d \in \omega\}$ is a collection of disjoint sets, and every $i \in A$ is in a U_d for some d. Since $\overline{A} \approx \beta A$, there is an extension $F : \overline{A} \to [0,1]$ of f^*. For each $d \in \omega$ $d \in \overline{U_d \cap \omega}$, this implies that $D \subseteq \overline{A}$, and therefore $\overline{D} \subseteq \overline{A}$. Note: If $d \in \omega$ then $f^* = f(d)$ on a dense subset of U_d. Therefore $F \restriction U_d = f(d)$. In particular $F(d) = f(d)$ for all $d \in \omega$. Therefore $f \restriction_{\overline{D}}$ extends f to a continuous function on $\overline{D}$. By corollary 13, $\overline{D} \approx \beta D$.$\square$

Corollary 34. *$\beta\omega$ contains no nontrivial convergent sequence.*

Proof. Suppose towards a contradiction that $D = \{x_i : i \in \omega\}$ is a nontrivial convergent sequence in $\beta\omega$ converging to the point p. Then $\{x_i : i \in o\}$ would be a relatively discrete countably infinite set whose closure is $\{p\}$ contrary to the fact that the closure is homeomorphic to βD. $\square$.

Lemma 35. *Every infinite Hausdorff space contains an infinite relatively discrete subset.*

Proof. Let X be an infinite Hausdorff space. Without loss of generality, assume X has no isolated points, and so every nonempty open subset of X is infinite. Choose $x_0 \neq x'_0 \in X$. Since X is Hausdorff, let $U_0, U'_0 \subseteq X$ be disjoint and open with $x_0 \in U_0$ and $x'_0 \in U'_0$ respectively. Inductively, for $\omega > i > 0$ since $U'_{i-1} \neq \emptyset$, it is infinite, so choose $x_i \neq x'_i \in U'_{i-1}$ and disjoint open $U_i, U'_i \subseteq U'_{i-1}$ with $x_i \in U_i$ and

$x_i' \in U_i'$ respectively. Let $Y = \{x_i : i < \omega\}$. Then since $\{U_i : i < \omega\}$ is a pairwise disjoint collection of open sets, Y is relatively discrete. □

The following says that closed subsets of $\beta\omega$ are either small (finite) or very big (cardinality $2^{\mathfrak{c}}$).

Theorem 36. *Every infinite closed subset of* $\beta\omega$ *(or* ω^**) contains a copy of* $\beta\omega$*.*

Proof. This theorem follows from Theorem 33 and Lemma 35. □

Lemma 37. *If* $A_0, A_1, A_2, ...$ *are infinite subsets of* ω *and* $A_{n+1} \subset^* A_n$ *for all* $n \in \omega$*, then there is an infinite* $A \subset \omega$ *such that* $A \subset^* A_n$ *for all* $n \in \omega$*.*

Proof. First, let us see that $\subset^*$ is transitive. Suppose $A \subset^* B$ and $B \subset^* C$, then $A \setminus C \subseteq [A \setminus B] \cup [B \setminus C]$ which is finite, and so $A \subset^* C$. Next, let us see that $\cap^n_{i=o} A_i$ is infinite for every $n < \omega$. $An \setminus \cap^{n-1}_{i=0} A_i \subseteq \cup^{n-1}_{i=0} [A_n \setminus A_i]$ which is finite by transitivity of $\subset^*$, and so $\cap^n_{i=0} A_i$ is infinite for every $n < \omega$. To finish off the lemma, Let $x_0 \in A_0$. By induction, for $n > 0$, $[\cap^n_{i=0} A_i] \setminus \{x_i : i < n\} \neq \emptyset$ by above, so pick $x_n \in [\cap^n_{i=0} A_i] \setminus \{x_i : i < n\}$. Let $A = \{x_n : n < \omega\}$, then A is infinite and for every $n < \omega$, $A \setminus A_n \subseteq \{x_i : i < n\}$ which is finite and hence $A \subset^* A_n$ for every $n < \omega$. □

Remark. An A satisfying the conclusion of Lemma 37 is sometimes called a *pseudo-intersection* of the A_n's.

Theorem 38. *Every non-empty* G_δ*-set in* ω^* *has non-empty interior.*

Proof. Let $U_0, U_1, U_2, \ldots$ be open sets in ω^* s.t. $\bigcap_{n\in\omega} U_n \neq \emptyset$. Let $p \in \bigcap_{n\in\omega} U_n$. Let A_0^* be a basic open set such that $p \in A_0^* \subseteq U_0$. Then $p \in A_0^* \cap U_1$. Let $A_1^* \subseteq A_0^* \cap U_1$. If $A_0^*, A_1^*, \ldots, A_n^*$ have been defined let A_{n+1}^* be a basic open set such that $p \in A_{n+1}^* \subseteq \left(\bigcap_{i=1}^{n} A_i^*\right) \bigcap U_{n+1}$. Then $A_0^* \supseteq A_1^* \supseteq \cdots$. So, $A_0 \supseteq^* A_1 \supseteq^* \cdots$. By Lemma 37, there is an infinite $A \subseteq \omega$ such that for all $n \in \omega A \subseteq^* A_n$. Therefore $(\forall n \in \omega)(A^* \subseteq A_n^*)$, hence $A^* \subseteq \bigcap U_n$. □

Definition. A point p in a space X is called a *$P-point$* if every G_δ-set containing p is a neighborhood of p. (Equivalently, the intersection of countably many neighborhoods of p is again a neighborhood of p.)

Theorem 39. *Assume the Continuum Hypothesis (CH). Then* ω^* *has a P-point.*

Proof. Assuming CH, $2^\omega = \omega_1$. Let $\{A_\alpha\}_{\alpha<\omega_1}$ be an enumeration of all infinite $A_\alpha \subset \omega$. Let $B_\alpha = \omega \setminus A_\alpha$. Define a collection of basic open sets $\{C_\alpha^*\}_{\alpha<\omega_1}$ as follows:

Let $C_0^* = A_0^*$. Assuming C_β^* has been defined for all $\beta < \alpha$ such that $\cap_{\beta<\alpha} C_\beta^* \neq \emptyset$, let C_α^* be a basic open set such that $C_\alpha^* \subset A_\alpha^* \cap (\cap_{\beta<\alpha} C_\beta^*)$ if this set is nonempty or $C_\alpha^* \subset B_\alpha^* \cap (\cap_{\beta<\alpha} C_\beta^*)$ otherwise. Note that since $A_\alpha \cup B_\alpha = \omega$, one of these is always possible.

$\{C_\alpha^*\}_{\alpha<\omega_1}$ is a decreasing sequence of basic open sets and since these sets are clopen, there is a point $p \in \cap_{\alpha<\omega_1} C_\alpha^*$. If there is a collection of open sets $\{U_i \subset \omega^* | p \in U_i\}_{i<\omega}$, choose $\alpha(i)$ such that $p \in A_{\alpha(i)}^* \subset U_i$ for every $i < \omega$. It follows that $p \in C_{\alpha(i)}^*$ for every $i < \omega$. Then if $\gamma = \sup_{i<\omega}\{\alpha(i)\}$, $p \in C_\gamma^* \subset \cap_{i<\omega} C_{\alpha(i)}^* \subset \cap_{i<\omega} U_i$, so $\cap_{i<\omega} U_i$ is a neighborhood of p and p is a P-point. □

Lemma 40. *Every infinite compact* T_2 *space has a non-P-point.*

Proof. Let X be a space as in the hypothesis. Then by Lemma 35 there exists a relatively discrete subset $Y = \{x_n : n < \omega\}$ of X. Now, $\cap_{n<\omega} X \setminus \{x_n\} = X \setminus Y$ is a G_δ-set. Since X is compact and Y is relatively discrete, there exists $x \in \overline{Y} \setminus Y$. And so $x \in X \setminus Y$ but x is not in the interior of $X \setminus Y$, and thus is not a $P - point$. □

Definition. A space X is said to be *homogeneous* if for any $x, y \in X$, there is a homeomorphism $h : X \to X$ such that $h(x) = y$.

Theorem 41. *Assume CH. Then* ω^* *is not homogeneous.*

Proof. Immediate from Theorem 39 and Lemma 40. □

Lemma 42. *Suppose* X, $Y \subset \beta\omega$ *and* $X \cap Y = \omega$. *Then* $X \times Y$ *is not countably compact.*

Proof. Look at $N = \{(n, n) : n \in \omega\}$. Now, $X \times Y \subseteq \beta\omega \times \beta\omega$. If N has a limit point, then that point will be on the diagonal, (since the diagonal in a T_2-space is closed). But (n, n) are isolated points. □

Definition. If $X \subset \mathbb{R}$, then to say that X is a *Bernstein set* means that if K is an uncountable closed set in $\mathbb{R}$, then X and $\mathbb{R} \setminus X$ meet K.

Lemma 42.1. *If* $H \subset \mathbb{R}$ *is uncountable and closed, then* $|H| = 2^\omega$

Proof. Let $\mathcal{U} = \{U \subset H\}$ such that U is relatively open and countable in H. Being a subset of $\mathbb{R}$, H has a countable basis and so there exists $\mathcal{V}$ a countable subset of $\mathcal{U}$ such that $\cup\mathcal{V} = \cup\mathcal{U}$. It follows that $\cup\mathcal{U} = \cup\mathcal{V}$ is countable and open in H; therefore, $H \setminus (\cup\mathcal{U})$ is a closed uncountable subset of $\mathbb{R}$ with no countable neighborhoods (and thus, no isolated points). Let $K = H \setminus (\cup\mathcal{U})$.

Note, that if K contains a nondegenerate interval in $\mathbb{R}$, then $|K| = 2^\omega$, so presume that K contains no such interval; hence, K has a base of clopen sets. Let $U(\emptyset)$ be a bounded clopen set in K and build a Cantor Tree $\{U(\sigma)\}_{\sigma \in 2^{<\omega}}$ of nonempty relatively clopen sets in K such that $\forall \sigma \in 2^{<\omega}$,

$$\begin{aligned} U(\sigma^\frown 0) \cup U(\sigma^\frown 1) &= U(\sigma) \\ U(\sigma^\frown 0) \cap U(\sigma^\frown 1) &= \emptyset \end{aligned}$$

If $f \in 2^\omega$, $\cap_{n\in\omega} U(f \restriction_n) \neq \emptyset$ because $\{U(f \restriction_n) : n \in \omega\}$ is a decreasing collection of nonempty closed (in K and $\mathbb{R}$) and bounded sets in $\mathbb{R}$, thus we can choose $x_f \in \cap_{n\in\omega} U(f \restriction_n)$. $\{x_f : f \in 2^\omega\} \subset K$ and has a one-to-one correspondence with 2^ω, therefore $|K| = 2^\omega$. □

Lemma 42.2. *There are* 2^ω *many closed uncountable subsets of* $\mathbb{R}$

Proof. $\mathbb{R}$ has a countable base, hence $|\mathcal{T}| = 2^\omega$, where $\mathcal{T}$ is the topology on $\mathbb{R}$. Because each open set corresponds to a single closed set, the collection of closed subsets of $\mathbb{R}$ also has cardinality 2^ω.

We know there are at least 2^ω uncountable closed subsets because $|\{[0, a] : a > 0\}| = 2^\omega$. □

Theorem 42.3. *There is a Bernstein set*

Proof. Let $\{H_\alpha\}_{\alpha \in 2^\omega}$ denote the collection of all closed uncountable subsets of $\mathbb{R}$. Inductively, choose $x_\alpha, y_\alpha \in H_\alpha \setminus \cup_{\beta<\alpha}\{x_\beta, y_\beta\}$ such that $x_\alpha \neq y_\alpha$ (we know that two such points exist because H_α has size 2^ω and $\cup_{\beta<\alpha}\{x_\beta, y_\beta\}$ has size less than $2^\emptyset$). Let $X = \{x_\alpha\}_{\alpha \in 2^\omega}$ and $Y = \{y_\alpha\}_{\alpha \in 2^\omega}$. By definition, $X \cap Y = \emptyset$, and if

$\alpha \in 2^\omega$, then both X and Y meet H_α; thus, $Y \subset \mathbb{R} \setminus X$ and both X and $\mathbb{R} \setminus X$ meet every uncountable closed subset of $\mathbb{R}$. ☐

Corollary 42.4. *$\mathbb{R}$ contains a non-measurable set.*

Proof. Let B be a Bernstein set and let $C = B \cap [0,1]$. Recall that if $A \subset [0,1]$, then the outer measure of A is defined as

$$\lambda(A) = inf(\{\lambda(U) : A \subset U,\ U \text{ is open in } [0,1]\}).$$

If U is open and $C \subset U$, then $[0,1] \setminus U$ misses C; thus, because $[0,1]$ is closed, and C is a closed subset of a Bernstein set and $[0,1]$, $[0,1] \setminus U$ is countable meaning $\lambda(U) = 1$.

Similarly, $[0,1] \setminus C$ is the subset of a Bernstein set $(\mathbb{R} \setminus B)$ and [0,1] so if U is open and contains $[0,1] \setminus C$, $\lambda(U) = 1$.

It follows that C and $[0,1]\setminus C$ are disjoint subsets of $[0,1]$ and $\lambda(C)+\lambda([0,1]\setminus C) = 2 > \lambda([0,1])$, therefore, C is not measurable. ☐

The *Michael line* X is the real line $\mathbb{R}$ with the irrationals isolated, and rationals having their usual Euclidean neighborhoods.

Example 42.5. *Let X be the Michael line, and let $\mathbb{P}$ be the irrationals with the usual Euclidean topology. Then X is a paracompact Hausdorff space whose product with the separable (complete) metric space $\mathbb{P}$ is not normal.*

Proof. It is easy to see that X is Hausdorff, that it has a base of clopen sets, and that these two facts imply X is regular. Thus, to show X is paracompact, it suffices (by a result in first year topology) to show that every open cover $\mathcal{U}$ of X has a refinement $\mathcal{V} = \bigcup_{n\in\omega} \mathcal{V}_n$, where each $\mathcal{V}_n$ is locally finite. Given $\mathcal{U}$, there are countably many members, say $V_1, V_2, \ldots$ such that $\mathbb{Q} \subset \bigcup_{n\in\omega\setminus\{0\}} V_n$. For $n > 0$, let $\mathcal{V}_n = \{V_n\}$, for let $\mathcal{V}_0 = \{\{x\} : x \in \mathbb{R} \setminus \bigcup_{n\in\omega\setminus\{0\}} V_n$. It is easy to check that each $\mathcal{V}_n$ is locally finite, and that $\mathcal{V} = \bigcup_{n\in\omega} \mathcal{V}_n$ refines $\mathcal{U}$. So X is paracompact.

To see that $X \times \mathbb{P}$ is not normal, let $H = \{(x,x) : x \in \mathbb{P}\}$ and $K = \mathbb{Q} \times \mathbb{P}$. It is straightforward to check that H and K are disjoint closed sets in $X \times \mathbb{P}$. Suppose U is an open set containing H. We'll show that $\overline{U} \cap K \neq \emptyset$, thus proving $X \times \mathbb{P}$ is not normal.

The proof uses the fact that $\mathbb{P}$ (with the usual topology) is Baire (being completely metrizable). For each $x \in \mathbb{P}$, there is some $n_x \in \omega$ such that $\{x\} \times (x - 1/n_x, x + 1/n_x) \subset U$. For each n, let $A_n = \{x \in \mathbb{P} : n_x = n\}$. By the Baire property, there are an integer k and an interval (a,b) such that A_k is dense in (a,b). Choose $q \in (a,b) \cap \mathbb{Q}$ and $p \in (a,b) \cap \mathbb{P}$ with $|q - p| < 1/k$. There are $p_n \in A_k$ with $p_n \to q$ and $|p_n - p| < 1/k$ for each n. Then the point $\langle p_n, p\rangle$ is in $\{p_n\} \times (p_n - 1/k, p_n + 1/k) \subset U$ for each n, while $\langle p_n, p\rangle \to \langle q, p\rangle \in K$. Thus $\overline{U} \cap K \neq \emptyset$. ☐

Example 42.6. *There is a regular Lindeloff space X, and a separable Baire metric space M, such that $X \times M$ is not normal.*

Proof. Let B be a Bernstein subset of $\mathbb{R}$. Define X to be the real numbers with points of B isolated and all other points having usual neighborhoods.

Claim 1. *X is regular Lindelof.*

Proof of Claim 1. X is regular, because B is dense in X and thus the set $\{(a,b) : a < b,\ a, b \in B\} \cup \{\{b\} : b \in B\}$ provides a base of clopen sets for X.

To show that X is Lindelof, let $\mathcal{U}$ be an open cover of X. If $x \in X \setminus B$, let $U_x \in \mathcal{U}$ that contains x and let O_x be a Euclidean open neighborhood of x such that $O_x \subset U_x$. $\cup\{O_x : x \in X \setminus B\}$ is open in the Euclidean sense, so we can choose $\mathcal{A}$, a countable subset of $\{O_x : x \in X \setminus B\}$, such that $\cup\mathcal{A} = \cup\{O_x : x \in X \setminus B\}$. $\cup A$ is an open set that contains $X \setminus B$, therefore $\mathbb{R} \setminus \cup\mathcal{A}$ is a closed subset of $\mathbb{R}$ as well as a subset of B. $\mathbb{R} \setminus \cup\mathcal{A}$ must be countable because no uncountable closed subset of $\mathbb{R}$ can be contained in B. Thus, $\{\{b\} : b \in \mathbb{R} \setminus \cup A\}$ is a countable collection and is in fact a collection of open sets (recall that $\{b\}$ is open in X if $b \in B$). So we have that $\mathcal{A} \cup \{\{b\} : b \in \mathbb{R} \setminus \cup\mathcal{A}\}$ is countable open refinement of $\mathcal{U}$ covering X.

Claim 2. *$X \times B$, with B having the Euclidean topology, is not normal*

Proof of Claim 2. The proof of this claim is similar to the proof that the Michael line crossed with the irrationals is not normal. All that must be shown is that B with Euclidean topology is a Baire space. See Theorem 70 for the proof of this. □

Definition. A subset A of the irrationals $\mathbb{P}$ is said to be *concentrated* about the rationals $\mathbb{Q}$ if every open superset of $\mathbb{Q}$ contains all but countably many points of A.

Lemma 42.7. *Assume the Continuum Hypothesis. Then there is an uncountable subset A of the irrationals $\mathbb{P}$ which is concentrated about the rationals $\mathbb{Q}$.*

Lemma 42.8. *Let A be an uncountable subset of the $\mathbb{P}$ which is concentrated about $\mathbb{Q}$, and let $X = \mathbb{Q} \cup A$ with points of A isolated. Then $X \times \mathbb{P}$ is not normal.*

Proof. Let $H = \{(a, a) | a \in A\}$ and $K = \mathbb{Q} \times \mathbb{P}$. Let U be an open set containing H. We show $\overline{U} \cap K \neq \emptyset$. For every $a \in A$, choose an $n_a \in \omega$ such that $\{a\} \times (a - \frac{1}{n_a}, a + \frac{1}{n_a}) \subset U$. Let $A_k = \{a \in A | n_a = k\} \subset \omega$, and fix a $k \in \omega$ such that A_k is uncountable. Since A is concentrated A_k has a limit point in $\mathbb{Q}$, for if not there would be an open set about $\mathbb{Q}$ missing uncountably many points in A. Choose $q \in \mathbb{Q}$ and $a_n \in A_k$ such that $a_n \to q$. Then there is some $p \in \mathbb{P}$ in $(q - \frac{1}{k}, q + \frac{1}{k})$ and for such a p, $(q, p) \in \overline{U} \cap K$. Thus $X \times \mathbb{P}$ is not normal. □

Corollary 42.9. *Assume the Continuum Hypothesis. Then there is a regular Lindelöf space X whose product with the irrationals is not normal.*

Proof. Let X be the space of Lemma 42.8, which by Lemma 42.7 exists assuming CH. It is easy to check that X is a regular Lindelöf space. □

Lemma 43.

(a) *There is a subset X of ω^* such that X and $\omega^* \setminus X$ meet every infinte closed subset of $w*$.*

(b) *If X is as in (a), then $\omega \cup X$ is countably compact.*

Proof.

(a) Note that $\{A* | A \subset \omega\}$ is a base for ω^* of cardinality 2^ω and so there are at most $2^{2^\omega} = 2^c$ many open subsets of ω^*. Thus if $\mathcal{H} = \{$ Infinite closed subsets of $\omega^*\}$, we may enumerate $\mathcal{H} = \{H_\alpha\}_{\alpha<\kappa}$ for some $\kappa \leq 2^c$.

Choose distinct $x_0, y_0 \in H_0$. Proceeding inductively, assume $\{x_\beta, y_\beta\}_{\beta<\alpha}$ are all distinct and $x_\beta, y_\beta \in H_\beta$ for every $\beta < \alpha$. By theorem 36, $|H_\alpha| = 2^c$ and since $|\cup_{\beta<\alpha}\{x_\beta, y_\beta\}| < 2^c$ we may choose distinct $x_\alpha, y_\alpha \in H_\alpha \setminus \cup_{\beta<\alpha}\{x_\beta, y_\beta\}$. Let $X = \{x_\alpha\}_{\alpha<\kappa}$ and $Y = \{y_\alpha\}_{\alpha<\kappa}$. Then $Y \subset \omega^* \setminus X$ and for every $\alpha < \kappa$, $X \cap H_\alpha$ and $Y \cap H_\alpha$ are nonempty.

(b) If $Y \subset X$ is infinite then there is some countably infinite relatively discrete $Z \subset Y$ by lemma 35. $\overline{Z}$ is infinite and closed and thus uncountable by theorem 36. Thus $Z' = \overline{Z} \setminus Z$ is also closed and infinite and so $X \cap Z' \neq \emptyset$. For any $p \in X \cap Z'$, p is a limit point of Z and thus also a limit point of Y in X. ☐

Theorem 44. *There are countably compact spaces X, $Y \subset \beta\omega$ such that $X \cap Y = \omega$ and hence $X \times Y$ is not countably compact.*

Proof. Immediate from lemma 42 and lemma 43. ☐

Theorem 45. *If X_n, $n \in \omega$, are sequentially compact, so is $\Pi_{n\in\omega} X_n$.*

Proof. This follows immediately from Theorem 47 since $\omega < \omega_1 \leq \mathfrak{t}$. ☐

Definition. Let α be an ordinal. A sequence A_β, $\beta < \alpha$, of distinct infinite subsets of ω is called a (decreasing) *tower* of length α if $\beta < \gamma < \alpha$ implies $A_\gamma \subset^* A_\beta$. If an infinite set A has the property that $A \subset^* A_\beta$ for every $\beta < \alpha$, we call A a *pseudo-intersection* of the tower $\{A_\beta : \beta < \alpha\}$. We then define $\mathfrak{t}$ to be least cardinal κ such that there is a tower of length κ with no infinite pseudo-intersection.

Obviously, $\mathfrak{t} \leq \mathfrak{c}$. It follows from Lemma 37 that $\mathfrak{t} \geq \omega_1$.

Theorem 46. *Assume Martin's Axiom. Then* $\mathfrak{t} = \mathfrak{c}$

Proof. Assume Martin's Axiom. It has been shown in Lemma 37 that $\mathfrak{t} > \omega$. Clearly $\mathfrak{t} \leq \mathfrak{c}$. We will show that $\mathfrak{t} = \mathfrak{c}$, by showing that any tower of length $\kappa \in \mathfrak{c}$ has a pseudo-intersection.

Suppose $\kappa \in \mathfrak{c}$ is an infinite ordinal. Then MA(κ). Let $A_1 \supseteq^* A_2 \supseteq^* \cdots$ is a tower of length κ. Let $\mathcal{P} = \{(\overline{c}, F) : \overline{c}$ is a finite sequence of distinct elements in ω, $F \subseteq \kappa$ is finite$\}$. Define a relation $\leq$ on $\mathcal{P}$ in the following manner. For any pair of elements $(\overline{c}, F), (\overline{d}, G) \in \mathcal{P}$ we say $(\overline{c}, F) \leq (\overline{d}, G)$ iff the following three properties are satisfied:

(i) $\overline{c}$ extends $\overline{d}$
(ii) $G \subseteq F$
(iii) If $i \in (\text{dom } \overline{c}) \setminus (\text{dom } \overline{d})$ then $c_i \in A_\alpha$ for all $\alpha \in G$.

Claim: $(\mathcal{P}, \leq)$ is a ccc partially ordered set. Antisymmetry and reflexivity of $\leq$ are clear. To show transitivity: Suppose $(\overline{c}, F) \leq (\overline{c'}, F') \leq (\overline{c}'', F'')$. Note: $\overline{c}$ extends $\overline{c}''$ and $F \supseteq F''$. Suppose $i \in (\text{dom } \overline{c}) \setminus (\text{dom } \overline{c}'')$. If $i \in \text{dom } \overline{c}'$ then since $(\overline{c}', F') \leq (\overline{c}'', F'')$ we have that $c_i \in A_\alpha$ for all $\alpha \in F''$. If $i \notin \text{dom } \overline{c}'$ then since $(\overline{c}, F) \leq (\overline{c}', F')$, we have that $c_i \in A_\alpha$ for all $\alpha \in F' \supseteq F''$. Hence $\leq$ is transitive and a partial order. To show that it's ccc, suppose $\{(\overline{c}_\alpha, F_\alpha)\}_{\alpha\in\Lambda}$ is an uncountable collection of elements from $\mathcal{P}$. Since there are only a countable many finite sequences of ω, it follows that there is a $\beta, \gamma \in \mathcal{L}$ such that $\beta \neq \gamma$ and $\overline{c}_\beta = \overline{c}_\gamma$. The element $(\overline{c}_\beta, F_\beta \cup F_\gamma)$ is an element in $\mathcal{P}$ which is less than both $(\overline{c}_\gamma, F_\gamma)$ and $(\overline{c}_\beta, F_\beta)$. It follows that the uncountable collection can't be an anti-chain.

Define the following families of subsets of $\mathcal{P}$. For each $i \in \omega$, let $D_i = \{(\overline{c}, F) : i \in \text{dom } \overline{c}\}$. For each $\alpha \in \kappa$, let $V_\alpha = \{(\overline{c}, F) : \alpha \in F\}$. We will show that each of these is dense. Let $(\overline{c}, F) \in \mathcal{P}$, and let $\alpha \in k$. Then $(\overline{c}, F \cup \{\alpha\}) \in V_\alpha$ and $(\overline{c}, F \cup \{\alpha\}) \leq (\overline{c}, F)$. It follows that for every $\alpha \in \kappa$ V_a is dense. Let $(\overline{c}, F) \in \mathcal{P}$ and $i \in \omega$. If $i \notin \text{dom } \overline{c}$ then extend $\overline{c}$ be choosing distinct elements from the infinite set $\bigcup_{\alpha\in F} A_\alpha$, call such an extension $\overline{c}'$. Then $(\overline{c}', F) \leq (\overline{c}, F)$ and $(\overline{c}', F) \in D_i$. It follows that for all $i \in \omega$, D_i is dense. Note: the collection of all D_i's and V_α's is a collection of $\leq \kappa$ many dense sets. Let $\mathcal{F}$ be a filter in $(\mathcal{P}, \leq)$ which intersection each of them.

Note: If $(\overline{c}, F), (\overline{d}, G) \in \mathcal{F}$ then either $\overline{c}$ is an extension of $\overline{d}$ or $\overline{d}$ is an extension of $\overline{c}$ for the following reason; Since $\mathcal{F}$ is a filter there is an element $(\overline{z}, K)$ which is less than both... hence the sequence $\overline{z}$ extends both $\overline{d}$ and $\overline{c}$, so one must be an initial segment of the other. Define a sequence as follows: For each $i \in \omega$ let $(\overline{c}, F) \in D_i \cap \mathcal{F}$ and define x_i to be the i^{th} term in $\overline{c}$. Such a sequence is well-defined by the previous remark.

Claim: $\{x_i : i \in \omega\}$ *is a pseudo-intersection of the tower.* This set is clearly infinite, since each finite sequence consisted of distinct elements in ω. We will show that for any $\alpha \in \kappa$ the sequence $\{x_i\}$ is eventually in A_α. Let $\alpha \in \kappa$. Let $(\overline{b}, G) \in V_\alpha \cap \mathcal{F}$. Let $N = |\text{dom } \overline{b}|$. Let $(\overline{c}, F) \in D_{N+1} \cap \mathcal{F}$. Let $(\overline{a}, K)$ be an element less than both $(\overline{c}, F)$ and $(\overline{b}, G)$. Then for any $i > N$ $i \in \text{dom } \overline{a} \setminus \text{ dom } \overline{b}$ so $c_i \in A_\gamma$ for all $\gamma \in G$. In particular, since $\alpha \in G$, $c_i \in A_\alpha$. It follows that the tail of $\{x_i\}$ is in A_α. So $\{x_i\}$ is almost contained in each member in the tower, hence the tower has a pseudo-intersection.

It follows that $\mathfrak{t} \neq \kappa$. Thus $\mathfrak{t} = \mathfrak{c}$. $\square$

The following is a strengthening of Theorem 45.

Theorem 47. *The product of fewer than* $\mathfrak{t}$*-many sequentially compact spaces is sequentially compact, and the product of* $\mathfrak{t}$*-many sequentially compact spaces is at least countably compact.*

Proof. Let $\kappa \leq \mathfrak{t}$. For $i < \omega$ let $\mathbf{x}_i = (x_{i\alpha}) \in \Pi_{\alpha<\kappa} X_\alpha$. By sequential compactness of X_0 there is some infinite $A_0 \subset \omega$ such that $(x_{i0})_{i \in A_0} \to y_0$ for some y_0 in X_0. Similarly by sequential compactness of X_1, there is some infinite $A_1 \subset A_0$ such that $(x_{i1})_{i \in A_1} \to y_1$ for some y_1 in X_1. Consequentially, $(x_{i0})_{i \in A_1} \to y_0$ as well.

Proceeding by transfinite induction, assume A_β has been defined for all $\beta < \alpha < \kappa$ such that $A_\gamma \subset^* A_\beta$ when $\beta < \gamma < \alpha$ and $(x_{i\beta})_{i \in A_\beta} \to y_\beta \in X_\beta$ (and consequentially, $(x_{i\beta})_{i \in A_\gamma} \to y_\beta$). If $\alpha = \gamma + 1$ then since A_γ is infinite, by sequential compactness of X_α there is an infinite $A_\alpha \subset A_\gamma$ such that $(x_{i\alpha})_{i \in A_\alpha} \to y_\alpha$ for some $y_\alpha \in X_\alpha$. If $\alpha < \mathfrak{t}$ is a limit ordinal then by definition there is some infinite $A \subset^* A_\beta$ for every $\beta < \alpha$ and by sequential compactness of X_α there is an infinite $A_\alpha \subset A$ such that $(x_{i\alpha})_{i \in A_\alpha} \to y_\alpha$ for some $y_\alpha \in X_\alpha$. This defines A_α for all $\alpha < \kappa$.

If $\kappa < \mathfrak{t}$ then as above, there is some $A \subset^* A_\alpha$ for all $\alpha < \kappa$. Thus if $\mathbf{y} = (y_\alpha)$, $(\mathbf{x}_i)_{i \in A} \to \mathbf{y}$ and so $\Pi_{\alpha<\kappa} X_\alpha$ is sequentially compact. If $\kappa = \mathfrak{t}$, let O be a basic open neighborhood of $\mathbf{y}$. Then $O = \Pi_{\alpha \in F} O_\alpha \times \Pi_{\alpha \notin F} X_\alpha$ for some open neighborhood $O_\alpha \subset X_\alpha$ of y_α and finite $F \subset \kappa$. Then $\sup F < \mathfrak{t}$ and so there is an infinite $A \subset \omega$ such that $(x_{i\alpha})_{i \in A} \to y_\alpha$ for all $\alpha \leq \sup F$. Thus O contains some (in fact, infinitely many) $\mathbf{x}_i$ and so $\mathbf{y}$ is a limit point of $\{\mathbf{x}_i | i < \omega\}$. Thus $\Pi_{\alpha<\kappa} X_\alpha$ is countably compact. $\square$

In particular, since $\mathfrak{t} \geq \omega_1$, the product of ω_1-many sequentially compact spaces is always countably compact. The following related problem is a well-known unsolved problem (the "Scarborough-Stone problem"):

Problem. *Is the product of any family of regular sequentially compact spaces countably compact?*

The regular assumption is in there because there is known to be a collection of Hausdorff sequentially compact spaces whose product is not countably compact.

Definition. A subset F of a space X is called a *zero-set* in X if $F = g^{-1}(0)$ for some continuous $g : X \to \mathbb{R}$. (It is equivalent to say, instead of $F = g^{-1}(0)$, that $F = g^{-1}(C)$ for some closed $C \subset \mathbb{R}$.) The complement of a zero-set is called a *cozero-set*.

Remark. In a perfectly normal space, every closed set is a zero-set (by a theorem from first-year topology–see Theorem 42 in my MH 7500 theorem list).

Lemma 48.

(a) *Each point of a completely regular space has a neighborhood base of zero-sets and an open neighborhood base of co-zero sets;*
(b) *Every zero-set F in X is closed and a regular G_δ set;*
(c) *In a normal space, the zero-sets are precisely the closed G_δ sets.*

Proof. (a) Suppose X is completely regular and $x \in X$. For every closed set C such that $x \notin C$, let $f_c : X \to [0,1]$ be continuous such that $f_c(x) = 0$ and $f_c(C) \subseteq \{1\}$. We will show the collection $\mathcal{Z} = \{f_c^{-1}([-1/2,1/2]) : x \notin C$ closed $\}$ is a base at x. Let V be open and $x \in V$. Let $K = X \setminus V$, then $x \in f_K^{-1}((-1/2,1/2) \subseteq f_k^{-1}([-1/2,1/2])$. Since $f_K^{-1}([-1/2,1/2]) \cap (X \setminus V) \subseteq f_K^{-1}([-1/2,1/2]) \cap f^{-1}(1) = \emptyset$, it follows that $x \in f_K^{-1}([-1/2,1/2]) \subseteq V$. This also shows that $\{f_C^{-1}((-1/2,1/2)) :$ $x \notin C$ closed $\}$ is an open neighborhood base of co-zero sets.

(b) Suppose F is a zero set. Let $f : X \to \mathbb{R}$ such that $f^{-1}(0) = F$. For each $i \in \omega$ let $U_i = f^{-1}((-1/i,1/i))$. Clearly $F \subseteq \bigcap_{i\in\omega} U_i \subseteq \bigcap_{i\in\omega} \overline{U_i}$. To show the other inclusion, suppose $x \notin F$. Then $f(x) > 0$ and there is an i such that $x \notin f^{-1}([-1/i,1/i]) \supseteq \overline{U_i} \supseteq U_i$. Thus $\bigcap_{i\in\omega} U_i \subseteq \bigcap_{i\in\omega} \overline{U_i} \subseteq F$.

(c) By (b), every zero-set is closed and G_δ. Suppose X is normal, and F is closed G_δ. Say $F = \bigcap_{n\in\omega} U_n$, where U_n is open. Let $f_n : X \to [0,1]$ be continuous such that $F \subset f_n^{-1}(0)$ and $X \setminus U_n \subset f_n^{-1}(1)$. Let $f = \Sigma_{n\in\omega} f_n/2^n$. Then f is continuous and it is easy to check that $F = f^{-1}(0)$. So F is a zero-set. ☐

In the following results when relevant, we always assume $X \subset \beta X$.

Lemma 49

(a) If F, G and zero-sets, so is $F \cup G$;
(b) If F_n is a zero-set for each $n \in \omega$, then $F = \cap_{n\in\omega} F_n$ is a zero-set;
(c) If F is a zero-set in X and $Y \subset X$, then $F \cap Y$ is a zero-set in Y;
(d) If F and G are disjoint zero-sets, then there is a continuous $f : X \to [0,1]$ with $F = f^{-1}(0)$ and $G = f^{-1}(1)$;
(e) Disjoint zero-sets in X have disjoint closures in βX.

Proof. Observe that if F is a zero-set in X then we can assume that there is an $f : X \to \mathbb{R}$ such that $F = f^{-1}(0)$ and that $f(X) \subseteq [0,c)$ for any $c > 0$.

(a) If $F = f^{-1}(0)$ and $G = g^{-1}(0)$ where $f, g : X \to \mathbb{R}$ are continuous, then fg is continuous on X and $F \cup G = (fg)^{-1}(0)$

(b) Assume $f_n : X \to \mathbb{R}$ is continuous with $f_n(X) \subseteq [0, 2^{-n})$ for every $n \in \omega$ and $F_n = f_n^{-1}(0)$ for every $n \in \omega$, then $f = \sum_{n\in\omega} f_n$ exists and is continuous on X and $\cap_{n\in\omega} F_n = f^{-1}(0)$.

(c) Let f be continuous such that $F = f^{-1}(0)$. Then if $Y \subseteq X$, $f|_Y : Y \to \mathbb{R}$ is continuous and $F \cap Y = (f|_Y)^{-1}(0)$

(d) Let $f, g : X \to \mathbb{R}$ be continuous with $F = f^{-1}(0)$ and $G = g^{-1}(0)$. Define $h = \frac{f}{f+g}$, then since $F \cap G = \emptyset$, h is defined everywhere on X and is continuous.

It is also clear that $h(x) = 0$ iff $x \in F$ and $h(x) = 1$ iff $x \in G$, so $F = h^{-1}(0)$ and $G = h^{-1}(1)$. (SORRY for not writing up your function James)

(e) Let F, G be disjoint zero-sets in X and h be as in (d). Then there exists $H : \beta X \to \mathbb{R}$ extending h and so $F \subseteq H^{-1}(0)$ and $G \subseteq H^{-1}(1)$, thus $\overline{F}^{\beta X} \cap \overline{G}^{\beta X} \subseteq \overline{H^{-1}(0)} \cap \overline{H^{-1}(1)} = H^{-1}(0) \cap H^{-1}(1) = \emptyset$. □

Definition. A *z-filter* in X is a collection $\mathcal{F}$ of zero-sets satisfying:

(i) $\emptyset \notin \mathcal{F}$;
(ii) $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \in \mathcal{F}$;
(iii) If $F \in \mathcal{F}$ and G is a zero-set containing F, then $G \in \mathcal{F}$.

A maximal z-filter is called a *z-ultrafilter.*

Theorem 50. *Let X be a completely regular space, and for each $p \in X^*$, let*

$$\mathcal{F}_p = \{F \subset X : F \text{ is a zero set in } X \text{ and } p \in \overline{F}^{\beta X}\},$$

Then:

(a) $\mathcal{F}_p$ is a z-ultrafilter in X;
(b) If F is a zero set in X, then $\overline{F}^{\beta X} = F \cup \{p \in X^* : F \in \mathcal{F}_p\}$;
(c) The collection $\{\beta X \setminus \overline{F}^{\beta X} : F$ is a zero set in $X\}$ is a base for the topology of βX;
(d) For each cozero set U in X, let

$$Ex(U) = U \cup \{p \in X^* : \exists F \in \mathcal{F}_p (F \subset U)\},$$

then $Ex(U) = \beta X \setminus \overline{(X \setminus U)}^{\beta X}$, and $\{ExU : U \subset X,\ U$ is cozero$\}$ forms a base for βX.
(e) For a cozero subset U of X, $Ex\,U$ is the largest open subset of βX whose intersection with X equals U.

Proof. (a) First check that $\mathcal{F}_p$ is a z-filter in X. It is clear that conditions (i) and (iii) are satisfied. To see (ii) is satisfied, let $F_1, F_2 \in \mathcal{F}_p$. By 49(b) F is a zero-set in X. Suppose $p \notin \overline{F_1 \cap F_2}^{\beta X}$, then by 48(a) let Z be a zero-set neighborhood of p in βX with $Z \cap \overline{F_1 \cap F_2}^{\beta X} = \emptyset$. By 49(C) $Z \cap X$ is a nonempty zero-set in X and we have $Z \cap X \cap F_1 \cap F_2 = (Z \cap F_1) \cap (Z \cap F_2) = \emptyset$ each of which are zero sets in X by 49(b). However, $p \in \overline{Z \cap F_1}^{\beta X} \cap \overline{Z \cap F_2}^{\beta X}$, a contradiction and so $p \in \overline{F_1 \cap F_2}^{\beta X}$ hence $F_1 \cap F_2$ is in $\mathcal{F}_p$ and $\mathcal{F}_p$ is a z-filter. Now to see that $\mathcal{F}_p$ is maximal, let $\mathcal{F}$ be filter with $\mathcal{F}_p \subset \mathcal{F}$ and $G \in \mathcal{F}$. Let U be open in βX with $p \in U$ and by 48(a) let F' be a zero-set neighborhood of p contained in U. Then $F = F' \cap X \neq \emptyset$ is a zero-set in X and is clearly in $\mathcal{F}_p$ and so is in $\mathcal{F}$. So $\emptyset \neq F \cap G \subset U \cap G$, i.e. $p \in \overline{G}^{\beta X}$ and $G \in \mathcal{F}_p$. Hence $\mathcal{G} \subset \mathcal{F}_p$ and $\mathcal{F}_p$ is a z-ultrafilter on X.

(b) $\overline{F}^{\beta X} = F \cup \{p \in \beta X \setminus X : p \in \overline{F}^{\beta X}\} = F \cup \{p \in \beta X \setminus X : F \in \mathcal{F}_p\}$.

(d) Let U be cozero, and let $U' = \beta X \setminus \overline{X \setminus U}^{\beta X}$. If $p \in Ex(U)$ ($p \notin X$), then there is $F \in \mathcal{F}_p$ such that $F \subset U$. $X \setminus U$is a zero set (because U is cozero), thus F and $X \setminus U$ are disjoint zero sets in X, which means $\overline{F}^{\beta X} \cap \overline{X \setminus U}^{\beta X} = \emptyset$. Hence, $p \in U'$. $\therefore Ex(U) \subset U'$.

If $p \in U'$ then there is F', a zero set neighborhood of p in βX such that $F' \subset U'$ (By Theorem 48). Let $F = F' \cap X$; hence, F is a zero set in X (by Theorem 48)

and $\overline{F}^{\beta X}$ contains p because $F = F' \cap X$ is dense in $Int_{\beta X}(X)$ and $p \in Int_{\beta X}(X)$. Lastly, $F \subset X \cap U' \subset X \setminus \overline{X \setminus U} \subset U$, thus $p \in Ex(U)$. $\therefore U' \subset Ex(U)$
$\therefore Ex(U) = U'$.

Because $\{\beta X \setminus \overline{X \setminus U}^{\beta X} : U$is cozero in $X\} = \{\beta X \setminus \overline{F}^{\beta X} : F$ is a zero-set in X$\}$ it follows from part (c) that this forms a base for βX.

An alternative proof for part (d) not requiring cozero property of U (ie $Ex(U) = \beta X \setminus \overline{(X \setminus X)}^{\beta X} \forall U(U$ open in X).

(d) Again, let $U' = \beta X \setminus \overline{(X \setminus U)}^{\beta X}$. If $p \in U'$, then there is F, a zero neighborhood of p such that $F \subset U'$. Let $F' = F \cap X$, then F' is a zero-set in X, and

$$F' \subset X \cap U' \subset X \setminus \overline{(X \setminus U)} \subset U.$$

Because $p \in Int_{\beta X}(F)$ and F' is dense in $Int_{\beta X}(F)$, $\overline{F'}^{\beta X}$ contains p, which means $F' \in \mathcal{F}_p$. $\therefore U' \subset Ex(U)$.

If $p \in Ex(U)$, there is $F \in \mathcal{F}_p$ such that $F \subset U$. Let $f : X \to [0,1]$ such that $f^{-1}(0) = F$. $X \setminus U$ is closed in X and $f(x) > 0 \forall x \in X \setminus U$, thus $glb(f(X \setminus U)) > 0$

$$\Rightarrow glb(\pi_f(\overline{X \setminus U}^{\beta X})) = glb(f(X \setminus U)) > 0 = \pi_f(\{p\}),$$

which means $p \notin \overline{X \setminus U}^{\beta X} \Rightarrow p \in U'$. $\therefore Ex(U) \subset U'$.
$\therefore Ex(U) = U'$.

(e) If U is a cozero subset of X and V is an open subset of βX such that $V \supsetneq$ ExU then by part d, $V \supsetneq \beta X \setminus \overline{(X \setminus U)}^{\beta X}$. Taking complements, $\beta X \setminus V \subseteq \overline{X \setminus U}^{\beta X}$ and so since $\beta X \setminus V$ is closed, $\beta X \setminus V \not\supset X \setminus U$. Thus there exists some $x \in X \setminus U$ such that $x \notin \beta X \setminus V$ and so $x \in V$. Thus $V \cap X \neq U$. □

Definition. A *continuum* is a compact connected Hausdorff space. A continuum is *indecomposable* if it cannot be written as the union of two proper subcontinua.

Remark. Many topologists include the adjective “metrizable” in the definition of a continuum. The Knaster continuum, or buckethandle continuum, constructed from the Cantor set by joining certain pairs of points with semicircles, is an example of an indecomposable continuum in the plane.

Theorem 51. *A continuum is indecomposable if and only if every proper subcontinuum is nowhere dense.*

Proof. First, show the if direction by contraposition. Assume X is a decomposable continuum and that $X = H \cup K$ where H and K are proper subcontinua of X. Then $\emptyset \neq X \setminus K \subset H$, hence H is not nowhere dense in X. Now to show the only if direction, by way of contradiction, assume $H \subset X$ is a proper subcontinuum of X with $int(H) \neq \emptyset$. Since X is indecomposable, $X \setminus int(H)$ is not connected, so let $X \setminus int(H) = M \cup N$ where M and N are mutually separated. Note that each M and N are closed and so $H \cup M$ and $H \cup N$ are closed and $X = (H \cup M) \cup (H \cup N)$. Look at each connected component of M in $X \setminus int(H)$. By the ”To the Boundary” theorem, each of these components intersect $bd(int(H)) \subset H$ and so $H \cup M$ is the union of connected subsets each having nonempty intersection with H which is connected and so $H \cup M$ is connected, hence a subcontinuum of X. Similarly, $H \cup N$ is a subcontinuum of X, a contradiction, hence every subcontinuum is nowhere dense. □

Let $\mathbb{H}$ be the half-line $[0, \infty)$. By Theorem 23, $\mathbb{H}^* = \beta\mathbb{H} \setminus \mathbb{H}$ is a continuum. Our goal now is to prove that it is indecomposable.

Lemma 52. *Let* O *be open in* $\mathbb{H}$ *such that* $0 \notin O$ *and* $\mathbb{H} \setminus \overline{O}$ *is unbounded. Then there are* $a_n, b_n \in \mathbb{H}$ *with* $a_0 < b_0 < a_1 < b_1 < a_2 < \dots$ *such that* $O \subset \bigcup_{n\in\omega}(a_n, b_n)$.

Proof. This is clear if O is bounded, so assume that O is unbounded. Let $a_0 = 0$, $z_0 \in O$ such that $z_0 > a_0$. Pick $c_0 \in \mathbb{H} \setminus \overline{O}$ such that $c_0 > \max\{z_0, 1\}$. Since $c_0 \in \mathbb{H} \setminus \overline{O}$ there is an open interval (b_0, a_1) such that $c_0 \in (b_0, a_1) \subset \mathbb{H} \setminus \overline{O}$. Proceeding inductively, choose $z_n \in O$ with $z_n > a_n$, $c_n \in \mathbb{H} \setminus \overline{O}$ with $c_n > \max\{z_n, n+1\}$, and $b_n, a_{n+1} \in \mathbb{H} \setminus \overline{O}$ with $c_n \in (b_n, a_{n+1}) \subset \mathbb{H} \setminus \overline{O}$. Then $O \subset \bigcup_{n\in\omega}(a_n, b_n)$. $\square$

Lemma 53. *Let* K *be a proper subcontinuum of* $\mathbb{H}^*$. *Then there are* $a_n,\ b_n \in \mathbb{H}$ *with* $a_n < b_n < a_{n+1}$ *for all* n *such that:*

(a) *If* $O = \bigcup_{n\in\omega}(a_n, b_n)$, *then* $K \subset$ *Ex* O;

(b) *If* $\mathcal{U} = \{A \subset \omega :\ K \subset cl_{\beta\mathbb{H}}(\bigcup_{n\in A}[a_n, b_n])\}$, *then* $\mathcal{U}$ *is an ultrafilter on* ω.

Proof. (a). If $O = \displaystyle\bigcup_{n\in\omega}(a_n, b_n)$ then $K \subset$Ex O. Let $x \in \mathbb{H}^* \smallsetminus K$ and U, V open in $\beta\mathbb{H}$ such that $K \subset U$, $x \in V$ and $\overline{U} \cap \overline{V} = \emptyset$. Let $U' = U \cap \mathbb{H}$ and $V' = V \cap \mathbb{H}$. Now V' is dense in V, so $\overline{V'}^{\beta\mathbb{H}} = \overline{V}^{\beta\mathbb{H}}$, which implies $\overline{V'}^{\beta\mathbb{H}} \cap \mathbb{H}^* \neq \emptyset$, and hence V' is unbounded in $\mathbb{H}$.
Now $V \subset \beta\mathbb{H} \smallsetminus \overline{U}^{\beta\mathbb{H}}$, so $V' \subset \mathbb{H} \smallsetminus \overline{U'}^{\mathbb{H}}$, which implies $\mathbb{H} \smallsetminus \overline{U'}^{\mathbb{H}}$ is unbounded in $\mathbb{H}$. WLOG assume $O \notin U$. By Lemma 52, there exists $a'_0 < b'_o < a'_1 < \dots$ such that $U' \subset \displaystyle\bigcup_{n\in\omega}(a'_n, b'_n)$. Set $O = \displaystyle\bigcup_{n\in\omega}(a'_n, b'_n)$. Then $U \subset$ Ex (O) (by lemma 50e.), and hence $K \subset$ Ex (O).

(b). If $A \subset \omega$, the define $C_A = \displaystyle\bigcup n \in A[a_n, b_n]$. Then show $\mathcal{U} = \{A \subset \omega : K \subset \overline{C_A}^{\beta\mathbb{H}}\}$ is an ultrafilter.

(i.) $\emptyset \notin \mathcal{U}$ This is clear.

(ii.) $\mathcal{U}$ is closed under supersets. This is also clear.

(iii.) $\mathcal{U}$ is closed under intersections.

Suppose A and B are elements of $\mathcal{U}$. Since $(A \smallsetminus B) \cup (A \cap B) = A$, then $K \subset \overline{C_{A\smallsetminus B}}^{\beta\mathbb{H}} \cup \overline{C_{A\cap B}}^{\beta\mathbb{H}}$. So K must be in one or the other since $C_{A\smallsetminus B}$ and $C_{A\cap B}$ are closed and disjoint. (Therefore $\overline{C_{A\smallsetminus B}}^{\beta\mathbb{H}} \cap \overline{C_{A\cap B}}^{\beta\mathbb{H}} = \emptyset$. K is connected, so K is in exactly one of them.)
If $K \subset \overline{C_{A\smallsetminus B}}$, then $K \not\subset \overline{C_B}$, becuase $C_{A\smallsetminus B}$, and C_B are closed disjoint, hence $B \notin \mathcal{U}$, a contradiction. So therefore $K \subset \overline{C_{A\cap B}}^{\beta\mathbb{H}}$, and therefore $A \cap B \in \mathcal{U}$.

(iv.) $\mathcal{U}$ is an ultrafilter.
If $A \subset \omega$, then $K \subset \overline{C_A}^{\beta\mathbb{H}} \cup \overline{C_{\omega\smallsetminus A}}^{\beta\mathbb{H}} = \overline{O}^{\beta\mathbb{H}}$. Because C_A and $C_{\omega\smallsetminus A}$ are closed disjoint, then $\overline{C_A}^{\beta\mathbb{H}} \cap \overline{C_{\omega\smallsetminus A}}^{\beta\mathbb{H}} = \emptyset$. So K is in exactly one, because

K is connected, so for every $A \subset \omega$ either $A \in \mathcal{U}$ or $\omega \smallsetminus A \in \mathcal{U}$. Hence $\mathcal{U}$ is an ultrafilter.

Theorem 54. $\mathbb{H}^*$ *is indecompossable.*

Proof. To show $\mathbb{H}^*$ is indecompossable it will be shown that each proper subcontinuum of $\mathbb{H}^*$ has empty interior.

Suppose that K is a proper subcontinuum of $\mathbb{H}^*$. Recall from Theorem 52 that it may be supposed for each $i \in \omega$ that $a_i, b_i \in \mathbb{H}$ such that $a_i < b_i < a_{i+1}$ and $\mathcal{O} = \cup_{i\in\omega}(a_i, b_i)$ is a cozero set in $\mathbb{H}$ such that $K \subset Ex(\mathcal{O})$. For each $A \subset \omega$, let $C_A = \cup_{i\in A}[a_i, b_i]$. From Theorem 53, it is known that $\mathcal{U} = \{A \subset \omega : K \subset \overline{C_A}^{\beta\mathbb{H}}\}$ is a free ultra filter on ω.

Because $\overline{\mathcal{O}} = \cup_{i\in\omega}[a_i, b_i]$ is closed in H, $\overline{\mathcal{O}}^{\beta\mathbb{H}}$ is essentially $\beta\overline{\mathcal{O}}$; because $K \subset Ex(O) \setminus O \subset \overline{\mathcal{O}}^*$ and $Ex(O) \setminus O$ is open in $\mathbb{H}^*$, it follows that $Int_{\overline{\mathcal{O}}^*}(K) = Int_{\mathbb{H}^*}K$). Therefore we may restrict our focus to $\beta\overline{\mathcal{O}}$ and view K as a subcontinuum of $\overline{\mathcal{O}}^*$

It will be shown that if the interior of K is not empty, then $\beta\overline{\mathcal{O}}$ is not regular, and thus not T_2 and compact.

Suppose that $Int_{\overline{\mathcal{O}}^*}(K) \neq \emptyset$ and let $p \in Int_{\overline{\mathcal{O}}^*}$. Let $Q = \overline{\mathcal{O}}^* \setminus Int_{\overline{\mathcal{O}}^*}(K)$. $\overline{\mathcal{O}}^*$ is closed in $\beta\overline{\mathcal{O}}$ and Q is closed in $\overline{\mathcal{O}}^*$, therefore, Q is closed in $\beta\overline{\mathcal{O}}$ and $p \notin Q$. Choose U and V to be sets open in $\beta\overline{\mathcal{O}}$ such that $p \in U$ and $Q \subset V$.

Let $A = \{i \in \omega : U \cap [a_i, b_i] \neq \emptyset\}$.

Claim 1: A is unbounded. $U \cap \mathbb{H} \subset C_A \Rightarrow Ex(U) \subset \beta C_A$. $p \in U \subset \beta C_A \Rightarrow \beta C_A \cap \mathbb{H}^* \neq \emptyset \Rightarrow C_A$ is unbounded in $\mathbb{H} \Rightarrow A$ is unbounded in ω. □

Claim 2: There is $B \subset \omega$ such that $B \notin \mathcal{U}$ and $B \cap A$ is infinite. Let A' and B be two infinite sets that partition A. $A' \cup B = A \in \mathcal{U}$ and $\mathcal{U}$ is an ultrafilter, which means $A' \in \mathcal{U}$ or $B \in \mathcal{U}$. Without loss of generality, assume $A' \in \mathcal{U}$. Because $A' \cap B = \emptyset$, $B \notin \mathcal{U}$, hence, $B \subset \omega$ such that $B \notin \mathcal{U}$ and $B \cap A = A$ is infinite. □

Let B be as described in Claim 2. Because $\mathcal{U}$ is an u.f. and $B \notin \mathcal{U}$, $\omega \setminus B \in \mathcal{U}$ $\Rightarrow K \subset \overline{C_{\omega\setminus B}}^{\beta\overline{\mathcal{O}}} \Rightarrow K \cap \overline{C_B}^{\beta\overline{\mathcal{O}}} = \emptyset$

$\Rightarrow \overline{C_B}^{\beta\overline{\mathcal{O}}} \cap \overline{\mathcal{O}}^* \subset \overline{\mathcal{O}}^* \setminus K \subset Q$. For each $i \in B$, $U \cap [a_i, b_i] \neq \emptyset \Rightarrow U \cap C_B$ is unbounded $\Rightarrow \overline{U \cap C_B}^{\beta\overline{\mathcal{O}}} \cap \overline{\mathcal{O}}^*$ is a nonempty subset of $Q \Rightarrow U \cap Q \neq \overline{\mathcal{O}} \Rightarrow U \cap V \neq \emptyset$.
□

Let $\mathbb{I}$ be the unit interval $[0,1]$, and let $\mathbb{M} = \omega \times \mathbb{I}$, where ω has the discrete topology (so, $\mathbb{M}$ is homeomorphic to a topological sum of countably many unit intervals). Note that $\mathbb{M}$ is homeomorphic to a closed subspace of $\mathbb{H}$. It follows that $\beta\mathbb{M}$ and $\mathbb{M}^* = \beta\mathbb{M} \setminus \mathbb{M}$ are homeomorphic to subspaces of $\beta\mathbb{H}$ and $\mathbb{H}^*$, respectively.

Now let u be a free ultrafilter on ω, and let

$$\mathbb{I}_u = \bigcap_{A\in u} \overline{A \times \mathbb{I}}^{\beta\mathbb{M}}.$$

Theorem 55. *Let u be an ultrafilter on ω, and let* $\mathbb{I}_u = \bigcap_{A\in u} \overline{A \times \mathbb{I}}^{\beta\mathbb{M}}$, *then:*

(a) $\mathbb{I}_u$ *is connected*;

(b) *Every proper subcontinuum K of $\mathbb{H}^*$ is contained in a subcontinuum of $\mathbb{H}^*$ homeomorphic to $\mathbb{I}_u$ for some ultrafilter u.*

Proof. (a) Suppose not. Then $\mathbb{I}_u = H \cup K$ for some non-empty, closed, disjoint subsets H and K of $\beta\mathbb{M}$. Now let O_1 and O_2 e disjoint open sets in $\beta\mathbb{M}$ such that $H \subset O_1$ and $K \subset O_2$, and look at $O_1 \cap \mathbb{M}$ and $O_2 \cap \mathbb{M}$.

Claim: $A_1 = \{n : O_1 \cap (\{n\} \times \mathbb{I}) \neq \emptyset\} \in u$.
If $A_1 \notin u$, then $\omega \setminus A_1 \in u$, since u is an ultrafilter, and $\mathbb{I}_u \subseteq \overline{(\omega \setminus A_1) \times \mathbb{I}}^{\beta\mathbb{M}}$. Also $O_1 \cap \mathbb{M} \subseteq A_1 \times \mathbb{I}$, so $O_1 \subset \overline{O_1 \times \mathbb{M}}^{\beta\mathbb{M}} \subset \overline{A_1 \times \mathbb{I}}^{\beta\mathbb{M}}$. So $\overline{A_1 \times \mathbb{I}}^{\beta\mathbb{M}} \cap \mathbb{I}_u \neq \emptyset$, which implies $A_1 \in u$. Similarly $A_2 = \{n : O_2 \cap (\{n\} \times \mathbb{I}) \neq \emptyset\} \in u$.
Let $A_3 = A_1 \cap A_2$. For each $n \in A_3$, pick $< n, x_n > \in (\{n\} \times \mathbb{I}) \setminus (O_1 \cup O_2)$. Take $\bigcap_{A \in u;\ A \subset A_3} \overline{\{< n, x_n >\}_{n \in A}}^{\beta\mathbb{M}}$. This is non-empty, so pick $x \in \bigcap_{A \in u;\ A \subset A_3} \overline{\{< n, x_n >\}_{n \in A}}^{\beta\mathbb{M}}$.
Then $x \in \mathbb{I}_u \setminus (O_1 \cup O_2)$, which is a contradiction, since $\mathbb{I}_u$ is a subset of $O_1 \cup O_2$. So $\mathbb{I}_u$ is connected.

(b.) Let K be a proper subcontinuum of $\mathbb{H}^*$. Let a_n, b_n be as in Lemma 53, and u as in Lemma 53b. Then by Lemma 53b, $K \subseteq \bigcap_{A \in u} \overline{\bigcup_{n \in A} [a_n,\ b_n]}^{\beta\mathbb{M}} \cong \mathbb{I}_u$. ☐

Definition. Let $\langle x_n \rangle_{n \in \omega}$ be a sequence of points in a space X, and let u be a free ultrafilter on ω. We say that a point p in X is a *u-limit* of $\langle x_n \rangle_{n \in \omega}$ if for every neighborhood N of p, we have

$$\{n : x_n \in N\} \in u.$$

In this case, we write $p = u - \lim_n x_n$.

Theorem 56. *Let X be a compact Hausdorff space, and let u be an ultrafilter on ω. Then every sequence $\langle x_n \rangle_{n \in \omega}$ in X has a unique u-limit in X.*

Proof. Equivalently let $s : \omega \to X$ be a sequence, we need to show that s has a unique u-limit in X. Now, suppose X has no u-limit. Then for every $x \in X$, let U_x be an open set such that $x \in U_x$ and $s^{-1}(U_x) \notin u$. Then $X \setminus [s^{-1}(u)] = s^{-1}(U_x^c) \in u$. So $\{U_x\}_{x \in X}$ is an open cover of compact space X. So let $\{V_i\}_{i=1}^{N}$ be the finite subcover. So since $V_1 \cup V_2 \cup \ldots \cup V_N = X$, then $V_1^c \cap V_2^c \cap \ldots \cap V_N^c = \emptyset$. So $\emptyset = s^{-1}(V_1^c \cap V_2^c \cap \ldots \cap V_N^c) = s^{-1}(V_1^c) \cap s^{-1}(V_2^c) \cap \ldots \cap s^{-1}(V_n^c) \in u$, which is a contradiction. So s has a u-limit in X.

For uniqueness, suppose z_1 and z_2 are both u-limits of s in X. Since X is T_2, let U_1 and U_2 be open, $U_1 \cap U_2 = \emptyset$ with $z_1 \in U_1$, $z_2 \in U_2$. Now $\emptyset = s^{-1}(U_1 \cap U_2) = s^{-1}(U_1) \cap s^{-1}(U_2)$ and since both $s^{-1}(U_1)$ and $s^{-1}(U_2)$ are in u, so is their intersection, so $\emptyset \in u$, a contradiction. Hence the u-limit is unique. ☐

Some facts on $\mathbb{I}_u$...

Now that we know that $\{\vec{x_u} : u$ is an ultra filter in $\omega\}$is the collection of all cut points of $\mathbb{I}_u$, define the following order on $\mathbb{I}_u$: $\vec{x}_u < \vec{y_u}$ if and only if $\{n : x_n < y_n\} \in u$. It follows that

$$A_{\vec{u}} = \{\vec{x}_u :< n, x_n >_{n \in u} \in M\},$$

is dense in $\mathbb{I}_u$ and its relative topology is the order topology (as defined above). Interestingly, no countable sequence in $A_{\vec{u}}$ can limit to $\vec{1}_u =< n, 1 >_{n \in u}$.

Theorem 57. *For a sequence* $< x_n >_{n\in\omega}$ *of points in* $\mathbb{I}$*, let* $\vec{x} =< (n, x_n) >_{n\in\omega}$ *be the corresponding sequence in* $\mathbb{M}$*, and for an ultrafilter* u *on* ω*, let* $\vec{x}_u$ *be the* u*-limit of* $\vec{x}$ *in* $\beta\mathbb{M}$*. Then* $\vec{x}_u \in \mathbb{I}_u$ *and* $\vec{x}_u$ *is a cut point of* $\mathbb{I}_u$ *if and only if* $\{n \in \omega : 0 < x_n < 1\} \in u$*.*

Proof. Let $A \in u$ and $\vec{x}_u \in O$, open. Since $A' = \{n : (n, x_n) \in O\} \in u$, there is some $i \in A \cap A'$ and so $(i, x_i) \in O \cap \{i\} \times \mathbb{I} \subset A \times \mathbb{I}$, i.e., $\vec{x}_u \in \overline{A \times \mathbb{I}}^{\beta\mathbb{M}}$. This is true for all $A \in u$, hence $\vec{x}_u \in \mathbb{I}$. Let

$$\mathbb{I}'_u = \cap_{A\in u}\overline{[\cup_{n\in A}\{n\} \times [0, x_n)]}^{\beta\mathbb{M}} \subset \mathbb{I}_u$$

and

$$\mathbb{I}''_u = \cap_{A\in u}\overline{[\cup_{n\in A}\{n\} \times (x_n, 1]]}^{\beta\mathbb{M}} \subset \mathbb{I}_u$$

To see that $\mathbb{I}'_u \cup \mathbb{I}''_u = \mathbb{I}$, assume by way of contradiction that $p \in \mathbb{I}_u$ and not in $\mathbb{I}'_u \cup \mathbb{I}''_u$. Then there is some V open with $p \in V$ and $V \cap (\mathbb{I}'_u \cup \mathbb{I}''_u) = \emptyset$. From this there is some $A \in u$ such that $V \cap (\overline{[\cup_{n\in A}\{n\} \times [0, x_n)]}^{\beta\mathbb{M}} \cup \overline{[\cup_{n\in A}\{n\} \times (x_n, 1]]}^{\beta\mathbb{M}}) = \emptyset$. Since $[\cup_{n\in A}\{n\} \times [0, x_n)] \cup [\cup_{n\in A}\{n\} \times (x_n, 1]]$ is dense in $A \times \mathbb{I}$, $V \cap \overline{A \times I}^{\beta\mathbb{M}} = \emptyset$, i.e. $p \notin \mathbb{I}_u$, a contradiction. To see that these sets are mutually separated (save for $\vec{x}_u$), let $p \in \mathbb{I}_u$ be distinct from $\vec{x}_u$. Let O be open with $p \in O$, $\overline{O}^{\beta\mathbb{M}} \cap \overline{\vec{x}}^{\beta\mathbb{M}} = \emptyset$, and let $A = \{n \in \omega : O \cap \{n\} \times \mathbb{I} \neq \emptyset\} \in u$. Then for each $n \in A$ there are some x'_n and x''_n such that $x'_n < x_n < x''_n$ and $O \cap \{n\} \times \mathbb{I} \subset \{n\} \times ([0, x'_n) \cup (x''_n, 1])$. Note that since $\mathbb{M}$ is normal and $O' = \cup_{n\in A}\{n\} \times [0, x'_n]$ and $\cup_{n\in A}\{n\} \times [x_n, 1]$ are disjoint closed sets in $\mathbb{M}$ they have disjoint closures in $\beta\mathbb{M}$ and hence $\overline{O'}^{\beta\mathbb{M}} \cap \mathbb{I}''_u = \emptyset$. Similarly, $\overline{O''}^{\beta\mathbb{M}} \cap \mathbb{I}'_u = \emptyset$ where O'' is defined in a similar way to O'. From this $\overline{O'}^{\beta\mathbb{M}} \cap \overline{O''}^{\beta\mathbb{M}} = \emptyset$ and it is clear that $O \subset \overline{O'}^{\beta\mathbb{M}} \cup \overline{O''}^{\beta\mathbb{M}}$. So p is in one and only one. Assume $p \in \overline{O'}^{\beta\mathbb{M}}$, then $p \in O \setminus \overline{O''}^{\beta\mathbb{M}}$ which is open and misses $\mathbb{I}''_u$, and $p \in \mathbb{I}'_u$. Similarly, if $p \in \overline{O''}^{\beta\mathbb{M}}$ then $p \in \mathbb{I}''_u$ and not a limit point of $\mathbb{I}'$. Finally, we shall show that both $\mathbb{I}'_u$ and $\mathbb{I}''_u$ are both nonempty if and only if $A = \{n : 0 < x_n < 1\} \in u$. Let $A \in u$. Then it is clear that the u-limit of $< (n, 0) >_{n\in\omega} \in \mathbb{I}'_U$ and the u-limit of $< (n, 1) >_{n\in\omega} \in \mathbb{I}''_U$, hence both are nonempty. Now for the other direction, assume by way of contradiction that $A \notin u$. Then since u is an ultrafilter, either $A' = \{n : 0 = x_n\}$ or $A'' = \{n : x_n = 1\}$ is in u. Without loss, assume $A' \in u$. Then $\mathbb{I}'_u \subset \cup_{n\in A'}\{n\} \times \emptyset = \emptyset$. □

Reminder: when we talk about a compactification αX of X, we continue to assume that $X \subset \alpha X$ and that the map $\alpha : X \to \alpha X$ is the identity on X.

Definition. A completely regular space X is *Čech complete* if X is a G_δ-set in some compactification αX of X (equivalently, $\alpha X \setminus X$ is an F_σ-set in αX).

Remark: Every locally compact completely regular space is Čech complete, since locally compact spaces are open in any compactification. The irrationals are G_δ in the one point compactification of the reals, so they are Čech complete. We will soon see that in fact any completely metrizable space is Čech complete.

Lemma 58. *If* αX *and* γX *are compactifications of* X*, and* $f : \alpha X \to \gamma X$ *is a continuous function with* $f|_X = id_X$*, then* $f^{-1}(\gamma X \setminus X) = \alpha X \setminus X$*.*

Proof. $f^{-1}(\gamma X \setminus X) \subset \alpha X \setminus X$ is clear since $f|_X = id_X$. Given $x \in X$, suppose there is some $y \in f^{-1}(x)$ with $y \in \alpha X \setminus X$. By T_2, there are disjoint open $U, V \subset \alpha X$ with $x \in U$ and $y \in V$. Then there is an open $W \subset \gamma X$ such that $X \cap W = X \cap U$

but for any $x' \in V \cap X$, $x' \notin U$ and so $x' \notin W$, contradicting the continuity of f. Thus for any $x \in X$, $f^{-1}(x) = x$. ☐

Theorem 59. *If $f : X \to Y$ is continuous onto, then TFAE*

(a) f is closed.

(b) If $y \in Y$ and U is open in X such that $f^{-1}(y) \subset U$, then there is V open in Y such that $y \in V$ and $f^{-1}(V) \subset U$.

Proof. (b) → (a): Let C be closed in X. If $y \notin f(C)$, then $f^{-1}(y) \subset X \setminus C$, which is open, and by our assumption there is V open in Y such that $y \in V$ and $f^{-1}(V) \subset X \setminus C \Rightarrow V \cap f(C) = \emptyset$. Thus $Y \setminus f(C)$ is open in Y meaning $f(C)$ is closed in Y.

(a) → (b): Suppose f is closed. Let $y \in Y$ and let U be an open set in X such that $f^{-1}(y) \subset U$. Thus $X \setminus U$ is closed in X meaning $f(X \setminus U)$ is closed in Y and $y \notin f(X \setminus U)$.

Let V be open in Y such that $y \in V$ and $V \cap f(X \setminus U) = \emptyset$. Thus $f^{-1}(y) \subset f^{-1}(V) \subset U$. ☐

Definition. A continuous mapping $f : X \to Y$ is called a *perfect* mapping if f is closed, onto, and $f^{-1}(y)$ is compact for every $y \in Y$.

Theorem 60. *If $f : X \to Y$ is perfect, then $f^{-1}(K)$ is compact when K is compact.*

Proof. Let $\mathcal{U}$ be an open cover of $f^{-1}(K)$. If $y \in K$, then $f^{-1}(y)$ is compact and we can choose $\mathcal{U}_y$ to be a finite open subcover of $f^{-1}(y)$ from $\mathcal{U}$. $f^{-1}(y) \subset \cup\mathcal{U}_y$ and f is closed, so we can choose V_y open in Y such that $y \in V_y$ and $f^{-1}(V_y) \subset \cup\mathcal{U}_y$. Because $\{V(y) : y \in K\}$ covers K and K is compact, we can choose $y_1, y_2, \ldots, y_n$ such that $\{V_{y_1}, V_{y_2}, \ldots, V_{y_n}\}$. Let $\mathcal{U}' = \{U \in \mathcal{U} : U \in \mathcal{U}_{y_i}$ for some $i = 1, 2, \ldots, n\}$. $\mathcal{U}'$ is a finite subset of $\mathcal{U}$ and if $x \in f^{-1}(K)$, then $f(x) \in K \Rightarrow f(x) \in V_{y_1}$ for some $i = 1, 2, \ldots n \Rightarrow x \in f^{-1}(V_{y_i}) \subset \cup\mathcal{U}_{y_i} \subset \cup\mathcal{U}$; thus $\mathcal{U}'$ covers $f^{-1}(K)$.

$\therefore$ $f^{-1}(K)$ is compact. ☐

Some notes on perfect maps...

Perfect maps preserve paracompactness in both directions. In fact if $f : X \to Y$ is closed, then Y is paracompact if X is paracompact. To go the forward direction, a result of Michael is used, which states a regular space is paracompact iff every open cover has a closure-preserving closed refinement.

For the reverse direction, suppose $\mathcal{U}$ is an open cover of X. If $y \in Y$ we can choose V_y, a neighborhood of y , such that $f^{-1}(V_y) \subset \cup\mathcal{U}_y$ and $\mathcal{U}_y \subset \mathcal{U}$ is finite. Applying paracompactness of Y, we see that there is a locally finite open cover of $\mathcal{V}$ of Y such that if $V \in \mathcal{V}$, then $f^{-1}(V)$ is contained in the union of a finite subset $\mathcal{U}(V)$ of $\mathcal{U}$. Then $\{f^{-1}(V) \cap U : V \in \mathcal{V}, \textit{ and } U \in \mathcal{U}(V)\}$ is a locally finite open refinement of $\mathcal{U}$.

Theorem 61. *The following are equivalent for a completely regular space X:*

(a) X is Čech complete;

(b) X is a G_δ-set in βX;

(c) X is a G_δ-set in every compactification of X;

(d) X is a G_δ-set in every completely regular space in which it is densely embedded.

Proof. (d) → (c) → (b) → (a) is obvious

(a) → (b): If X is G_δ in some compactification αX, let $X = \cap_{i<\omega} U_i$ where each U_i is open in αX. If βX is the Stone-Čech compactification of X, then there is a map $f : \beta X \to \alpha X$ such that $f|_X = \mathrm{id}_X$. Let $V_i = f^{-1}(U_i)$. Then V_i is open in βX and $X \subset V_i$ for every i. If $b \in \beta X \setminus X$ then $f(b) \in \alpha X \setminus X$ by Lemma 58 and so there is some k such that $f(b) \notin U_k$. Thus $b \notin V_k$ and so $b \notin \cap_{i<\omega} V_i$. Thus $X = \cap_{i<\omega} V_i$ and so X is G_δ in βX.

(b) → (c): If X is G_δ in βX, let $X = \cap_{i<\omega} U_i$ where each U_i is open in βX. If αX is any other compactification of X, then there is a map $f : \beta X \to \alpha X$ such that $f|_X = \mathrm{id}_X$. Let $V_i = \alpha X \setminus f(\beta X \setminus U_i)$. Then V_i is open since βX is compact T_2, and $X \subset V_i$ since $f(\beta X \setminus U_i) \subset \alpha X \setminus X$ by Lemma 58. If $a \in \alpha X \setminus X$, then $a = f(b)$ for some $b \in \beta X \setminus X$ by surjectivity of f and Lemma 58. Thus there is some k such that $b \notin U_k$ and so $a \in f(\beta X \setminus U_k)$ and $a \notin \cap_{i<\omega} V_i \subset V_k$. Thus $X = \cap_{i<\omega} V_i$ and so X is G_δ in αX.

(c) → (d): Let $X \subset Y$ be dense and Y be completely regular. Then Y has a compactification C and considering $X \subset Y \subset C$, C is also a compactification of X. Thus there are open sets U_i in C such that $X = \cap_{i<\omega} U_i$. Letting $V_i = Y \cap U_i$, V_i is open in Y and $\cap_{i<\omega} V_i = Y \cap (\cap_{i<\omega} U_i) = Y \cap X = X$. □

Theorem 62. *The following are equivalent for a completely regular space X:*

- *(a) X is Čech complete;*
- *(b) There is a sequence $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X such that: whenever $\mathcal{F}$ is a filter of closed sets such that for each n there is some $F_n \in \mathcal{F}$ with $F_n \subset U$ for some $U \in \mathcal{U}_n$, then $\bigcap \mathcal{F} \neq \emptyset$.*

Proof. (a)⇒(b). Let X be be Čech complete. Then X is G_δ in some compactification αX. Let $X = \bigcap\limits_{i\in\omega} U_i$ where $U_i \subset \alpha X$ is open. Let $\mathcal{U}_i$ be a cover of X such that for every $U \in \mathcal{U}_i$, $\overline{U} \subset U_i$.
If $\mathcal{F}$ is such a filter, then by compactness of αX, there exists $p \in \bigcap\limits_{F\in\mathcal{F}} \overline{F}^{\alpha X}$. Note $p \in \overline{F_n}^{\alpha X} \subset U_i$, so $p \in X$, which implies $p \in \bigcap \mathcal{F}$.

(b)⇒(a). By contraposition, assume X is not Cech complete, and for every $i \in \omega$ let $\mathcal{U}_i$ be an open cover of X. For every $i \in \omega$ and $U \in \mathcal{U}_i$ let $U' \subset \beta X$ be open such that $U' \cap X = U$. For every $i \in \omega$ let $V_i = \cup_{U \in \mathcal{U}_i} U'$, then by assumption there exists $p \in \cap_{i\in\omega} V_i \setminus X$. Look at $\mathcal{F} = \mathcal{F}_p \cup \{C \subset X : C$ is closed and C contains a member of $\mathcal{F}_p\}$, a filter of closed sets on X. Fix $i \in \omega$, then since $p \in V_i$, there exists $U_i \in \mathcal{U}_i$ such that $p \in U_i'$. Since βX is completely regular, p has a neighborhood basis of zero-sets, and so there exists a zero set $F_i' \subset \beta X$ such that $p \in F_i' \subset U_i'$ and so, $F_i = F_i' \cap X \subset U_i$ and $F_i \in \mathcal{F}$. So, for every $i \in \omega$ there is some $F \in \mathcal{F}$ such that for some $U \in \mathcal{U}_i$ $F \subset U$, but $\cap \mathcal{F} \subset \cap \mathcal{F}_p = \emptyset$. Hence, no sequence of covers as in (b) exist for X.

Exercise I. *The Sorgenfrey line is not Čech complete.*

Proof. Let S denote the Sorgenfrey line and $\mathcal{U}_i$ be an open cover of S for every $i \in \omega$. We claim there are points $p \in S$ and $a_i \in S$ and open sets $U_i \in \mathcal{U}_i$ such that $[a_i, p) \subset U_i$ for every $i \in \omega$, which may be found as follows:

Pick any $U_0 \in \mathcal{U}_0$. Since U_0 is open there are $a_0, b_0 \in S$ such that $[a_0, b_0) \subset U_0$. Proceeding inductively, if $i \in \omega$ and a_i, b_i have been defined, since $\mathcal{U}_{i+1}$ covers S, there is some $U_{i+1} \in \mathcal{U}_{i+1}$ such that $(a_i, b_i) \cap U_{i+1}$ is nonempty and open. Choose a_{i+1}, b_{i+1} such that $[a_{i+1}, b_{i+1}) \subset (a_i, b_i) \cap U_{i+1}$. This defines a_i and b_i for all $i \in \omega$. Since $\{[a_i, b_i] \mid i \in \omega\}$ is a nested collection, there is some $p \in \cap_{i\in\omega}[a_i, b_i]$. It follows that $[a_i, p)$ is a nonempty subset of U_i for all $i \in \omega$.

Let $\mathcal{F} = \{\text{closed } C \subset S \mid C \supset [x, p) \text{ for some } x < p \in S\}$. Then $\mathcal{F}$ is a filter of closed sets satisfying the property of Theorem 62, but $\cap\mathcal{F} \subset \cap_{x<p}[x, p) = \emptyset$. Thus, by Theorem 62, S is not Čech complete. ☐

Excercise II. *The tangent disk space is Čech complete.*

Proof. Let X denote the tangent disk space and $\mathcal{U}$ be a cover of X by basic open sets and $\mathcal{U}_i = \mathcal{U}$ for each $i \in \omega$. Let $\mathcal{F}$ be a filter of closed sets such that for every $n \in \omega$ there is some $F_n \in \mathcal{F}$ with $F_n \in U_n \in \mathcal{U}_n$. There are two cases:

- If there is some $G \in \mathcal{F}$ such that $G \cap \mathbb{R} = \emptyset$, then for every $F \in \mathcal{F}$, $F \cap G$ is a nonempty bounded (and thus compact) subset of $\mathbb{R}^2$. Thus $\cap\mathcal{F} \neq \emptyset$.

- If not, then for every $F \in \mathcal{F}$, $F \cap \mathbb{R} \neq \emptyset$. Since $F_0 \subset U_0$, $F_0 \cap \mathbb{R} = \{p\}$ and since $F \cap F_0 \in \mathcal{F}$, $F \cap \mathbb{R} \supset \{p\}$ as well. Thus $p \in \cap\mathcal{F}$. ☐

Theorem 63. *Suppose X is metrizable. Then X is Cech complete iff X is completely metrizable.*

Proof: ($\Rightarrow$) Suppose X is Cech complete. Then X is a dense subset of a completely metrizable space $\widetilde{X}$. By $61(d)$ a Cech complete space is G_d in any completely regular space it's densely embedded in. So X is G_δ in $\widetilde{X}$, hence X is complete.

($\Leftarrow$) Suppose X is completely metrizable. For any $n \in \omega$ let $\mathcal{U}_n$ be an open cover of X by sets of diameter $\leq 1/n$. Suppose $\mathcal{F}$is a filter of closed sets such that for any $n \in \omega$, $\exists F_n \in \mathcal{F}$ such that $F_n \subseteq U \in \mathcal{U}_n$, for some U in $\mathcal{U}_n$. Let $G_n = \cap_{i\leq n} F_i$. Then $G_0 \supseteq G_1 \supseteq \cdots$ is a decreasing sequence of closed sets and diameter of $G_n \to 0$. So $\bigcap_{n\in\omega} G_n \neq \emptyset$. In fact there is a $p \in X$ such that $\{p\} = \cap G_n = \cap F_n$. We will show that for every $F \in \mathcal{F}$ that $p \in F$. Suppose that there is an $F \in \mathcal{F}$ such that $p \notin F$. Since $p \in G_n$ and the diameters of the $G_n's$ approach 0, there is an $m \in \omega$ such that $G_n \cap F = \emptyset$. But both G_n and F are in $\mathcal{F}$. ☐

Corollary 64. *Let X be completely metrizable, and $A \subseteq X$. Then A is completely metrizable iff A is G_δ in X.*

Proof: ($\Leftarrow$) Done in Math 7500. ($\Rightarrow$) Suppose $A \subseteq X$ is completely metrizable. Then A is dense in $\overline{A}$. Therefore it's G_δ in $\overline{A}$, and since every closed subset is G_δ, it follows that $\overline{A}$ is G_δ in X. So A is G_δ in X. ☐

Theorem 65 (a) *If X is closed or a G_δ subspace of a Cech complete space Y, then X is Cech complete.*

Proof: Suppose X is G_δ in Y. Since Y is Cech complete, Y is G_δ in βY. Thus X is G_δ in $cl_{\beta Y}(X)$. Hence X is Cech complete. Suppose X is closed in Y. There exists open sets $\{U_i\}_{i\in\omega}$ of βY, such that $Y = \bigcap U_i$. Then $\bigcap(U_i \cap cl_{\beta Y}(X)) = X$ so X is G_δ in $cl_{\beta Y}(X)$. So X is Cech complete.

Theorem 65. *(b) If X_n is Čech complete for each $n \in \omega$, then $\Pi_{n\in\omega} X_n$ is Čech complete.*

Proof. Let Y_n be a compactification of X_n for each n. Then there are open sets $U_{ni} \subset Y_n$ for each $i \in \omega$ such that $\cap_{i\in\omega} U_{ni} = X_n$. Also, $\Pi_{n\in\omega} Y_n$ is a compactification of $\Pi_{n\in\omega} X_n$. Let $\pi_i : \Pi_{n\in\omega} Y_n \to Y_i$ be the i^{th} projection and $V_{ni} = \pi_n^{-1}(U_{ni})$.

Then V_{ni} is open and $\cap_{i\in\omega}V_{ni} = \Pi_{k<n}Y_k \times X_n \times \Pi_{k>n}Y_n = \pi_n^{-1}(X_n)$. Thus $\cap_{i,j\in\omega}V_{ij} = \Pi_{n\in\omega}X_n$ so $\Pi_{n\in\omega}X_n$ is Čech complete. □

Theorem 66. *ω^{ω_1} is not Cech Complete.*

Proof. It will be shown that ω^{ω_1} as a subset of the compact space $(\omega+1)^{\omega_1}$ is not G_δ, and therefore not Cech complete.

Suppose that $U_i \subset (\omega+1)^{\omega_1}$ is open for each $i \in \omega$. Pick $f \in \cap_{i\in\omega}U_i$ and for each $i \in \omega$ let B_i be a basic open set such that $f \in B_i \subset U_i$.

For each $i \in \omega$, let α_i denote the least upper bound of the support of B_i. The support for a basic open set is finite, thus $\alpha_i < \omega_1$ for each $i \in \omega$ and $\gamma = \sup(\{\alpha_i : i \in \omega\}) < \omega_1$. Let $G \subset (\omega+1)^{\omega_1}$ to which g belongs if and only if $g_\alpha = f_\alpha$ for each $\alpha \leq \gamma$. $G \subset \cap_{i\in\omega}B_i$ and G contains the point h, where $h_\alpha = f_\alpha$ if $\alpha \leq \gamma$ and $h_\beta = \omega$ if $\beta > \gamma$. Because a coordinate of h has value ω, $h \notin \omega^{\omega_1}$; hence, $\cap_{i\in\omega}B_i \subset \cap_{i\in\omega}U_i$ and $\cap_{i\in\omega}B_i$ is not a subset of ω^{ω_1} (that is $\cap_{i\in\omega}U_i \neq \omega^{\omega_1}$). □

Theorem 67. *Every Cech complete space is a Baire space.*

Proof. The proof of Theorem 66 follows from 61, 67, and 71(a) each of which is proved independently of Theorem 66. If X is Cech complete, then X is a G_δ subspace of βX and X (by 61) and X is dense in βX by definition. βX is compact (and thus countably compact) so by 67, βX is Baire. It follows from 71(a) that X is Baire. □

Remark re Theorem 62: There is no "decreasing sequence version" (as there is for complete metric spaces) of condition 62(b) that is equivalent to Čech completeness. To see this, note that if there were, then any completely regular countably compact space would be Čech complete (since in a countably compact space, every decreasing sequence of closed sets has non-empty intersection). But the countably compact space we encountered in Lemma 43(a) is not Čech complete:

Theorem 68 *Let X be a subspace of ω^* such that X and $\omega^* \setminus X$ meet every infinite closed subset of ω^*. Then X is completely regular and countably compact but not Čech complete.*

Proof. X is countably compact by earlier problems. Now if X were countable, then $\overline{X} \subsetneq \omega^*$ so $\omega^* \setminus \overline{X}$ contains a basic open set A^* But $A*$ is closed infinite. Similarly $\omega^* \setminus X$ is also uncountable, a contradiction.

If X is G_δ in ω^*, then $X = \bigcap_{i\in\omega} U_i$, U_i open in ω^*. So $\omega^* \setminus U_i$ is a closed set missing X, so it is finite. But $\bigcup_{i\in\omega} \omega^* \setminus U_i = \omega^* \setminus \bigcap U_i = \omega^* \setminus X$ is countable, a contradiction. □

Theorem 69 *Every regular countably compact space is Baire.*

Proof: Let X be a regular countably compact space, and $\{U_i\}_{i<\omega}$ be a collection of dense open sets. Suppose U is an arbitrary non-empty open set. We will show that there is a point in $U \cap \bigcap U_i$. Let V_0 be a non-empty open set such that $V_0 \subseteq \overline{V_0} \subseteq (U_0 \cap U)$; this is possible since X is regular. If V_i has been defined for all $i \leq n$, let V_n be an open non-empty set such that $V_n \subseteq \overline{V_n} \subseteq (U_n \cap V_{n-1})$. Note: since $\overline{V_i} \subseteq U_i$ and $\overline{V_i} \subseteq U$ for each $i \in \omega$, it follows that $\bigcap \overline{V_i} \subseteq \bigcap U_i \cap U$. Also, $\overline{V_0} \supseteq \overline{V_1} \supseteq \overline{V_2} \supseteq \cdots$ is a decreasing chain of non-empty closed sets in a countably compact space, so $\bigcap \overline{V_i} \neq \emptyset$. Let $v \in \bigcap \overline{V_i} \subseteq U \cap \bigcap U_i$. We have shown that $\bigcap U_i$ is dense in X. Hence X is Baire. $\square$

There are separable metrizable spaces that are Baire but not Čech complete (equivalently, not completely metrizable):

Theorem 70. *Let $B \subset \mathbb{R}$ be a Bernstein set. B is Baire and not completely metrizable.*

Proof. Let $\mathcal{B} \subset \mathbb{R}$ be a Bernstein set. B is not completely metrizable,

To show B is Baire, suppose U_n is a dense open subset of B for each $n \in \omega$. Let I be an open interval in $\mathbb{R}$. If $n \in \omega$, let U'_n be open in $\mathbb{R}$ such that $U'_n \cap B = U_n$; $B \subset U'_n$, thus U_n is dense in $\mathbb{R}$. Pick open sets V_0 and V_1 such that $\overline{V_0} \cap \overline{V_1} = \emptyset$ and $\overline{V_0} \cup \overline{V_1} \subset I \cap U'_0$. Inductively, if $\sigma : n \to 0, 1$ and V_σ is defined, choose open sets $V_{\sigma^\wedge 0}$ and $V_{\sigma^\wedge 1}$ such that $\overline{V_{\sigma^\wedge 0}} \cap \overline{V_{\sigma^\wedge 1}} = \emptyset$ and $\overline{V_{\sigma^\wedge 0}} \cup \overline{V_{\sigma^\wedge 1}} \subset V_\sigma \cap U'_0$.

$\cup\{\cap \overline{V_{f\restriction_n}} : f : \omega \to \{0,1\}\}$ is an uncountable closed subset of $I \cap (\cap_{i\in\omega} U_i)$, and therefore both sets meet B; hence $(I \cap B) \cap (\cap_{n\in\omega} U_n) = I \cap B \cap (\cap_{n\in\omega} U'_n) \neq \emptyset$. Thus $\cap_{n\in\omega} U_n$ is dense in B. $\square$

Theorem 71.

(a) If $D \subset X$ is a dense G_δ set and X is Baire, then D is Baire.

(b) If X has a dense Baire subspace, then X is Baire.

Proof. Part (a): Let $D = \cap_{i\in\omega} U_i$ be dense G_δ in the Baire space X. Suppose $\{W_i\}_{i\in\omega}$ be a collection of dense open subsets of D and A is a nonempty open subset of D. Let $\{V_i\}_{i\in\omega}$ be a collection of sets open in X such that $V_i \cap D = W_i$, for each $i \in \omega$ and let B be an open subset of X such that $B \cap D = A$.

For each $i \in \omega$, V_i is an open set containing a dense subset of a dense subspace of X, thus $U_i \cap V_i$ is a dense open set in X. Hence, $Q = \cap_{i\in\omega}(U_i \cap V_i)$ is dense in X, and there is

$$p \in B \cap (\cap_{i\in\omega}(U_i \cap V_i)) = B \cap D \cap (\cap_{i\in\omega} V_i) = A \cap (\cap_{i\in\omega} W_i)$$

Part (b): The proof of (b) is similar to that of (a). Let D be a dense Baire subspace of X. Choose $\{W_i\}_{i\in\omega}$ first (where W_i is dense open in X), and then let $V_i = D \cap W_i$. It follows that $\{V_i\}_{i\in\omega}$ is dense open in D, hence $\cap_{i\in\omega} V_i$ is dense in D. D is dense in X, thus if B is open and nonempty in X, then $D \cap B$ is open and nonempty in B, which means $\cap_{i\in\omega} V_i$ meets $D \cap B$. $\cap_{i\in\omega} V_i \subset \cap_{i\in\omega} W_i$ and $B \cap D \subset B$, thus $\cap_{i\in\omega} W_i$ meets B. $\square$

Example 72. *There is a subspace $X \subseteq \mathcal{R}$ which is Baire but has a close (hence non-G_δ) non-Baire subspace.*

Example: Let C be the Cantor set, let $I = [0, 1]$. Note: $I \setminus C$ is Baire by theorem 71a. Let E be the collection of endpoints of the intervals that generate C. Consider $X = (I \setminus C) \cup E$. Then $\overline{X}$ is Baire since $I \setminus C$ is a dense Baire subspace of X. E is closed in X since $E^c = (I \setminus C)$ is open. Each point of E is a limit point. E is a countable union of singletons (which are no-where dense sets). Hence E is not Baire. $\square$

Theorem 73. *If X is Baire and Y is Baire and second-countable, then $X \times Y$ is Baire*

Proof. Let $\{O_i\}_{i\in\omega}$ be a collection of dense open sets in $X\times Y$. Let $\{B_j\}_{j\in\omega}$ be a basis for Y. Then the sets $U_{ij} = \prod_x(O_i\cap(X\times B_j))$ are dense open in X: open since $\prod_x$ is an open projection; it is dense sinceif $U\subset X$, $U\times B_j$ is open in $X\times Y$ so $(U\times B_j)\cap O_i\neq\emptyset$.

Since X is Baire, there exists $x_0\in\bigcap\limits_{i,j\in\omega}U_{ij}$. Define $V_i = \{y|(x_0,y)\in O_i\}$ is dense open in Y. It is open since if $y\in V_i$, then $(x_0,y)\in O_i$, so there exists a basic open set with $(x_0,y)\in U\times V\subset O_i$ which implies $y\in V\subset V_i$. It is dense since if B_j is basic open then since $x_0\in\mathcal{U}_{ij}$, there exists $(x_0,y)\in O_i\cap(X\times B_j)$ so $y\in V_i\cap B_j$. Thus there exists $y_0\in\bigcap_{i\in\omega}V_i$ which implies $(x_0,y_0)\in\bigcap O_i$. □

Theorem 74. *If X is Baire, and Y is regular and countably compact or Čech complete, then $X\times Y$ is Baire.*

Proof. (for countably compact) Let $O_n,\ n\in\omega$ be dense open in $X\times Y$. We need to show that the intersection of O_n's is dense.

Let $U\times V$ be a basic open set in $X\times Y$. Since X is Baire, U is Baire. Let $\{U_\alpha\}$ be a maximal disjoint collection of subsets of U such that for each α there exists an open V_α with $\overline{V_\alpha}\subset V$ and $U_\alpha\times V_\alpha\subset O_0$.

Claim: $\bigcup_\alpha U_\alpha$ is dense in U. Suppose not: Then there exists $W\subset U$, W open, such that $W\cap\bigcup U_\alpha = \emptyset$. But then $W\times V$ is a nonempty open set such that $W\times V\cap O_0\neq\emptyset$. and choosing $(x,y)\in(W\times V)\cap O_0$, then there exists an open W_x,V_y such that $\overline{V_y}\subset V$, $(x,y)]inW_x\times V_y$, and $W_x\times V_y\subset O_0$. But W_x contradicts the maximality of U_α's ✓

Now for every α let $\{U_{\alpha\beta}\}_\beta$ be a maximal disjoint collection of open subsets of U_α such that for each β there exists an open $V_{\alpha\beta}$ such that $\overline{V_{\alpha\beta}}\subset V_\alpha$ and $U_{\alpha\beta}\times V_{\alpha\beta}\subset O_1$. Note: $\bigcup\limits_\beta U_{\alpha\beta}$ is dense open in U_α and $\bigcup\limits_{\alpha,\beta}U_{\alpha\beta}$ is dense open in U. For each $\alpha,\ \beta$ let $\{U_{\alpha\beta\gamma}\}_\gamma$ be a maximal disjoint collection of open subsets of $U_{\alpha\beta}$ such that ..., $\overline{V_{\alpha\beta\gamma}}\subset V_{\alpha\beta}$... continue on in this way...

Now, assume we have defined all U's and V's. Let $W_n = \bigcup\limits_{\alpha_0,\ \alpha_1,\dots,\ \alpha_n}U_{\alpha_0,\ \alpha_1,\dots,\alpha_n}$. Then W_n is dense open in U. U is Baire, so there exists $x\in\bigcap_{n\in\omega}W_n$. So $x\in U_{\alpha_0}$, and $x\in U_{\alpha_0,\alpha_1}$, and ... i.e. x is in some infinite sequence $\alpha_0,\alpha_1,\alpha_2,\dots$ with $x\in U_{\alpha_0,\alpha_1,\alpha_2,\dots,\alpha_n}$ for all n. We also have $V_{\alpha_0,\alpha_1,\alpha_2,\dots,\alpha_n}$ such that $U_{\alpha_0,\alpha_1,\alpha_2,\dots,\alpha_n}\times V_{\alpha_0,\alpha_1,\alpha_2,\dots,\alpha_n}\subset O_n$ and $\overline{V_{\alpha_0,\alpha_1,\alpha_2,\dots,\alpha_n}}\subset V_{\alpha_0,\alpha_1,\alpha_2,\dots,\alpha_n}$ so Y is countably compact, so there exists $y\in\bigcap_{n\in\omega}V_{\alpha_0,\alpha_1,\alpha_2,\dots,\alpha_n}$. Therefore $x,y\in\bigcap_{n\in\omega}O_n\cap(U\times V)$. □

Let X be a space. Define a two-person infinite game on X, called the *Choquet game* and denoted $Ch(X)$, as follows. There are two players, Empty (E) and Nonempty (NE). In the first round, E chooses a nonempty open set U_0, and NE then chooses a nonempty open subset V_0 of U_0. In the n^{th} round, E chooses a nonempty open subset U_n of V_{n-1}, and NE responds with a nonempty open $V_n\subset U_n$. There is a round for every $n\in\omega$. A play of the game results in a decreasing sequence

$$U_0\supseteq V_0\supseteq U_1\supseteq V_1\supseteq U_2\supseteq\dots$$

of nonempty open sets. We say that NE *wins* the game if $\bigcap_{n\in\omega} U_n \neq \emptyset$ (note that $\bigcap_{n\in\omega} U_n = \bigcap_{n\in\omega} V_n$, since the sequence is decreasing). Otherwise E wins, i.e., E wins if $\bigcap_{n\in\omega} U_n = \emptyset$.

A *strategy* for player E is a function σ from finite decreasing sequences of nonempty open sets (including the empty sequence) of the form

$$\langle U_0, V_0, U_1, V_1, \dots, U_n, V_n\rangle$$

to the collection of all nonempty open sets such that, for each $n \in \omega$,

$$\sigma(\langle U_0, V_0, U_1, V_1, \dots, U_n, V_n\rangle) \subset V_n.$$

Such a strategy σ is called a *winning strategy* for E if whenever

$$U_0, V_0, U_1, V_1, U_2, \dots$$

is an infinite decreasing sequence of open sets such that $U_0 = \sigma(\emptyset)$ and

$$U_{n+1} = \sigma(\langle U_0, V_0, U_1, V_1, \dots, U_n, V_n\rangle)$$

for each $n \in \omega$, then $\bigcap_{n\in\omega} U_n = \emptyset$. A strategy and winning strategy for NE is defined analogously.

Obviously, E and NE cannot both have a winning strategy. But it is possible that neither does; in such a case, we say the game is *undetermined.*

Theorem 75. *A space X is a Baire space if and only if E has no wining strategy in $Ch(X)$.*

Proof. ($\Leftarrow$) Suppose X is not Baire. Let $\{U_i\}_{i\in\omega}$ be a family of open dense subsets of X such that $\bigcap\limits_{i\in\omega} U_i$ is not dense. Let W be an open, set such that $W \cap \bigcap\limits_{i\in\omega} U_i = \emptyset$.

Have E choose $W_0 = W$. Let W_1 be an open set such that $W_1 \subset U_0 \cap V_0$ (V_0 having been chosen by NE). If V_i has been chosen for $i \in \{0, 1, ..., n\}$ choose W_{n+1} open such that $W_{n+1} \subset V_n \cap U_n$.

Then $\bigcap\limits_{i\in\omega} W_i \subset \bigcap\limits_{i\in\omega} U_i$ and $\bigcap_{i\in\omega} W_i \subset W_0 = W$. Hence $\bigcap_{i\in\omega} W_i = \emptyset$. So this is a winning strategy for E.

($\Rightarrow$) Suppose X is Baire, and σ is a strategy for E. We'll show that σ can be defeated.

Let $U_\emptyset$ be E's first choice using σ. Then let $\{U_\alpha\}_\alpha$ be a maximal collection of disjoint open subsets of $U_\emptyset$ such that, for each α, there is an open subset V_α of $U_\emptyset$ such that $U_\alpha = \sigma(U_\emptyset, V_\alpha)$. Suppose $\alpha_0\alpha_1...\alpha_n$ is some sequence of ordinals, and $U_{\alpha_0\alpha_1...\alpha_i}$ as well as $V_{\alpha_0\alpha_1...\alpha_i}$ have been defined for each $i \leq n$. Then let $\{U_{\alpha_0\alpha_1...\alpha_n\alpha}\}_\alpha$ be a maximal disjoint collection of open subsets of $U_{\alpha_0\alpha_1...\alpha_n}$ such that, for each α, there is some open subset $V_{\alpha_0\alpha_1...\alpha_n\alpha}$ of $U_{\alpha_0\alpha_1...\alpha_n}$ such that

$$U_{\alpha_0\alpha_1...\alpha_n\alpha} = \sigma(U_\emptyset, V_{\alpha_0}, U_{\alpha_0}, V_{\alpha_0,\alpha_1}, ..., U_{\alpha_0\alpha_1...\alpha_n}, V_{\alpha_0\alpha_1...\alpha_n\alpha}).$$

Claim. *For each n,*

$$W_n = \bigcup \{U_\tau : \tau \textit{ is a sequence of length } n\}$$

is dense in $U_\emptyset$.

We prove the claim by induction. Clear for $n = 0$. Suppose true for n. Let O be any open subset of $U_\emptyset$, Then by the inductive hypothesis, $O \cap W_n \neq \emptyset$, so

$V = O \cap U_{\alpha_0\alpha_1...\alpha_{n-1}} \neq \emptyset$ for some sequence $\alpha_0, \alpha_1, ...\alpha_{n-1}$ of length n. Suppose V misses W_{n+1}. Then $V \cap U_{\alpha_0\alpha_1...\alpha_{n-1}\alpha} = \emptyset$ for every α. But then taking

$$U = \sigma(U_\emptyset, V_{\alpha_0}, U_{\alpha_0}, V_{\alpha_0,\alpha_1}, ..., U_{\alpha_0\alpha_1...\alpha_{n-1}}, V),$$

we get a contradiction to maximality of the collection $\{U_{\alpha_0\alpha_1...\alpha_{n-1}\alpha}\}_\alpha$. This proves the claim.

Since X is Baire, so is $U_\emptyset$, so we can find $x \in \bigcap_{n\in\omega} W_n$. Since $x \in W_1 = \bigcup_\alpha U_\alpha$ and the U_α's are disjoint, there is a unique α_0 with $x \in U_{\alpha_0}$. Then since $x \in W_2$ and $\{U_{\alpha_0\alpha}\}_\alpha$ is pairwise disjoint, there is a unique α_1 with $x \in U_{\alpha_0\alpha_1}$. Continuing in this way, we see that there is a unique sequence $\alpha_0, \alpha_1, \ldots$ such that $x \in U_{\alpha_0\alpha_1...\alpha_n}$ for each n. But then

$$U_\emptyset, V_{\alpha_0}, U_{\alpha_0}, V_{\alpha_0\alpha_1}, U_{\alpha_0\alpha_1}, V_{\alpha_0\alpha_1\alpha_2}, \ldots$$

is a play of the game with E using σ in which E loses. So σ is not a winning strategy. □

Theorem 76. *If X is either Čech complete, or regular and countably compact, then NE has a winning strategy in Ch(X).*

Proof. If X is regular, countably compact, then: If U_n has been chosen by E let NE pick some open V_n with $\overline{V_n} \subset U_n$. Then $\bigcap \overline{V_n}$ is non-empty and so is $\bigcap V_n = \bigcap U_n$.

If X is Čech complete, let $\mathcal{U}_i$ be a sequence of open covers as in Theorem 62. If E chooses U_n, since $\mathcal{U}_n$ is an open cover, then there exists $U \in \mathcal{U}_n$ with $U \cap U_n \neq \emptyset$. Let NE choose V_n such that $\overline{V_n} \subset U \cap U_n$. Let $\mathcal{F} = \{C \subset X | C \textit{ is closed}, V_n \subset C \textit{ for some } n\}$. Then $\mathcal{F}$ is a filter of closed sets and for each n there exists $\overline{V_n} \in \mathcal{F}$ with $\overline{V_n} \subset U \in \mathcal{U}_n$. So $\bigcap \mathcal{F} \neq \emptyset$ by Theorem 62, and $\bigcap \mathcal{F} \subset \bigcap \overline{V_n} = \bigcap V_n$ so $\bigcap V_n \neq \emptyset$.□

Theorem 77. *If B is a Bernstein subset of the real line, then the game $Ch(B)$ is undetermined.*

Proof. Since B is Bernstein, B is Baire and hence E has no winning strategy. It will be shown that a winning strategy for NE implies that there is a winning strategy for E in the game $Ch(\mathbb{R}\setminus B)$, which contradicts the fact that $\mathbb{R}\setminus B$ is also a Bernstein set and hence Baire (having no winning strategy).

For ease, C will refer to the set $\mathbb{R} \setminus B$, and using the result of Theorem (???) proved independently of 70, it will be assumed that players restrict their choice to sets of the form $(a, b) \cap B$ or $(a, b) \setminus B$ (where $a, b \in \mathbb{R}$ and $a < b$), which forms a base for B and $C = \mathbb{R} \setminus B$, respectively.

Suppose that γ is a winning strategy for NE in $Ch(B)$. Let $S_0 = (a_0, b_0) \cap B$, and denote $\gamma(S_0)$ by $T_0 = (c_0, d_0) \cap B$. In a game for $Ch(C)$, let σ be a strategy for E defined in the following manner: $\sigma(\emptyset) = U_0 = (c_0, \frac{c_0+d_0}{2}) \setminus B$; if $U_0, V_0, U_1, V_1, \ldots U_n$ is a sequence of play in $Ch(C)$ determined by σ, and $S_0, T_0, S_1, T_1, \ldots S_n, T_n$ is a sequence of play in $Ch(B)$ determined by γ such that $T_i = (c_i, d_i) \cap B$ and $U_i = (c_i, \frac{c_i+d_i}{2}) \setminus B$ and $S_{i+1} = (a_{i+1}, b_{i+1}) \setminus V_i$, then if $V_n = (a_{n+1}, b_{n+1}) \setminus B$ is an open subset of U_n let $S_{n+1} = (a_{n+1}, b_{n+1}) \setminus V_n$ and let $T_{n+1} =$

$(c_{n+1}, d_{n+1}) = \gamma(S_0, T_0, \ldots S_n, T_n, S_{n+1})$ and define $\sigma(U_0, V_0, U_1, V_1, \ldots, U_n, V_n)$ as $(c_{n+1}, \frac{c_{n+1}+d_{n+1}}{2}) \setminus B$.

It follows that if $U_0, V_0, \ldots$ is a sequence of play in $Ch(C)$ determined by σ then there is $S_0, T_0, \ldots$ a sequence of play in $Ch(B)$ determined by γ such that $T_i = (c_i, d_i) \setminus U_i$ and $S_{i+1} = (a_{i+1}, b_{i+1}) \setminus V_i$. Because it is supposed that γ is a winning strategy for NE, $\cap_{i\in\omega} T_i \neq \emptyset$. However, since

$$diam(T_{i+1}) \leq diam(S_{i+1}) = diam(V_i) \leq diam(U_i) = \frac{1}{2} diam(T_i),$$

$$\lim_{i\to\infty} diam(T_i) = 0$$

and therefore $\cap_{i\in\omega} T_i = \{b\}$ for some $b \in B$; furthermore, because the prior intersection is a singleton, $\cap_{i\in\omega}(c_i, d_i) = \cap_{i\in\omega} T_i$. Of course $\cap_{i\in\omega} U_i \subset \cap_{i\in\omega}(c_i, d_i) \setminus B$, which means $\cap_{i\in\omega} U_i = \emptyset$; thus σ is a winning strategy for E in $Ch(C)$. □

It is sometimes useful to know that one may assume a player is choosing from a certain base $\mathcal{B}$ for X. Let $Ch(X, \mathcal{B})$ denote the game in which both players always choose a member of $\mathcal{B}$. The next theorem implies that the game $Ch(X, \mathcal{B})$ is equivalent to the original game $Ch(X)$ in the sense that a player has a winning strategy in one game iff he/she has one in the other.

Theorem 78. *Let $\mathcal{B}$ and $\mathcal{C}$ be bases for X. E has a winning strategy in $Ch(X, \mathcal{B})$ iff he has one in $Ch(X, \mathcal{C})$.*

Proof: Let $\sigma^{\mathcal{B}}$ be a winning strategy for E in Ch$(X, \mathcal{B})$. We will define a winning strategy for E in Ch$(X, \mathcal{C})$. Let U_0' denote $\sigma^{\mathcal{B}}(\emptyset)$. Choose an open set $U_0 = \sigma^{\mathcal{C}}(\emptyset) \in \mathcal{C}$ contained in U_0'. Suppose $V_0 \in \mathcal{C}$ is any NE's response to U_0. Let $V_0' \in B$ nonempty such that $V_0' \subseteq V_0$. Choose $U_1 = \sigma^{\mathcal{C}}(U_0, V_0)$ as a nonempty set in $\mathcal{C}$ contained in $U_1' = \sigma^{\mathcal{B}}(U_0', V_0')$. Suppose $U_1, U_2, \ldots, U_n$ and $V_1, V_2, \ldots, V_{n-1}$ have been defined. Let V_n be any response by NE U_n. Let $V_n' \in \mathcal{B}$ such that $V_n' \subseteq V_n$. Choose a basic open set $U_{n+1} \in \mathcal{C}$ contained in the open set $U_{n+1}' = \sigma^{\mathcal{B}}(U_0', V_0', \ldots, U_n', V_n')$, and let $\sigma^{\mathcal{C}}(U_0, V_0, U_1, V_1, \ldots, U_n, V_n)$. Then $\bigcap U_i \subseteq \bigcap U_i'$ and it follows that $\sigma^{\mathcal{C}}$ is a winning strategy for E in Ch$(X, \mathcal{C})$. The other implication is shown similarly. □

Definition. A space X is said to be *Choquet* if NE has a winning strategy in the game $Ch(X)$.

Clearly, Choquet spaces are Baire. Sometimes Choquet spaces are called "weakly α-favorable". This terminology stems in part from the fact that in the literature the players are sometimes denoted β and α instead of E and NE.

Theorem 79. *The product of Choquet spaces is Choquet.*

Proof. Let $\{X_a\}_{\alpha\in\lambda}$ each be a family of Choquet spaces and let $X = \pi_{\alpha\in\lambda} X_a$. We will show that NE has a winning strategy in $Ch(X, \mathcal{B})$ where $\mathcal{B}$ is the typical product base for X.

For each $\alpha \in \lambda$, let σ_α be a winning strategy for NE in $Ch(X_\alpha)$. Suppose U^0 is the first choice for E in $Ch(X, \mathcal{B})$. Let F^0 be the support of U^0 and for each $\alpha \in F^0$ let $V^0_\alpha = \sigma_\alpha(U^0_\alpha)$, where $U^0_\alpha = \pi_\alpha(U^0)$, and let V^0 be the corresponding basic open set in X with support F^0. Continue in this manner, so that if $< U^0, V^0, \ldots, U^n >$ is a play in $Ch(X, \mathcal{B})$ and F^i is the support of U^i, then for each $\alpha \in F^j \setminus (\cup_{i<j} F^i)$, define V^n_α as NE's response to $< U^j_\alpha, V^j_\alpha, \ldots, U^n_\alpha >$ in $Ch(X_\alpha)$ using σ_α; this is a well defined play, for if $i < n$, then $F^i \subset F^{i+1}$ (since $U^{i+1} \subset U^i$), thus for each $\alpha \in F^n$

there is a unique $j \leq n$ such that $\alpha \in F^j \setminus (\cup_{i<j} F^i)$. Let V^n be the corresponding basic open set in X with support F^n and let $\sigma(U^0, V^0, \ldots, U^n) = V^n$.

To show that σ is a winning strategy for NE, suppose that $< U^0, V^0, \ldots, >$ is a sequence of play determined by σ. Let $F = \cup_{i<\omega} F^i$, where F^i is the support for U^i (and subsequently V^i). If $\alpha \in F$, then there is $j \in \omega$ such that $\alpha \in F^j \setminus \cup_{i<j} F^i$ and thus $< U^j_\alpha, V^j_\alpha, U^{j+1}_\alpha, \cdots >$ is a sequence of play determined by σ_α, which means that $\cap_{i \geq j} V^i_\alpha \neq \emptyset$; because for each $i < j$, $V^i_\alpha = U^i_\alpha = X_\alpha$, it follows that $\cap_{i\in\omega} V^i_\alpha = (\cap_{i<j} X_a) \cap (\cap_{i \geq j} V^i_\alpha) \neq \emptyset$. Furthermore, if $\alpha \in \lambda \setminus F$, then $V^i_\alpha = X_\alpha$ for each $i \in \omega$, and $\cap_{i\in\omega} V^i_\alpha = X_\alpha \neq \emptyset$.

Since $\{V^i\}_{i\in\omega}$ is a decreasing collection of basic open sets in X, we have that

$$\cap_{i\in\omega} V^i = \pi_{\alpha\in\lambda}(\cap_{i\in\omega} V^i_\alpha) \neq \emptyset$$

□

Corollary 80. *The product of Cech complete spaces is Baire.*

Proof. By Theorem 76, a Cech complete space is Choquet, thus a product of Cech complete spaces is a product of Choquet spaces, which is Choquet by the previous theorem. A Choquet space is Baire, hence a product of Cech complete spaces is Baire. □

Remark. Note that by Theorem 65(b) and Example 66, while a countable product of Čech complete spaces is always Čech complete, an uncountable product need not be Čech complete. But Corollary 80 says it will still be Baire.

It's time we saw an example of Baire spaces X and Y whose product is not Baire. By various results above, neither X nor Y can be Čech complete, regular countably compact, or 2^{nd}-countable Baire. But we will presently see that there are (necessarily non-separable) metrizable Baire spaces X and Y such that $X \times Y$ is not Baire. The example is a nice application of stationary sets.

First let's fix some notation. The spaces are going to be subspaces of ω_1^ω, where the set ω_1 of countable ordinals is given the discrete topology. So points in ω_1^ω are functions from ω to ω_1, or, equivalently, infinite sequences (in order type ω) of countable ordinals. Let $\omega_1^{<\omega}$ denote the set of all finite sequences of countable ordinals. If $\sigma \in \omega_1^{<\omega}$, let

$$[\sigma] = \{f \in \omega_1^\omega : \sigma \subset f\}.$$

Note that $\sigma \subset f$ simply means that the infinite sequence f extends the finite sequence σ. It is easy to check that

$$\mathcal{B} = \{[\sigma] : \sigma \in \omega_1^{<\omega}\}$$

is a base for ω_1^ω. Also, for each $f \in \omega_1^\omega$, the collection $\{[f \restriction n] : n \in \omega\}$ is a local base at the point f.

For $f \in \omega_1^\omega$, let $f^* = sup\{f(n) : n \in \omega\}$. Of course, f^* is a countable ordinal. For $A \subset \omega_1$, let $A^* = \{f \in \omega_1^\omega : f^* \in A\}$.

Theorem 81.

(a) If $A \subset \omega_1$ is uncountable, then A^* is dense in ω_1^ω.

(b) If A and B are uncountable disjoint subsets of ω_1, then $A^* \times B^*$ is not Baire.

Proof. Part (a):

Suppose $[\sigma]$ is a basic open set. Let S be the support for $[\sigma]$ (i.e. $S \subset \omega$ is finite and $\sigma : S \to \omega_1$). Let $\alpha = max(\sigma(S))$. A is uncountable so there is $\beta \in A$ such that $\beta > \alpha$; let $f : \omega \to \omega_1$ such that $f \restriction_S = \sigma$ and $f(\gamma) = \beta$ if $\gamma \in \omega_1 \setminus S$. Thus $\sigma \subset f$, and $f^* = \beta$, which means $f \in [\sigma] \cap A^*$.

Part (b):

Claim: For each $i \in \omega$, if $U_i = \{(f,g) \in \omega_1^\omega \times \omega_1^\omega : max(f(i),g(i)) < min(f^*,g^*)\}$, then U_i is open and dense in $\omega_1^\omega \times \omega_1^\omega$.

To show this, suppose that $(f,g) \in U_i$, and pick $j,k \in \omega$ such that $f(j) > max(f(i),g(i))$ and $g(k) > max(f(i),g(i))$. $[f \restriction_{i+j}] \times [g \restriction_{i+k}]$ contains (f,g) and if (f',g') is also in $[f \restriction_{i+j}] \times [f \restriction_{i+k}]$, then $f'(i) = f(i)$, $f'(i+j) = f(i+j)$, $g'(i) = g(i)$ and $g'(i+k) = g(i+k)$, thus $max(f'(i),g'(i)) < min(f'(i+j), g'(i+k)) \leq min(f'^*, g'^*) \Rightarrow (f',g') \in U_i \Rightarrow [f \restriction_{i+j}] \times [g \restriction_{i+k}] \subset U_i \Rightarrow U_i$ is open.

U_i is dense for if $\sigma, \gamma \in \omega_1^{<\omega}$, then there is $i \in dom(\sigma) \cap dom(\gamma)$ and there is $\alpha > max(\sigma(i), \gamma(i))$. Choosing $f, g \in \omega_1^\omega$ such that f and g extend σ and γ, respectively, by mapping all other elements in ω_1 to α, it follows that $(f,g) \in [\sigma] \times [\gamma] \cap U_i$; meaning U_i is dense in $\omega_1^\omega \times \omega_1^\omega$.

Continuing with the proof of (b). For each $i \in \omega$, let $V_i = U_i \cap (A^* \times B^*)$, where U_i is as defined in the claim. U_i is open in $\omega_1^\omega \times \omega_1^\omega$, meaning $A^* \times B^*$ is dense in U_i and V_i is open in $A^* \times B^*$. U_i is dense in $\omega_1^\omega \times \omega_1^\omega$ meaning that V_i is dense in $A^* \times B^*$; hence V_i is dense open in $A^* \times B^*$ for each $i \in \omega$. $\cap_{i \in \omega} V_i = \emptyset$, for if $(f,g) \in A^* \times B^*$, then $f^* \neq g^*$ (since A^*, B^* disjoint), and assuming that $f^* < g^*$ it follows that there is $i \in \omega$ such that $f^* \leq g(i)$, hence $max(f(i),g(i)) = g(i) \not> g^* = max(f^*, g^*)$, which means $(f,g) \notin V_i \Rightarrow (f,g) \notin \cap_{j \in \omega} V_j$ (a similar argument holds in the case that $f^* > g^*$). □

Lemma 82. *Let U be dense open in ω_1^ω. For each $\sigma \in \omega_1^{<\omega}$, choose an extension σ_U of σ such that $[\sigma_U] \subset U$. Then*

$$C_U = \{\delta \in \omega_1 | \forall \sigma \in \delta^{<\omega} (\sigma_U \in \delta^{<\omega})\}$$

is a club.

Proof. Let β be a limit point of C_U, $\sigma \in \beta^{<\omega}$, and $\alpha = \max_{n \in dom(\sigma)}\{\sigma(n)\}$. Note that $\alpha < \beta$ and since β is a limit point and $(\alpha, \beta]$ is open, there is some $\gamma \in C_U \cap (\alpha, \beta]$. Then $\sigma \in \gamma^{<\omega}$ and $\gamma \in C_U$ so $\sigma_U \in \gamma^{<\omega} \subset \beta^{<\omega}$. Thus $\beta \in C_U$ and C_U is closed.

Define a map $f : \omega_1 \to \omega_1$ by $f(\alpha) = \sup_{\sigma \in \alpha^{<\omega}}\{\max_{n \in dom(\sigma_U)}\{\sigma_U(n)\}\}$. By Thrm 44 from MATH 7550, $D' = \{\alpha | \forall \gamma < \alpha (f(\gamma) < \alpha)\}$ is a club. Thus $D = \{\alpha \in D' | \alpha \text{ is a limit ordinal}\}$ is a club as well. If $\alpha \in D$ and $\sigma \in \alpha^{<\omega}$ then $\max_{n \in dom(\sigma)}\{\sigma(n)\} = \beta < \beta + 1 < \alpha$ so $f(\beta+1) < \alpha$. Thus $\sigma \in (\beta+1)^{<\omega}$ and $\max_{n \in dom(\sigma_U)}\{\sigma_U(n)\} \leq f(\beta+1) < \alpha$ so $\sigma_U \in \alpha^{<\omega}$ and $\alpha \in C_U$. Thus C_U is closed and contains an unbounded set D, so C_U is a club. □

Theorem 83. *Let A be a stationary subset of ω_1. Then A^* is Baire.*

Proof. Let $V_i \subset A^*$ be dense open and $U_i \subset \omega_1^\omega$ be open such that $V_i = U_i \cap A^*$ for every $i \in \omega$. Let $[\sigma]$ be a basic open set in ω_1^ω. Note that C_{U_i} is a club for every $i \in \omega$ by Lemma 82 and so $\cap_{i \in \omega} C_{U_i}$ is a club as well. Choose a $\delta \in \cap_{i \in \omega} C_{U_i} \cap A \cap \{\alpha \in \omega_1 | \alpha \text{ is a limit ordinal}\}$ such that $\sigma \in \delta^{<\omega}$. Choose an increasing sequence $\{\alpha_i\}_{i \in \omega}$ such that $\alpha_i \to \delta$, and define a sequence $\{f_i\}_{i \in \omega}$ as follows:

Take $f_0 \in \delta^{<\omega}$ such that f_0 extends σ_{U_0} and $\alpha_0 \in range(f_0)$. Inductively, if $f_i \in \delta^{<\omega}$ has been defined for all $i < n$, $(f_{n-1})_{U_n} \in \delta^{<\omega}$ and $\alpha_i < \delta$ so choose

$f_n \in \delta^{<\omega}$ extending $(f_{n-1})_{U_n}$ such that $\alpha_i \in range(f_n)$. Note each f_n extends compatibly and so defines a function $f = \cup_{n\in\omega} f_n \in \delta^\omega \subset \omega_1^\omega$ and thus $f^* \leq \delta$. Since f extends σ and each $(f_i)_{U_{i+1}}$ $i \in \omega$, $f \in [\sigma] \cap \cap_{i\in\omega} U_i$. Also by construction, $\delta \leq f^*$ so $f^* = \delta \in A$. Thus $f \in [\sigma] \cap A^* \cap \cap_{i\in\omega} U_i = ([\sigma] \cap A^*) \cap (\cap_{i\in\omega} V_i) \neq \emptyset$. Thus A^* is Baire. □

Theorem 84. *There is a collection $\mathcal{M} = \{X_{\alpha,n} | \alpha < \omega_1, n < \omega\}$ of subsets of ω_1 such that*

(i) For each $n \in \omega$, $X_{\alpha,n} \cap X_{\beta,n} = \emptyset$ if $\alpha \neq \beta$;

(ii) For each $\alpha < \omega_1$, $\omega_1 \setminus \cup_{n\in\omega} X_{\alpha,n}$ is countable.

Proof. For every $\alpha < \omega_1$ let $f_\alpha : \alpha \to \omega$ be a one-to-one function. Define $X_{\alpha,n} = \{\rho > \alpha | f_\rho(\alpha) = n\}$. If $\alpha \neq \beta$, $X_{\alpha,n} \cap X_{\beta,n} = \{\rho > \alpha, \beta | f_\rho(\alpha) = f_\rho(\beta) = n\} = \emptyset$ since each f_ρ is one-to-one. If $\alpha < \omega_1$, $\cup_{n\in\omega} X_{\alpha,n} \supset \omega_1 \setminus (\alpha + 1)$ since for every $\rho > \alpha$, $f_\rho(\alpha) = m$ for some $m \in \omega$.

Remark. $\mathcal{M}$ is called an *Ulam matrix* for ω_1.

Corollary 85. *There is an uncountable pairwise-disjoint collection of stationary subsets of ω_1.*

Proof. For every $\alpha < \omega_1$ there is some n_α such that X_{α,n_α} is stationary, for if not there is a club C_n such that $C_n \cap X_{\alpha,n} = \emptyset$ for every $n \in \omega$ and if $C = \cap_{n\in\omega} C_n$, $C \subset \omega_1 \setminus \cup_{n\in\omega} X_{\alpha,n}$ is countable. Thus there is an uncountable $A \subset \omega_1$ and a fixed $n \in \omega$ such that $n_\alpha = n$ for every $\alpha \in A$. Thus $\{X_{\alpha,n} | \alpha \in A\}$ is an uncountable collection of disjoint stationary subsets of ω_1. □

Corollary 86. *There are metrizable Baire spaces X and Y such that $X \times Y$ is not Baire.*

Proof. This follows immediately from Theorems 81(b) and 83, and Corollary 85. □

Corollary. *There is no countably additive measure μ on $\{$subsets of $\omega_1\}$ such that $\mu(\{\alpha\}) = 0$ for every $\alpha \in \omega_1$ and $\mu(\omega_1) = 1$.*

Proof. Suppose μ is such a measure. Then for every $\alpha < \omega_1$ there is some n_α such that $\mu(X_{\alpha,n_\alpha}) > 0$. Thus there is some uncountable $A \subset \omega_1$ and $n \in \omega$ such that $\mu(X_{\alpha,n} > 0$ for every $\alpha \in A$. But $\{X_{\alpha,n} | \alpha \in A\}$ is a pairwise disjoint collection, so $\mu(\omega_1) \neq 1$. □

If X is a topological space, let $C(X)$ denote the set of all continuous real-valued functions on X. The *compact-open topology* on $C(X)$ is denoted by $C_k(X)$ and is defined as follows. Let $K \subset X$, K compact, and let U be open in $\mathbb{R}$. Let $[K, U] = \{f \in C(X) : f(K) \subset U\}$. Then

$$\mathcal{B} = \{\bigcap_{i\leq n} [K_i, U_i] : K_i \text{ compact in } X,\ U_i \text{ open in } \mathbb{R}\}$$

is a base for $C_k(X)$. (Alternatively, we could say that the $[K, U]$'s form a subbase for the topology–a subbase is a collection of sets such that the set of all finite intersections from the collection is a base.)

Let $f \in C(X)$, K compact in X, and $\epsilon > 0$. Let

$$B(f, K, \epsilon) = \{g \in C(X) : \forall x \in K(|f(x) - g(x)| < \epsilon)\}$$

and let

$$\mathcal{C} = \{B(f, K, \epsilon) : f \in C(X), K \text{ compact in } X, \epsilon > 0\}.$$

Theorem 87. *The two bases $\mathcal{B}$ and $\mathcal{C}$ are equivalent bases for $C_k(X)$.*

Proof. We first show that $\mathcal{C}$ is a base. Let $g \in B(f,K,e)$ and set $\delta = sup_{xinK}|f(x))-g(x)| < \epsilon$. Well, $B(g,K,\frac{\epsilon}{2}) \subset B(f,K,\epsilon)$ since if $h \in B(g,K,\epsilon-\delta)$ then for all $x \in K$, $|h(x)-f(x) \leq |h(x)-g(h)| + |g(x)-f(x)| < \epsilon-\delta+\delta = \epsilon$. So $\mathcal{C}$ is a base. ✓

Suppose that $f \in \bigcap_{i\leq n}[K_i,U_i] \in \mathcal{B}$; i.e. that $\forall i \leq n,\ \forall x \in K_i,\ f(x) \in U_i$. Let $K = \bigcap_{i\leq n} K_i$. This is compact since is a finite union of compact sets. Now, for each $i \leq n$ let $\epsilon_i > 0$ such that for all $x \in K_i$, we have $(f(x)-\epsilon_i,\ f(x)+\epsilon_i) \subset U_i$. (We can do this because of Lebesgue's Theorem.) Let $\epsilon = min_{i\leq n}\{\epsilon_i\}$.

Claim: $B(f,K,\epsilon) \subset \bigcap_{i\leq n}[K_i,U_i]$.

If $g \in B(f,K,\epsilon)$ then for every $x \in K$, $|g(x)-f(x)| < \epsilon$. Let $i \leq n$ and suppose $x \in K_i$. Then $|g(x)-f(x)| < \epsilon \leq \epsilon_i$. Therefore, for all i, $g(x) \in (f(x)-\epsilon_i,\ f(x)+\epsilon_i) \subset U_i$. Hence $g \in [K_i,U_i]$ for all i. So $g \in \bigcap_{i\leq n}[K_i,U_i]$.✓

Now suppose $f \in B(f,K,\epsilon)$. Note: $f(K)$ is compact. Then for all $x \in K$ let $U_x = (f(x)-\frac{\epsilon}{2},\ f(x)+\frac{\epsilon}{2})$. Let V_x be an open set such that $x \in V_x \subset \overline{V_X} \subset U_X$ (since $\mathbb{R}$ is regular). So $\{V_x\}_{x\in K}$ is an open cover of $f(K)$. Let $\{V_{x_1},\ V_{x_2},\ \ldots,\ V_{x_n}\}$ be a finite subcover. Note also that $f^{-1}(\overline{V_{x_i}}) \cap K \subset K$ and that $f^{-1}(\overline{V_X})$ is closed, so therefore so is the intersection. Then $K_i = f^{-1}(\overline{V_{x_i}} \cap K$ is compact and $\bigcup_{i\leq n} K_i = K$.

Claim: $\bigcap[K_i,U_{x_i}] \subset B(f,K,\epsilon)$.

Suppose $g \in \bigcap[K_i,U_{x_i}]$. Then let $x \in K$, and $i \leq n$ such that $x \in K_i$. Then $|f(x)-f(x_i)| < \epsilon$ and $|g(x)-f(x_i)| < \epsilon$. Now $f(K_i) \subset \overline{V_{X_i}} \subset U_{x_i} = (f(x_i)-\frac{\epsilon}{2},\ f(x_i)+\frac{\epsilon}{2})$. From the triangle inequality we have $|f(x)-g(x)| \leq |f(x)-f(x_i)| + |f(x_i)-g(x)| \leq \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$. ☐

Corollary 88. *Let X be compact. If $f,g \in C(X)$, let $d(f,g) = sup_{x\in X}|f(x)-g(x)|$. Then d is a metric on $C(X)$ which generates the compact-open topology. Hence $C(X)$ is completely metrizable.*

Proof. Well, $B_d(f,\epsilon) = \{g : d(f,g) < \epsilon\} = \{g : sup|f(x)-g(x)| < \epsilon\} = B(f,K,\epsilon)$. Every $B(f,K,\epsilon) \subset B(f,X,\epsilon)$. ✓ And the rest is clear... ☐

A mapping $f : X \to Y$ is said to be *compact-covering* if whenever K is a compact subset of Y, there is a compact $L \subset X$ with $f(L) = K$.

If $q : Y \to X$ is continuous, there is a natural induced mapping $\phi_q : C_k(X) \to C_k(Y)$ defined by $\phi_q(f) = f \circ q$.

Theorem 89: *Let $q : Y \to X$ be continuous and onto, and $\phi_q : C_k(X) \to C_k(Y)$ the induced map.*

(a) *ϕ_q is continuous and one-to-one*
(b) *If q is a quotient map, the $\phi_q(C_k(X))$ is closed in $C_k(Y)$;*

(c) *If* q *is compact covering, then* ϕ_q *is a homeomorphic embedding of* $C_k(X)$ *into* $C_k(Y)$

Proof.

(a) For one-to-one: Let $f, g \in C_k(X)$ such that $f \neq g$. So, there exists $x \in X$ such that $f(x) \neq g(x)$ which implies that since there exists $y \in Y$ such that $q(y) = x$ (because q is onto) that $(f \circ q)(y) \neq (g \circ q)(y)$, and hence $\phi_q(f) \neq \phi_q(y)$.

For continuity: Let $f \in C_k(X)$. Examine the basic open set $B(f \circ q, K, \epsilon)$. Since q is continuous, $q(K)$ is compact in X, so let $g \in B(f, q(K), \epsilon)$. If $y \in K$, then $q(y) \in q(K)$ and so $|f(q(y)) - g(q(y))| < \epsilon$. So $|\phi_q(f)(y) - \phi_q(g)(y)| < \epsilon$ so $\phi_q(g) \in B(f \circ q, K, \epsilon)$ and hence ϕ_q is continuous.

(b) Let $g \in C_k(Y) \setminus \phi_q(C_k(X))$. Since q is quotient, if there is an f that makes the diagram commute, then $g \in \phi_q(C_k(X))$. Suppose for every distinct $y, y' \in Y$ if $q(y) = q(y')$ then $g(y) = g(y')$ then define $f : X \to \mathbb{R}$ by $f(x) = g(y)$ where $g(y) \in q^{-1}(x)$. Then $f \circ q)(y) = f(q(y)) = g(y)$. So there exists $y, y' \in Y$ such that $g(y) \neq g(y')$ but $q(y) = q(y')$. Look at $B(g, \{y, y'\}, \frac{g(y) - g(y')|}{2})$, then if $|f(y) - g(y)| < \frac{|g(y)-g(y')|}{2}$ and $|f(y') - g(y')| < \frac{g(y)-g(y')|}{2}$. So $f(y) \neq f(y')$, and hence $f \notin \phi_q(C_k(X))$ which implies $B(g, \{y, y'\}, \frac{|g(y)-g(y')|}{2}) \subset C_k(Y) \setminus \phi_q(C_k(X))$. This set is open, so $\phi_q(C_k(X))$ is closed.

(c) From earlier, ϕ_q is continuous, and one-to-one. So is sufficient to show it is open.

Let $g \in \phi_q(B(f, K, \epsilon))$ then there exists $g' \in B(f, K, \epsilon)$ such that $g = g' \circ q$. So there exists $K' \subset Y$ such that K' is compact and $q(K') = K$. Let $h \in B(g, K', \epsilon - \delta) \cap \phi_q(C_k(X))$ (where $\delta = sup_{x \in K}\{|f(x) - g(x)|\}$.) This implies that there exists $h' \in C_k(X)$ such that $h = h' \circ q$

Let $x \in K$. (Note: there exists $y \in K'$ such that $q(y) = x$.)

so $|f(x) - h'(x)| \leq |f(x) - g'(x)| + |g'(x) - h'(x)| = |f(x) - g'(x)| + |g(y) - h(y)| < sup_{x \in K}\{|f(x) - g'(x)|\} + \epsilon - \delta = \epsilon$.

So $h' \in B(f, K, \epsilon)$, and hence $h \in \phi_q(B(f, K, \epsilon))$. so $B(g, K', \epsilon - \delta) \cap \phi_q(C_k(X)) \subset \phi_q(B(f, K, \epsilon))$. □

Recall that a space X is said to be σ-*compact* if X is a union of countably many of its compact subsets.

Lemma 90. *Let* X *be a locally compact Hausdorff* σ*-compact space. Then there are compact subsets* $\{K_i\}_{i \in \omega}$ *of* X *such that* $K_i \subseteq K_{i+1}^o$ *for each* $i \in \omega$*, and* $X = \bigcup K_i$*. Furthermore, every compact subset* C *of* X *is contained in a* K_j *for some* $j \in \omega$*.*

Proof. Let $\{C_i\}$ be a countable family of compact subsets of X such that $X = \bigcup C_i$. We will define the desired family of compact sets as follows. Let $K_0 = C_0$. For each $x \in K_0$ let U_x be an open neighborhood of x whose closure is compact. Then $\{U_x\}_{x \in K_0}$ is an open cover of X. Note: K_0 is compact, let $\{U_{x_0}, U_{x_1}, \ldots, U_{x_n}\}$ be finite subcover. Set $U_0 = \overline{\bigcup_{i=0}^n U_{x_i}} = \bigcup_{i=0}^n \overline{U_{x_i}}$. Then U_0 is compact (being the finite union of compact sets) and $K_0 \subseteq U_0^\omega$. Define $K_1 = C_1 \cup U_0$. It follows that $K_0 \subseteq K_1^o$. If $K_0, K_1, K_2, \ldots, K_n$ have been defined so that $K_i \subseteq K_{i+1}^o$ for all $i \in \{0, 1, \ldots, n-1\}$, then following a similar procedure as above let U_n be a compact set such that $K_n \subseteq U_n^o$. Define $K_{n+1} = C_{n+1} \cup U_n$. Continue by induction. Since $C_i \subseteq K_i$ for all $i \in \omega$, we have that $X = \bigcup C_i \subseteq \bigcup K_i$. Furthermore, for each $i \in \omega$, $K_i \subseteq K_{i+1}^o$. Suppose $C \subseteq X$ is compact. Then $\{K_i^o\}_{i\in\omega}$ is an open cover of C. Let $\{K_{i_0}^o, K_{i_1}^o, \ldots, K_{i_m}^o\}$ be a finite subcover. Set $N = \max\{i_0, i_1, \ldots, i_m\}$. Then $C = \bigcup_{j=0}^m K_{i_j}^o = K_N^o \subseteq K_N$. □

Arhangelskii's Theorem: *If X is a compact, T_2, first countable space then X has cardinality $|X| \leq 2^\omega$.*

Proof. Pick $x_0 \in X$, and let $X_0 = \{x_0\}$. Now, for each $x \in X$, let $\{b(x,n)\}_{n\in\omega}$ be a countable base at x. Suppose $\alpha < \omega_1$ and we have constructed closed sets X_β for all $\beta < \alpha$, that satisfy for all $\beta \leq \gamma < \alpha$:

i. $X_\beta \leq X_\gamma$
ii. $|X_\beta| \leq 2^\omega$

To define X_α:

Case 1: α a limit ordinal -
Let $X_\alpha = \bigcup\limits_{\beta<\alpha} X_\beta$. Then, by following lemma, $|X_\alpha| \leq 2^\omega$.

Lemma. *If X is a first countable, T_2 space, and $Y \subset X$, with $|Y| \leq 2^\omega$, then $|\overline{Y}| \leq 2^\omega$.*

Proof of Lemma. For all $p \in \overline{Y}$, since X is first countable, there exists a sequence $\{y_n^p\} \in Y$ such that p is the limit of the y_n^p's. Now, the function that sends $p \mapsto (y_n^p)$ maps $\overline{Y}$ one-to-one into Y^ω. But $|Y^\omega| \leq (2^\omega)^\omega = 2^\omega$. □

Case 2: α a successor ordinal, i.e. $\alpha = \gamma + 1$ for some γ.

Then X_γ is defined. To define $X_{\gamma+1}$, let $\mathcal{U}_\gamma = \{b(x,n) : x \in X_\gamma,\ n \in \omega\}$. (Note: $|\mathcal{U}_\gamma| \leq 2^\omega$, since X_γ is.
So $\mathcal{U}_\gamma$ has $\leq 2^\omega$ many countable open subsets. For each countable $\mathcal{V} \subset \mathcal{U}_\gamma$ such that $X_\gamma \subset \bigcup \mathcal{V}$, pick if possible a point $x_v \in X \setminus \bigcup \mathcal{V}$. Then let
$$X_\alpha = X_{\gamma+1} = \overline{X_\gamma \cup \{x_v : \mathcal{V} \subset \mathcal{U}_\gamma,\ |\mathcal{V}| \leq \omega,\ \text{and } X_\gamma \subset \bigcup \mathcal{V},\ X \setminus \mathcal{V} \neq \emptyset\}}$$

Let $X_{\omega_1} = \bigcup\limits_{\alpha<\omega} X_\alpha$. Now this is the union of $\leq c$ many sets $\leq \omega_1$, thus $|X_{\omega_1}| \leq 2^\omega$.

Claim 1: X_{ω_1} is closed.
Suppose x is a limit point of X_{ω_1}. Then there exists $x_n \in X_{\omega_1}$ such that $x_n \to x$, by first countability. Now $x_n \in X_{\alpha_n}$, $\alpha_n < \omega_1$. Let $\alpha > \alpha_n$ for all n, $\alpha < \omega_1$, then for all n, $x_n \in X_\alpha$, and X_α closed implies that $x \in X_\alpha \subset X_{\omega_1}$.

Claim 2: $X_{\omega_1} = X$
Suppose $p \in X \setminus X_{\omega_1}$. For all $x \in X_{\omega_1}$, choose n_x such that $p \notin b(x, n_x)$. Now the $b(x, n_x)$'s cover X_{ω_1}. X_{ω_1} is Lindelof, so countably many must cover X_{ω_1}. But this implies that there exists a countable $Y \subset X_{\omega_1}$ such that $\hat{\mathcal{V}} = \{b(y, n_y) : y \in Y\}$ covers X_{ω_1}.

For $y \in Y$, let $y \in X_{\alpha_y}$, $\alpha_y < \omega_1$. Let $\alpha < \omega_1$ such that $\alpha_y < \alpha$ for all $y \in Y$. Then $\hat{V}$ is a countable subset of $\{b(x,n) : \ x \in X, \ n \in \omega\}$, and $X \smallsetminus \bigcup \hat{\mathcal{V}} \neq \emptyset$. (since $p \in X \smallsetminus \bigcup \mathcal{V}$).
Therefore there exists $x_{\hat{\mathcal{V}}} \in X_{\alpha+1} \smallsetminus \bigcup \hat{\mathcal{V}}$. But $\bigcup \hat{\mathcal{V}} \supseteq X_{\omega_1}$, a contradiction. So $X_{\omega_1} = X$. $\Box$.

The *character*, $\chi(x,X)$, at x is the least cardinal of a base at x in X, and $\chi(X) = sup_{x\in X}\chi(x,X)$. The *Lindelof degree*, $L(X)$, of a space X is the least cardinal κ such that every open cover of X has a subcover of cardinality less than or equal to κ. By a similar proof, we can get:

Theorem. $|X| \leq 2^{\chi(X)L(X)}$.

Corollary. *If X is compact T_2 and 1^{st} countable, then $|X| \leq \omega$ or $|X| = 2^{\omega}$.*

Proof. Suppose $|X| > \omega$. Proceeding inductively, assume open sets U_σ has been defined for all $\sigma \in 2^{<n}$ such that $\overline{U_{\sigma\cap 0}} \cap \overline{U_{\sigma\cap 1}} = \emptyset$, $\overline{U_{\sigma\cap 0}} \cup \overline{U_{\sigma\cap 1}} \subset \overline{U_\sigma}$, and U_σ is uncountable. If $\sigma \in 2^n$, define $U_{\sigma\cap 0}$ and $U_{\sigma\cap 1}$ as follows: since $\overline{U_\sigma}$ is compact, T_2 and uncountable, there is some point p_0 such that every neighborhood of p_0 in U_σ is uncountable. Since X is first countable, p_0 has a local neighborhood base $\{B_n | n < \omega\}$. Since U_σ is uncountable, there is some k such that $\overline{U_\sigma} \setminus B_k$ is uncountable, so similarly since $\overline{U_\sigma} \setminus B_k$ is compact, T_2 and uncountable, there is some $p_1 \in \overline{U_\sigma} \setminus B_k$ such that every neighborhood of p_1 is uncountable. By T_2 there are open sets $U_{\sigma\cap 0}$ and $U_{\sigma\cap 1}$ such that $p_i \in U_{\sigma\cap i}$ and $U_{\sigma\cap i}$ is uncountable for $i = 0, 1$, and $\overline{U_{\sigma\cap 0}} \cap \overline{U_{\sigma\cap 1}} = \emptyset$ and $\overline{U_{\sigma\cap 0}} \cup \overline{U_{\sigma\cap 1}} \subset \overline{U_\sigma}$. This defines U_σ for every $\sigma \in 2^{<\omega}$ and choosing $x_f \in \cap_{n<\omega}\overline{U_{f|n}}$ for every $f \in 2^\omega$, $2^\omega = |\{x_f | f \in 2^\omega\}| \leq |X|$. $\Box$

Definition. Let $\{X_\alpha : \alpha < \kappa\}$ be a pairwise-disjoint collection of topological spaces, and let $X = \bigcup_{\alpha<\kappa} X_\alpha$. Define a topology on X by declaring $U \subset X$ to be open iff $U \cap X_\alpha$ is open in X_α for every $\alpha < \kappa$. X with this topology is called the *topological sum* of the X_α's, and we write $X = \oplus_{\alpha<\kappa} X_\alpha$. Note that each X_α is both closed and open in the topological sum X, and X_α as a subspace of X has the same topology as X_α by itself.

Often one also considers a topological sum of a collection $\{X_\alpha : \alpha < \kappa\}$ of not necessarily disjoint spaces by replacing each X_α with a space X'_α homeomorphic to X_α in such a way that the collection $\{X'_\alpha : \alpha < \kappa\}$ is pairwise-disjoint and then take the topological sum of the X'_α's. One convenient way to do this is to replace X_α with $\{\alpha\} \times X_\alpha$.

Lemma 91. *Let X be a locally compact σ-compact Hausdorff space, let $\{K_n\}_{n\in\omega}$ be as in Lemma 90, and let $Y = \oplus_{n\in\omega}\{n\} \times K_n$. Define $q : Y \to X$ by $q((n,x)) = x$. Then q is continuous, onto, quotient and compact covering.*

Proof. First off, q is clearly onto. So let us now see that it is continuous. Let $U \subset X$ be open, then $U \cap K_i$ is open in K_i for each i, and

$$q^{-1}(U) = \{(n,x) \in Y : x \in U\} = \cup_{n\in\omega}\{(i,x) : x \in K_i \cap U\}$$

which is open in Y. Now, let us see that q is a quotient map, suppose $U \subset X$ and $q^{-1}(U) = V$ is open in Y. For every $x \in U$ there is some $n \in \omega$ such that $x \in int(K_n)$, then $(n,x) \in V \cap (\{n\} \times int(K_n))$ and so $x \in \pi_n(V \cap (\{n\} \times int(K_n))) \subset U$ and hence U is open. Finally, to see that q is compact covering, let $K \subset X$ be compact, then there exists $n \in \omega$ such that $K \subset K_n$ and so $\{n\} \times K$ is compact in Y and $q(\{n\} \times K) = K$.

Lemma 92. *If* $Y = \oplus_{\alpha < \mathcal{K}} Y_\alpha$ *then* $C_K(Y) \cong \Pi_{\alpha<\kappa} C_K(Y_\alpha)$

Proof. Define $\phi : C_K(Y) \to \Pi_{\alpha<\kappa} C_K(Y_\alpha)$ by $\phi(f) = (f \restriction Y_\alpha)_{\alpha<\kappa}$ and $\psi : \Pi_{\alpha<\kappa} C_K(Y_\alpha) \to C_K(Y)$ by $\psi((f_\alpha)_{\alpha<\kappa})(y) = f_\alpha(y)$. Then ϕ and ψ are inverses. To see that ϕ is continuous, let $U = \cap_{\alpha\in F} \pi_\alpha^{-1}(B_\alpha)$ where $F \subset \kappa$ is finite and $B_\alpha = \cap_{i \le n_\alpha} [K_{\alpha i}, U_{\alpha i}] \subset C_K(Y_\alpha)$. Claim: If $f \in \phi^{-1}(U)$ then $f \in \cap_{\alpha\in F, i\le n_\alpha} [K_{\alpha i}, U_{\alpha i}] = V \subset \phi^{-1}(U)$. Proof: Since $f \in \phi^{-1}(U)$ then for every $\alpha \in F$ and $i \le n_\alpha$, $f(K_{\alpha i}) = f \restriction K_\alpha(K_{\alpha i}) \subset U_{\alpha i}$ and so $f \in V$. Now let $g \in V$, then for every $\alpha \in F$ and $i \le n_\alpha$ $g \restriction K_\alpha(K_{\alpha i}) = g(K_{\alpha i}) \subset U_{\alpha i}$ and so $\phi(g) \in U$. Now, to see that ψ is continuous, let $U = \cap_{i<n}[K_i, U_i]$ be open in $C_K(Y)$. Claim: If $\psi((f_\alpha)_{\alpha<\kappa}) \in U$ then $(f_\alpha)_{\alpha<\kappa} \in \cap_{\alpha\in F} \pi^{-1}(\cap_{i<n}[K_{\alpha i}, U_i]) \subset \psi^{-1}(U)$ where $K_{\alpha i} = K_i \cap Y_\alpha$ and $F = \{\alpha : Y_\alpha \cap K_i \neq \emptyset \text{ for some } i < n\}$ (note: F is finite). Proof: If $f = \psi((f_\alpha)_{\alpha<\kappa})$ the for every $i, \alpha \in F$, $f_\alpha(K_{\alpha i}) = f \restriction Y_\alpha(K_{\alpha i}) = f(K_{\alpha i}) \subset f(K_i) \subset U_i$, and so $(f_\alpha)_{\alpha<\kappa} \in \cap_{\alpha\in F} \pi^{-1}(\cap_{i<n}[K_{\alpha i}, U_i])$. If $(g_\alpha)_{\alpha<\kappa} \in \cap_{\alpha\in F} \pi^{-1}(\cap_{i<n}[K_{\alpha i}, U_i])$, then for every i, $\psi((g_\alpha)_{\alpha<\kappa})(K_i) = \cup_{\alpha\in F} g_\alpha(K_{\alpha I}) \subset U_i$ and so $\psi((g_\alpha)_{\alpha<\kappa}) \in \cap_{i<n}[K_i, U_i]$. ☐

Theorem 93. *If X is locally compact Hausdorff and σ-compact, then $C_K(X)$ is completely metrizable.*

Proof. Follows from Theorem 89, Lemma 90, Lemma 91, Lemma 92, taking $Y = \oplus_{i\in\omega} K_i$ where $\{K_i\}_{i\in\omega}$ is as in Lemma 90 and the fact that a countable product of completely metrizable spaces is completely metrizable and closed subsets of completely metrizable spaces are completely metrizable. ☐

Theorem 94. *If X is a paracompact locally compact Hausdorff space, then X is a topological sum of σ-compact spaces.*

Proof. Since X is paracompact and locally compact, there exists a locally finite cover $\mathcal{U}$ of open sets with compact closures. Note that for each $U \in \mathcal{U}$, U meets finitely other members of $\mathcal{U}$. For every $U \in \mathcal{U}$ define $\mathcal{C}(U) = \{V \in \mathcal{U} : V \text{ and } U \text{ are linked by a finite chain in } U\}$. Clearly $\cup\mathcal{C}(U)$ is open. Also, $\cup\mathcal{C}(U)$ is closed since if $x \in X \setminus \cup\mathcal{C}(U)$, then there is some $U' \in \mathcal{U}$ disjoint from every member in $\mathcal{C}(U)$ such that $x \in U' \subset X \setminus \cup\mathcal{C}(U)$. Each $\cup\mathcal{C}(U) = \cup_{V\in\mathcal{C}(U)} \overline{V}$ each of which is compact. Also, each $\mathcal{C}(U)$ is countable for if you look at the minimum length of linked chains to elements from $\mathcal{U}$, each level is finite, so X is the union of disjoint clopen subsets, each of which is σ-compact. ☐

Corollary 95. *Let X be a locally compact Hausdoff space. If X is paracompact, then $C_k(X)$ is a product of completely metrizable spaces, hence is Choquet.*

Proof. Suppose X is a paracompact locally compact Hausdorff space. By Theorem 94, X is a topological sum of locally compact σ-compact spaces, and then it follows from Lemma 92 and Theorem 93 that $C_k(X)$ is homeomorphic to a product of completely metrizable spaces. Such a product is Choquet by Theorem 79. ☐

Remark. Soon will get a converse to Corollary 95, i.e., for a locally compact Hausdorff space X, $C_k(X)$ is Choquet iff X is paracompact.

Recall that a collection $\{H_\alpha : \alpha \in \kappa\}$ of subsets of a space X is called a *discrete* collection if each point of X has a neighborhood which meets at most one member of the collection. We will also say that $\{H_\alpha : \alpha \in \kappa\}$ has a *discrete open expansion* if there are open sets O_α, $\alpha < \kappa$, such that $O_\alpha \supset H_\alpha$ and $\{O_\alpha : \alpha \in \kappa\}$ is a discrete collection.

Definition. We define the games $G_{K,L}(X)$ and $G^o_{K,L}(X)$ played on a space X as follows. There are two players, K and L. In the n^{th} round, K chooses a compact subset K_n of X, and L responds with a compact subset L_n of X such that $K_n \cap L_n = \emptyset$. We say that K *wins* the game $G_{K,L}(X)$ (resp., $G^o_{K,L}(X)$) if $\{L_n : n \in \omega\}$ is a discrete collection (resp., has a discrete open expansion).

For example, if $X = \mathbb{R}$, K has an easy winning strategy: she chooses $K_n = [-x_n, x_n]$, where x_n is $\geq n$ and large enough so that $[-x_n, x_n] \supset \bigcup_{i<n} L_i = \emptyset$. A similar idea shows that K has a winning strategy in any locally compact σ-compact Hausdorff space.

Recall that in normal spaces, countable discrete collections of closed sets have discrete open expansions. Hence, for normal spaces X, the two games are equivalent. $G_{K,L}(X)$ seems simpler and more natural than $G^o_{K,L}(X)$. The next theorem is the reason for the interest in the latter. First, a useful lemma.

Lemma 96. *Let X be completely regular, K and L disjoint compact sets, $f, g \in C(X)$, and $\epsilon, \delta > 0$. Then $B(f, K, \epsilon) \cap B(g, L, \delta) \neq \emptyset$.*

Proof. Because K and L are closed and disjoint, and because zero sets form a base for X (because X completely regular), for each $k \in K$ there is a zero-set neighborhood M_k of k such that $M_k \cap L = \emptyset$. K is compact, so there is a finite cover of K by such zero-set neighborhoods, and since the union of finitely many zero-sets is a zero set, there is a zero set neighborhood of K that misses L; call one such zero-set neighborhood M. M is closed and $M \cap L = \emptyset$, hence for each $l \in L$ there is a zero set neighborhood N_l of l such that N_l misses M; similarly, because L is compact, a zero-set neighborhood of L can be formed from the union of a finite number of elements in $\{N_l\}_{l \in L}$. Call one such zero-set neighborhood N; it follows that M misses N, $K \subset M$ and $L \subset N$. M and N are disjoint zero sets, hence there is $s : X \to [0, 1]$ such that s is continuous, $s^{-1}(0) = M$, and $s^{-1}(1) = N$. Let $h : X \to \mathbb{R}$ be defined as $h = (1 - s)f + (s)g$. h is the composition of functions in $C(X)$, thus h is in $C(X)$. If $k \in K \subset M$, then $s(k) = 0$, which means $h(k) = (1 - 0)f(k) + (0)g(k) = f(k)$ and $h \in B(f, K, \epsilon)$; similarly, if $l \in L$, then $s(l) = 1$ and $h(l) = (1-1)f(l) + (1)g(l) = g(l)$, meaning $h \in B(g, L, \delta)$. Thus, $B(f, K, \epsilon) \cap B(g, L, \delta) \neq \emptyset$. □

Theorem 97 *Let X be completely regular.*

(a) *If $C_k(X)$ is Baire, then L has no winning strategy in $G^O_{K,L}(X)$;*
(b) *If $C_k(X)$ is Choquet, then K has a winning strategy in $G^O_{K,L}(X)$.*

Proof of (a). Suppose L has a winning strategy, we will construct a strategy for E in Choquest game on $C_k(X)$.

E's first move: Chooses $C_k(X)$

NE chooses $B(f_0, K_0, \epsilon_0)$. Now have L_0 be L's response to K_0 in $G^0_{K,L}(X)$ using the winning strategy.

E replies: $B(f_0, K_0, \epsilon_0) \cap B(C_0, L_0, \frac{1}{3})$ where C_0 is constant 0.

Now NE: $B(f_1, K_1, \epsilon_1)$ and let L_1 be L's response to sequence (K_0, L_0, K_1) in $G^0_{K,L}(X)$

E Plays: $B(f_1, K_1, \epsilon_1) \cap B(C_1, L_1, \frac{1}{3})$ where C_1 is constant 1.

NE plays: $B(f_2, K_2, \epsilon_2)$. ...

Claim: E wins over NE

Suppose $f \in \bigcap_{i \in \omega} B(f_i, K_i, \epsilon_i) \cap B(c_i, L_i, \frac{1}{3})$

then $f(L_i) \subset (i - \frac{1}{2}, i + \frac{1}{2})$

If L played by a winning strategy then $\{L_i\}_{i \in \omega}$ does not have a discrete open expansion, so $f(L_i) \subset V_i = (i - \frac{1}{3}, i + \frac{1}{3})$ which implies that $L_i \subset f^{-1}(V_i)$. Now $\{V_i\}_{i \in \omega}$ is a discrete collection of open sets in $\mathbb{R}$, and hence $\{f^{-1}(V_i)\}_{i \in \omega}$ is discrete in X. Contradiction.

Proof of (b). This is similar to (a), except that now NE is playing by a winning strategy, and we use this to find a winning strategy for K in $G^o_{K,L}(X)$.

Let E's first move be $C_k(X)$, and let

NE's reply using a winning strategy be $B(f_0, K_0, \epsilon_0)$. Then K's first move in $G^o_{K,L}(X)$ is K_0.

Let L_0 be L's response.

E replies: $B(f_0, K_0, \epsilon_0) \cap B(c_0, L_0, \frac{1}{3})$ where c_0 is constant 0.

Now NE plays $B(f_1, K_1, \epsilon_1)$ using his winning strategy, and then K plays K_1 as his next move. Let L_1 be L's response to sequence (K_0, L_0, K_1) in $G^0_{K,L}(X)$.

Let E plays $B(f_1, K_1, \epsilon_1) \cap B(c_1, L_1, \frac{1}{3})$ where c_1 is constant 1.

NE, using his winning strategy, plays $B(f_2, K_2, \epsilon_2)$. ...

This process defines a strategy for K in $G^o_{K,L}(X)$. We claim that it is winning. Since NE was playing by a winning strategy, there is $f \in \bigcap_{i \in \omega} B(f_i, K_i, \epsilon_i) \cap B(c_i, L_i, \frac{1}{3})$. Then, as in (a), we see that the L_i's have a discrete open expansion, so K wins. □

It's an open question whether or not the converses of 97(a) and (b) hold. We will show that they do hold in case X is locally compact. First, a result which says that the two games are the equivalent in this case.

Theorem 98. *Suppose X is locally compact T_2. Then L has a winning strategy in $G_{K,L}(X)$ iff L has a winning strategy in $G^o_{K,L}(X)$.*

Proof: Let σ be a winning strategy for L in $G^o_{K,L}(X)$. We will construct a winning strategy μ for L in $G_{K,L}(X)$. Let K_0 be any opening choice for K in $G_{K,L}(X)$. Let $L_0 = \sigma(K_0)$. For each $x \in L_0$ choose a neighborhood U_x of x with compact closure that missed K_0. Then the $U'_x s$ form an open cover. Let U_0 be the union of a finite subcover of the $U'_x s$. Then $L_0 \subseteq U_0$ open, and $\overline{U_0}$ is compact and misses K_0. Define $L'_0 = \overline{U_0}$ and set $\mu(K_0) = L'_0$. Suppose $K_0, L_0, L'0, K_1, L_1, L'_1, \ldots, K_n, L_n, L'_n$ have been defined. Let K_{n+1} be any response by K in $G_{K,L}(X)$ to L'_n. Let $L_{n+1} = \sigma(K_0, L_0, K_1, L_1, \ldots, K_n, L_n, K_{n+1})$. As above let U_{n+1} be an open set such that $L_{n+1} \subseteq U_{n+1}$, $\overline{U_{n+1}}$ is compact and $\overline{U_{n+1}} \cap \bigcup_{i=0}^{n+1} K_i = \emptyset$. Define $L'_{n+1} = \overline{U_{n+1}}$, and set $\mu(K_0, L'_0, K_1, L'1, \ldots, K_{n+1}) = L'_{n+1}$. Continue inductively. Then since L has a winning strategy in $G^o_{K,L}(X)$ it follows that $\{L_i\}$ have no discrete open expansion; but $\{L'_i\}$ contains an open expansion of $\{L_i\}$ and hence it's not discrete. So μ is a winning strategy for L in $G_{K,L}(X)$. □

NOTICE: I use the following terminology in the next theorem. If σ is a strategy for K in the game $G_{K,L}(X)$, and $K_1, L_1, \dots, K_n, L_n, K_{n+1}$ is a sequence of compact sets in X, then this sequence is called a **valid sequence of play determined by** σ (or rather $\sigma - VSOP$) if $K_1 = \sigma(\emptyset)$ and for each $i \leq n$, L_i misses K_i and $K_{i+1} = \sigma(K_1, L_1, \dots, K_i, L_i)$.

Theorem 99. *If X is locally compact and Hausdorff, then the following are equivalent:*

- *(a) K has a winning strategy in $G_{K,L}(X)$;*
- *(b) K has a winning strategy in $G^o_{K,L}(X)$;*
- *(c) $C_k(X)$ is Choquet;*
- *(d) $C_k(X)$ is homeomorphic to a product of completely metrizable spaces;*
- *(e) X is paracompact.*

Proof. The implications $(e) \Rightarrow (d) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a)$ are fairly straight forward. $((e) \Rightarrow (d))$ Recall that a paracompact, locally compact space is the sum of a countable union of σ-compact spaces; thus $C_k(X)$ is homeomorphic to a product of completely metrizable spaces (by 95). $((d) \Rightarrow (c))$ The product of completely metrizable spaces is completely metrizable, and thus Choquet. $((c) \Rightarrow (b))$ This is a result of Theorem 97 (b).

Lastly, the implication $(a) \Rightarrow (e)$ will be shown.

Let $\mathcal{U}$ be an open cover of X with sets having compact closures. Suppose that σ is a winning strategy for K in $G_{K,L}(X)$.

1. If Q is a compact subset of X let $\mathcal{U}(Q)$ be a finite cover of Q from $\mathcal{U}$;
2. If $\mathcal{V} \subset \mathcal{U}$, let $\mathcal{V}^* = \{\overline{V \setminus \mathcal{W}} : V \in \mathcal{V},\ \textit{and}\ \mathcal{W} \text{ is a finite subset of } \mathcal{V}\}$.
3. If $\mathcal{V} \subset \mathcal{U}$, $\mathcal{V}$ is called $\sigma^* - closed$ means that
   - (i) $V \in \mathcal{V} \Rightarrow \mathcal{U}(\overline{V}) \subset \mathcal{V}$, and
   - (ii) if $L_1, L_2, \dots, L_n \in \mathcal{V}^*$ and $K_1, L_1, \dots, K_n, L_n, K_{n+1}$ is $\sigma - VSOP$, then $\mathcal{U}(K_{n+1}) \subset \mathcal{V}$.

Fact 1: If $\mathcal{V} \subset \mathcal{U}$ is infinite, then there is a $\sigma^* - closed$ set $\mathcal{V}' \subset \mathcal{U}$ such that $\mathcal{V} \subset \mathcal{V}'$ and $|\mathcal{V}'| = |\mathcal{V}|$.

To show this, let $\mathcal{V}_0 = \mathcal{V}$ and note that $\mathcal{V}_0^*$ has cardinality less than $\mathcal{V}$. Let

$$\begin{aligned}\mathcal{V}_1 \quad = \quad & \mathcal{V}_0 \cup (\cup\{\mathcal{U}(\overline{V}) : V \in \mathcal{V}_0\}) \\ & \cup\{\mathcal{U}(K_{m+1}) : \exists L_1, L_2, \dots, L_m \in V_0 \textit{ and } K_1, L_1, \dots, K_n, L_n, K_{m+1} \textit{ is } \sigma - VSOP\}.\end{aligned}$$

Inductively, if $\mathcal{V}_n$ is defined, if

$$\begin{aligned}\mathcal{V}_{n+1} \quad = \quad & \mathcal{V}_n \cup (\cup\{\mathcal{U}(\overline{V}) : V \in \mathcal{V}_n\}) \\ & \cup\{\mathcal{U}(K_{m+1}) : \exists L_1, L_2, \dots, L_m \in V_n \textit{ and } K_1, L_1, \dots, K_n, L_n, K_{m+1} \textit{ is } \sigma - VSOP\}.\end{aligned}$$

Let $\mathcal{V}' = \cup_{i \in \omega} \mathcal{V}_i$, in each step of the inductive definition, $|\mathcal{V}_{n+1}| = |\mathcal{V}_n|$, thus $|\mathcal{V}'| = |\mathcal{V}|$. $\mathcal{V}'$ is $\sigma^* - closed$, for if $V \in \mathcal{V}'$, then $V \in \mathcal{V}_n$ for some $n \in \omega$ and $U\overline{V} \in \mathcal{V}_{n+1} \subset \mathcal{V}'$; and if $L_1, \dots L_m$ is a sequence of compact sets in $\mathcal{V}'$ and $K_1, L_1, \dots, K_n, L_n, K_{n+1}$ is $\sigma - VSOP$, then there is $n \in \omega$ such that $L_1, \dots, L_m \in \mathcal{V}_n$ and hence $K_{n+1} \subset \mathcal{V}_{n+1} \subset \mathcal{V}'$. □

Fact 2. If $\mathcal{V}$ is σ^*-closed, $\cup\mathcal{V}$ is clopen.

Proof. $\cup\mathcal{V}$ is clearly open. We show it is closed. Let $x \in \overline{\cup\mathcal{V}} \setminus \cup\mathcal{V}$. Choose a compact neighborhood N of x. Let $K_0 = \sigma(\emptyset)$. By property *ii)* of σ^*-closed subcovers, $\mathcal{U}(K_0) \subset \mathcal{V}$ and by property *i)*, $\overline{V} \subset \cup\mathcal{V}$ for any $V \in \mathcal{U}(K_0)$. Thus

$\overline{\cup\mathcal{U}(K_0)} \subset \cup\mathcal{V}$ and so $x \notin \overline{\cup\mathcal{U}(K_0)}$. There is some $V_0 \in \mathcal{V}$ so that $V_0 \cap (N \setminus \overline{\cup\mathcal{U}(K_0)}) \neq \emptyset$. Let $L_0 = \overline{V_0 \setminus \cup\mathcal{U}(K_0)}$. Then $L_0 \in \mathcal{V}^*$, $L_0 \cap K_0 = \emptyset$ and $L_0 \cap N \neq \emptyset$.

Continue in this fashion, with K playing by the winning strategy with n^{th} choice K_n and L responding with an $L_n \in \mathcal{V}^*$ such that $L_n \cap K_n = \emptyset$ and $L_n \cap N \neq \emptyset$ for every $n \in \omega$. Then since N is compact, $\{L_n \mid n \in \omega\}$ is not discrete, contradicting K's strategy.

Fact 3. If $\mathcal{V}$ is σ^*-closed and $|\mathcal{V}| = \kappa > \omega$, then there is a collection $\{\mathcal{V}{-}\alpha \mid \alpha < \kappa\}$ such that

(a) $\mathcal{V} = \cup_{\alpha<\kappa}$;
(b) each $\mathcal{V}\alpha$ is σ^*-closed;
(c) $\alpha < \beta \Rightarrow \mathcal{V}_\alpha \subset \mathcal{V}_\beta$;
(d) $|\mathcal{V}_\alpha| \leq |\alpha| + \omega$;
(e) α a limit ordinal $\Rightarrow \mathcal{V}_\alpha = \cup_{\beta<\alpha}\mathcal{V}_\beta$.

Proof. Let $\mathcal{V} = \{V_\alpha \mid \alpha < \kappa\}$. Let $\mathcal{V}_0$ be any countable σ^*-closed collection with $V_0 \in \mathcal{V}_0$. Inductively, suppose $\mathcal{V}_\beta$ is defined for all $\beta < \alpha$. If α is a limit ordinal, let $\mathcal{V}_\alpha = \cup_{\beta<\alpha}\mathcal{V}_\beta$. If $\alpha = \gamma + 1$, let $\mathcal{V}_\alpha$ be a σ^*-closed collection containing containing $\mathcal{V}_\gamma \cup \{V_\gamma\}$. Then $\{\mathcal{V} - \alpha \mid \alpha < \kappa\}$ satisfies (a) - (e).

Fact 4. If $\mathcal{V}$ is σ^*-closed, there is a locally finite open refinement of $\mathcal{V}$ covering $\cup\mathcal{V}$.

Proof. By induction on $|\mathcal{V}|$. If $|\mathcal{V}| \leq \omega$, $\cup\mathcal{V} = \cup\{\overline{V} \mid V \in \mathcal{V}\}$ is σ-compact and thus Lindelöf and thus paracompact. If $|\mathcal{V}| = \kappa > \omega$ and this fact holds for all collections of cardinality $< \kappa$, let $\mathcal{V} = \cup_{\alpha<\kappa}\mathcal{V}_\alpha$ as in 3. By induction, each $\mathcal{V}_\alpha$ has a locally finite open refinement $\mathcal{W}_\alpha$. By 2, $\cup\mathcal{V}_{\alpha+1} \setminus \cup\mathcal{V}_\alpha$ is clopen and each $x \in \cup\mathcal{V}$ is in exactly one $\cup\mathcal{V}_{\alpha+1} \setminus \cup\mathcal{V}_\alpha$. Let $\mathcal{W}'_\alpha = \{W \setminus \cup\mathcal{V}_\alpha \mid W \in \mathcal{W}_{\alpha+1}\}$. Then $\mathcal{W}'_\alpha$ is a locally finite cover of $\cup\mathcal{W}'_\alpha = \cup\mathcal{V}_{\alpha+1} \setminus \cup\mathcal{V}_\alpha$. Thus $\mathcal{W} = \mathcal{W}_0 \cup \cup_{\alpha<\kappa}\mathcal{W}'_\alpha$ is a locally finite open refinement of $\mathcal{V}$. If $x \in \cup\mathcal{V}$ there is some least α so that $x \in \cup\mathcal{V}_\alpha$. By 3(e), α cannot be a limit ordinal, so if $\alpha = \gamma + 1$, $x \in \cup\mathcal{W}'_\alpha$. Thus, $\mathcal{W}$ is a locally finite open refinement of $\mathcal{V}$ and $\cup\mathcal{W} = \cup\mathcal{V}$.

Thus $\mathcal{U}$ has a locally finite open refinement covering $\cup\mathcal{U} = X$ and so X is paracompact. □

Remark. It is not known if (b) and (c) are equivalent for all completely regular spaces.

Theorem 100. *Suppose X is locally compact, then $C_k(X)$ is completely metrizable iff X is σ-compact.*

Proof. Suppose $C_k(X)$ is completely metrizable. By theorem 99, X is paracompact. So X is the topological sum of σ-compact spaces $X = \bigoplus_{\alpha\in\kappa} X_\alpha$. So $C_k(X) \cong \prod_{\alpha\in\kappa} C_k(X_\alpha)$. Since $\prod_{\alpha\in\kappa} C_k(X_\alpha)$ is metrizable it follows that $|C_k(X_\alpha)| \leq 1$ for all but countably many α. Hence $\kappa \leq \omega$. So X is the topological sum of countably many σ-compact spaces, hence it's σ-compact. The converse follows from theorem 93. □

Now we will go on to show that for locally compact spaces X, $C_k(X)$ is Baire iff L has no winning strategy in $G^o_{K,L}(X)$. But what we really would like is an internal property of X, not a game-theoretic property, such that X has this property iff $C_k(X)$ is Baire. To that end, we make the following definition.

Definition. A collection $\mathcal{L}$ of non-empty subsets of X is said to *move off the compact sets* if for every compact subset K of X, there is some $L \in \mathcal{L}$ with $K \cap L = \emptyset$. The space X is said to have the *Moving Off Property (MOP)* iff every collection $\mathcal{L}$ of compact sets which moves off the compact sets contains an infinite subcollection which has a discrete open expansion.

Remark. Note that any compact space has the MOP because no collection of non-empty compact sets moves off the compact sets. More generally:

Theorem 101. *If X is T_2, paracompact, and locally compact, then X has the MOP.*

Proof. If X is compact, then X has MOP.

Suppose that X is not compact and that $\mathcal{L}$ moves off compact sets in X. X is T_2, locally compact, and paracompact so X can be expressed as the sum of $\sigma-$compact spaces (Theorem 94). Let $X = \oplus_{\alpha \in \Lambda} X_\alpha$, where X_α is $\sigma-$compact.

Notice that if $L \in \mathcal{L}$, then L is compact and the set of $\alpha \in \Lambda$ such that $X_\alpha \cap L \neq \emptyset$ is finite; hence, if $L \in \mathcal{L}$, let $\Lambda(L)$ be a finite subset of Λ such that $L \subset \cup_{\alpha \in \Lambda(L)} X_\alpha$.

Pick $L_0 \in \mathcal{L}$ and let $X^0 = \cup_{\alpha \in \Lambda(L_0)} X_\alpha$. X^0 is the finite sum of $\sigma-$compact spaces and is therefore $\sigma-$compact. From results of Theorem 90, $\{C_i^0\}_{i \in \omega}$ can be chosen to be an increasing sequence of compact subsets of X^0 such that

(i) $\cup_{i \in \omega} C_i^0 = X^0$,
(ii) $C_i^0 \subset Int(C_{i+1}^0)$.

Without loss of generality we can assume that (iii) $L_0 \subset Int(C_0^0)$; this follows because a sequence with properties (i) and (ii) will have a least term that whose interior contains L_0, and the subsequence formed by dropping prior terms and renumbering will have property (iii).

Continuing inductively, if $n \in \omega$ and L_i and C_n^i are each defined for each $i \leq n$, then let $L_{n+1} \in \mathcal{L}$ such that L_{n+1} misses $\cup_{i \leq n} C_n^i$ (possible since $\mathcal{L}$ moves off compact sets). Let $X^{n+1} = \cup_{\alpha \in \Lambda(L)} X_\alpha$ and let $\{C_i^{n+1}\}_{i \in \omega}$ be an increasing sequence of compact sets such that

(i) $\cup_{i \in \omega} C_i^{n+1} = X^{n+1}$
(ii) $C_i^{n+1} \subset Int(C_{i+1}^{n+1}$, and
(ii) $L_{n+1} \subset Int(C_{n+1}^{n+1})$.

Claim: $\mathcal{L}' = \{L_i\}_{i \in \omega}$ discrete. Notice that $L_0 \subset Int(C_0^0)$ and if $n > 0$, then $L_n \subset Int(C_n^n) \setminus \cup_{i<n} C_{n-1}^i$; thus, $\mathcal{L}'$ is pairwise disjoint. To show $\mathcal{L}'$ is closure preserving, first note that if $x \notin \cup_{i \in \omega} X^i$, then x is not in the closure of $\cup \mathcal{L}'$ because $\cup_{i \in \omega} X^i = \cup\{X_\alpha : \alpha \in \cup_{i \in \omega} \Lambda(L_i)\}$ is clopen in X. If $x \in X'$, then there is $m, n \in \omega$ such that $x \in Int(C_n^m)$, thus $x \in Int(\cup_{i \leq n} C_n^i)$ and misses the closure of $\cup_{i > n} L_i$. It follows that there is an open neighborhood of x that intersects at most one element in $\{L_0, L_1, \ldots, L_n\}$.

$\therefore \mathcal{L}'$ is discrete. □

We will soon see that the MOP is equivalent to L having no winning strategy in $G^o_{K,L}(X)$. First, it will be useful to get a result about pseudocompact spaces with the MOP. Recall that X is *pseudocompact* if every continuous real-valued function on X is bounded.

Lemma 102. *Let X be a completely regular space. Then X is pseudocompact if and only if every discrete collection of open sets of X is finite.*

Proof. Assume there exists $\mathcal{D}$ an infinite discrete collection of open subsets of X. WLOG $\mathcal{D}$ is countable.

For every $D_i \in \mathcal{D}$ let $X_i \in D_i$. Let $f_i : X_i \to \mathbb{R}$ be such that $f_i(x_i) = i$ and $f(X \setminus D_i) \subset 0$.

Look at $f = \sum f_i$:

if $x \in X \setminus \bigcup \mathcal{D}$, then $f_i(x) = 0$ for all i. which implies $f(x) = 0$ if $x \in D_i$ for some i then $f_j(x) = 0$ for all $j \neq i$, and hence $f(x) = f_i(x)$.

f is unbounded (obviously)

For continuity: obviously for $x \in \bigcup \mathcal{D}$ and $x \notin \bigcup_{i\in\omega} \overline{D_i} = \overline{\bigcup_{i\in\omega} D_i}$ So suppose $x \in \overline{D_i} \setminus D_i$. Then there exists a $U_x \subset X$, open, such that U_x only meets D_i. Let $V \subset \mathbb{R}$ be open. Since f_i is continuous, there exists $U'_x \subset X$ such that $f_i(U'_x) \subset V$. If $y \in U_x \cap U'_x$ and if $Y \notin D_i$, then $y \notin D_j$ for all $j \neq i$, so $f(y) = 0 \in V$. If $y \in D_i$, then since $y \in U'_x$, $f(y) = f_i(y) \in V$ and so $f(U_x \cap U'_x) \subset V$.

For the other direction:

Claim: If there exists $f : \to \mathbb{R}$ that is continuous and unbounded, then there exists an infinite discrete collection of open sets.

Suppose $f : X \to [0, \infty)$ is unbounded. Let $y_0, y_1, y_2, \dots$ be an increasing unbounded set and for each $i \in \omega$, let $y_i \in f(X)$. Then let $V_0 = (y_0 - 1, y_0 + \frac{y_1 - y_0}{3})$, $V_1 = (y_1 - \frac{y_1 - y_0}{3}, y_1 + \frac{y_2 - y_1}{3})$, $V_n = (y_n - \frac{y_n - y_{n-1}}{3}, y_n + \frac{y_{n+1} - y_n}{3})$

Then if $i \neq j$, $\overline{V_i} \cap \overline{V_j} = \emptyset$. So $\{f^{-1}(V_i) : i \in \omega\}$ is a collection of non empty open sets. Let $x \in X$, and consider $f(x)$. There is an open set $U \subset \mathbb{R}$ such that $f(x) \in U$ and U meets at most one V_i Then $x \in f^{-1}(U)$ and $f^{-1}(U)$ meets at most one $f^{-1}(V_i)$. $\square$.

Theorem 103. *Suppose X is completely regular, psuedocompact and has MOP. Then X is compact*

Proof. Every collection of compact sets which moves off compact sets has an infinite subcollection with a discrete open expansion. By theorem 102, no infinite collection of open sets is discrete. So no collection of compact sets moves off. Let $L = \{\{x\} : x \in X\}$. There must be a compact K such that $K \cap \{x\} \neq \emptyset$. The only such K must be X. So X is compact. $\square$

Theorem 104 *The following are equivalent for a completely regular space X*

(a) X has the MOP
(b) Whenever $\{\mathcal{L}_i\}_{i\in\omega}$ is a sequence of compact sets each of which moves off the compact sets, then there are $L_i \in \mathcal{L}_i$ suh that $\{L_i\}_{i\in\omega}$ has a discrete open expanstion
(c) L has no winning strategy in $G^O_{K,L}(X)$

Proof. $(a) \Rightarrow (b)$:

Choose $\{x_i\}_{i\in\omega}$ that has a discrete open expansion and let $\mathcal{L} = \{\{x_n\} \cup \bigcup_{i\leq n}\{L_i : n \in \omega \textit{ and } L_i \in \mathcal{L}_i\}\}$.

Claim: $\mathcal{L}$ move off compact sets:

Suppose C is compact. Choose $n \in \omega$ such that $x_n \notin C$ and for each $i \leq n$ we can choose $L_i \in \mathcal{L}_i$ such that $L_i \cap C = \emptyset$. This implies $\{x_n\} \cup \bigcup_{i\leq n} L_i \in \mathcal{L}$ and misses C. Therefore $\mathcal{L}$ moves off compact sets. ✓

Now pick $\{L^i\}_{i\in\omega} \subset \mathcal{L}$ with a discrete open expansion. Let $L^n = \{x_{n(i)}\} \cup \bigcup_{j\leq n(i)} L^i_j$ with $n(i) \in \omega$. Note: $i \neq i'$ implies $n(i) \neq n(i')$ therefore we can find $i_o < i_1 < i_2 < \ldots <$ such that $n(i_0) < n(i_1) < \ldots <$

Then pick $L_k \in \mathcal{L}_k$ such that $L_k \subset L^{i_k}$. We can do this since $n(i_k) \geq k$ Then $\{L_k\}_{k\in\omega}$ has a discrete open expansion since $\{\mathcal{L}^i_k\}$ does.

$(c) \Rightarrow (a)$ Suppose X does not have MOP. Then there exists a collection $\mathcal{L}$ of compact sets which moves off compact sets but no infinite subcollection has a discrete open expansion. L wins by playing $L_n \in \mathcal{L}$ for all n.

(b)⇒(c) Let σ be a strategy for L. Note that $\{L : L = \sigma(K_0)$ for some $K_0\}$ is moving off, so by (b) there is an infinite subcollection which is discrete, i.e., there is some $(K_0^{<n>}, L_0^{<n>})_{n\in\omega}$ such that $L_0^{<n>} = \sigma(K_0^n)$ and $\mathcal{L}_\emptyset = \{L_0^{<n>}\}_{n\in\omega}$ is discrete, and hence moves off. For $i > 0$, continuing on this way for each $\tau \in \omega^i$ we can get $(K_i^{\tau^\frown m}, L_i^{\tau^\frown m})_{m\in\omega}$ such that $L_i^{\tau^\frown m} = \sigma(K_0^{\tau\restriction 1}, L_0^{\tau\restriction 1}, ..., K_i^{\tau^\frown m})$ and $\mathcal{L}_\tau = \{L_i^{\tau^\frown m}\}_{m\in\omega}$ is discrete, and moves off. For every $\tau \in \omega^{<\omega}$, $\mathcal{L}_\tau$ moves off, so by (b) for each τ there is some $m_\tau \in \omega$ such that $\mathcal{L} = \{L^{\tau^\frown m_\tau}_{dom\tau}\}$ has a discrete open expansion. Let $m_0 = m = \emptyset$, so $L_0^{<m_0>} \in \mathcal{L}$. Then there exists $m_1 \in \omega$ such that $L_1^{<m_0,m_1>} \in \mathcal{L}$, etc. So we get a collection $\mathcal{L}' = \{L_0^{<m_0>}, L_1^{<m_0,m_1>}, L_2^{<m_0,m_1,m_2>}, ..\} \subset \mathcal{L}$ corresponding to a play for L using strategy σ, but $\mathcal{L}'$ has a discrete expansion, hence L loses this play, and so σ is not a winning strategy for L. □

Lemma 105. *Let T be the space $\{\infty\} \cup (\omega \times \omega)$ where points of $\omega \times \omega$ are isolated and basic neighborhoods of ∞ have the form $\{\infty\} \cap ((\omega \setminus n) \times \omega)$ where $n \in \omega$. Then T does not have MOP.*

Proof. For each $i \in \omega$ define $\mathcal{L}_i = \{\{(i,n)\} : n \in \omega\}$. Since each element is a singleton, each is clearly compact. The $\mathcal{L}_i$'s move off since any compact set cannot contain an infinite number of elements from one column. For every $i \in \omega$ let $L_i = \{(i, n_i)\} \in \mathcal{L}_i$, then $\{L_i\}_{i\in\omega}$ is not discrete because every neighborhood of ∞ contains infinitely many columns, and hence infinitely many L_i's. Thus, by Theorem 104, T does not have MOP. □

Theorem 106. *Suppose X is regular and first countable. If X has the MOP, then X is locally compact.*

Proof. Suppose X has the MOP and X is not locally compact. Let $p \in X$ such that no neighborhood of p is compact and let $\{B_i\}_{i\in\omega}$ be a local base at p.

Note that a closed subset of X has the MOP as a subspace, hence $\overline{B_i}$ has MOP for each $i \in \omega$.

Let $n(0) = 0$. For each $i \in \omega$, if $n(i)$ is defined, let $\mathcal{L}_i = \{\{x\}\}_{x\in\overline{B_{n(i)}}}$, thus $\mathcal{L}_i$ moves off compact sets in $\overline{B_{n(i)}}$. $\overline{B_{n(i)}}$ is closed in X and therefore has the MOP, so $D_i = \{x^i_j\}_{j\in\omega}$ may chosen to be an infinite discrete subset of $\mathcal{L}_i$ that does not contain p.

Since D_i is closed in $\overline{B_{n(i)}}$, D_i is closed in X and $p \notin D_i$, hence there is $n(i+1) \in \emptyset$ such that $B_{n(i+1)} \cap D_i = \emptyset$.

Notice that p is the only limit point of $\cup_{i\in\omega}D_i$, and if U is an open set containing p, then there is $n \in \omega$ such that $\cup_{i\geq n}D_i \subset U$. It follows that $\{p\} \cup (\cup_{i\in\omega}D_i)$ is a closed subset of X and therefore has the MOP.

Define $h : T \to X$ such that $h(i, j) = x_i^j$ and $h(\infty) = p$. It follows that h is a homeomorphsm; however, this would mean that T has the MOP, which is not possible. □

Theorem 107. *The following are equivalent for a paracompact first-countable Hausdorff space X.*

(a) $C_k(X)$ *is Baire;*
(b) $C_k(X)$ *is Choquet.*
(c) X *is locally compact.*

Proof. $(b) \to (a)$ follows from theorem 75. $(c) \to (b)$ follows from theorem 99. Assume $C_k(X)$ is Baire. By theorem 97(a) is follows that L has no winning strategy in $G^o_{K,L}(X)$. By theorem 104, this implies that X has the MOP. By theorem 106 we get X is locally compact. □

Definition. A subspace Y of a space X is said to be *first category* in X if Y is contained in the union of countably many nowhere dense subsets of X. X is said to be *first category in itself* if X is the union of countably many of its nowhere dense subsets, or, equivalently, there is a countable collection of dense open sets in X whose intersection is empty. A set Y is *second category* in X if Y is not first category in X, and X is *second category in itself* if X is not first category in itself. Note that X is second category in itself iff the intersection of countably many dense open subsets of X is always nonempty. Thus every Baire space is second category in itself.

Recall that a space X is said to be homogeneous if for any two points $x, y \in X$, there is a homeomorphism $h : X \to X$ such that $h(x) = y$.

Theorem 108: *If X is homogeneous and second category in itself, then X is Baire.*

Proof. Suppose X is not Baire. Then there exists a dense open collection $\{U_n\}_{n\in\omega}$ such that $\bigcap U_n$ is not dense. Then there exists an open $O \neq \emptyset$ such that $O \cap \bigcap_{n\in\omega} U_n = \emptyset$. Now $O \cap U_n$ is dense open in O for all n, but $\bigcap(O \cap U_n) = \emptyset$, hence O is first category in itself.

Let $\{O_\alpha\}_\alpha$ be a maximal disjoint collection of open sets such that for all α, there exists $O_{\alpha n}$, $n \in \omega$ dense open in O_α such that $\bigcap_n O_{\alpha n} = \emptyset$.

Claim: $\bigcup_\alpha O_\alpha$ is dense open in X: Suppose not, then let $y \in X \setminus \overline{\bigcup_\alpha O_\alpha}$. Let $x \in O$. Now there exists a homeomorphism $h : X \to X$ such that $h(x) = y$. So there exists $O' \subset O$ with $x \in O'$, such that $h(O) \subset X \setminus \overline{\bigcup_\alpha O_\alpha}$. Well, $h(O')$ is open and misses all O_α's, and $h(O' \cap U_n)$ us dense in $h(O')$. Also, $\bigcap\limits_n h(O' \cap U_n) = h(\bigcap\limits_n O' \cap U_n) = h(\emptyset) = \emptyset$ since h is a homeomorphism. □.

Definition. A *topological group* is a triple $(G, \mathcal{T}, \cdot)$, where $(G, \cdot)$ is a group, and $\mathcal{T}$ is a topology on the set G such that the mapping from $G \times G$ to G defined by $(g, h) \mapsto g \cdot h^{-1}$ is continuous.

Theorem 109. *$C_k(X)$ is a topological group under addition.*

Proof. Clearly, $C_k(X)$ is a group with identity element the constant function 0. Let $\phi : C_k(X) \times C_k(X) \to C_k(X)$ by $(g, h) \mapsto g - h$. If $(g, h) \in \phi^{-1}(\mathcal{B}(f, K, \epsilon))$, then $|f(x) - (g - h)(x)| < \epsilon$ for every $x \in K$. Let $\delta = \max_{x \in K}\{|f(x) - (g - h)(x)|\} < \epsilon$. If $(g', h') \in \mathcal{B}(g, K, \frac{\epsilon-\delta}{2}) \times \mathcal{B}(h, K, \frac{\epsilon-\delta}{2})$ then for every $x \in K$, $|g(x) - g'(x)|, |h(x) - h'(x)| < \frac{\epsilon-\delta}{2}$ and so for every $x \in K$, $|f(x) - (g' - h')(x)| \leq |f(x) - (g - h)(x)| + |(g - h)(x) - (g' - h')(x)| \leq |f(x) - (g - h)(x)| + |g(x) - g'(x)| + |h(x) - h'(x)| < \delta + \frac{\epsilon-\delta}{2} + \frac{\epsilon-\delta}{2} = \epsilon$ and so ϕ is continuous. □

Lemma 110. *In the definition of topological group, it is equivalent to replace the condition that the map* $(g, h) \mapsto g \cdot h^{-1}$ *is continuous with the condition that the two maps* $(g, h) \mapsto g \cdot h$ *and* $h \mapsto h^{-1}$ *are continuous.*

Proof. $\Rightarrow$: If $\phi : (g, h) \mapsto g \cdot h^{-1}$ is continuous, then if $i : G \to G \times G$ by $g \mapsto (e, g)$, $\psi = \phi \circ i$ is continuous and $\psi(h) = \phi(e, h) = h^{-1}$. Thus $\phi \circ (id_G \times \psi)$ is continuous and $\phi \circ (id_G \times \psi)(g, h) = \phi(g, h^{-1}) = g \cdot h$.

$\Leftarrow$: If $\phi : (g, h) \mapsto g \cdot h$ and $\psi : h \mapsto h^{-1}$ are continuous, then $\phi \circ (id_G \times \psi)$ is continuous and $\phi \circ (id_G \times \psi)(g, h) = \phi(g, h^{-1}) = g \cdot h^{-1}$. □

Theorem 111. *Every topological group is homogeneous.*

Proof. Let G be a topological group and $\phi : G \times G \to G$ be multiplication. For any $g \in G$, $i_g : G \to G \times G$ by $h \mapsto (g, h)$ is continuous, so $\phi_g = \phi \circ i_g$ is continuous. Note $\phi_g(h) = g \cdot h$ and so $\phi_g \circ \phi_{g^{-1}} = \phi_{g^{-1}} \circ \phi_g = id_G$. Thus for any $g \in G$, ϕ_g is a homeomorphism. Let $x, y \in G$. Then $h = \phi_y \circ \phi_{x^{-1}}$ is a homeomorphism and $h(x) = y \cdot x^{-1} \cdot x = y$. Thus G is homogeneous. □

Corollary 112. $C_k(X)$ *is homogeneous.*

Theorem 113. *The following are equivalent for a locally compact Hausdorff space* X*:*

(a) $C_k(X)$ *is Baire;*

(b) X *has the MOP;*

(c) L *has no winning strategy in* $G^o_{K,L}(X)$*;*

(d) L *has no winning strategy in* $G_{K,L}(X)$*.*

Proof. (c) $\iff$ (d) by Theorem 98

(b) $\iff$ (c) by Theorem 104 (noting that locally compact Hausdorff implies completely regular)

(a) $\Rightarrow$ (c) by Theorem 97

We will show that (d) $\Rightarrow$ (a) by showing that if L has no winning strategy in $G_{K,L}(X)$ then $C_k(X)$ is second category in itself. It then follows from Corollary 112 and Theorem 108 that $C_k(X)$ is Baire.

Consider a play of $G_{K,L}(X)$ as follows: K chooses some compact K_0. Let A_0 be compact with $int(A_0) \supset K_0$. Consider $\mathcal{B}(f_0, A_0, 1)$ where f_0 is the constant function 0. We will show that $\mathcal{B}(f_0, A_0, 1)$ is Baire by showing E has no winning strategy in $Ch(\mathcal{B}(f_0, A_0, 1))$. Suppose E plays $\mathcal{B}(f'_0, K'_0, \epsilon'_0) \subset \mathcal{B}(f_0, A_0, 1)$. Note that $K'_0 \supset A_0$. Let C_0 be compact with $int(C_0) \supset K'_0$ and let L respond with $L_0 = C_0 \setminus int(A_0)$

In the next round, K chooses some compact K_1. Let A_1 be compact with $int(A_1) \supset K_1 \cup C_0$. Let NE play $\mathcal{B}(f_1, A_1, \epsilon_1) \subset \mathcal{B}(f'_0, K'_0, \epsilon'_0)$ with $\epsilon_1 \leq \frac{1}{2}$ and $f_1(A_1 \setminus int(C_0)) \subset (-\frac{1}{2}, \frac{1}{2})$. Such a function exists since $\mathcal{B}(f'_0, K'_0, \epsilon'_0) \cap \mathcal{B}(f_0, A_1 \setminus int(C_0), \frac{1}{2}) \neq \emptyset$ by Lemma 96. E responds with some $\mathcal{B}(f'_1, K'_1, \epsilon'_1) \subset \mathcal{B}(f_1, K_1, \epsilon_1)$. Let C_1 be compact with $int(C_1) \supset K'_1$ and let L respond with $L_1 = C_1 \setminus int(A_1)$.

Continue this process. In the end, NE has played

$$\mathcal{B}(f_0, A_0, 1) \supset \mathcal{B}(f_1, A_1, \epsilon_1) \supset \mathcal{B}(f_2, A_2, \epsilon_2) \supset ...$$

so that $f_n(A_n \setminus int(C_{n-1})) \subset (-\frac{1}{2^n}, \frac{1}{2^n})$ and $\epsilon_n < \min\{\frac{\epsilon_{n-1}}{2}, \frac{1}{2^n}\}$. Since L has no winning strategy, there is some collection of plays for K in $G_{K,L}(X)$ so that $\{L_i\}_{i\in\omega}$ is not discrete. We will define a function $h : X \to \mathbb{R}$ so that $h \in \cap_{n\in\omega}\mathcal{B}(f_n, A_n, \epsilon_n)$. It follows that E lost the corresponding play of $Ch(\mathcal{B}(f_0, A_0, 1))$.

Fix $k \in \omega$. $\{f_n|_{A_k} \mid n \in \omega\}$ is uniformly convergent on A_k, since for any $x \in A_k$ and $n \geq m \geq k$, $\mathcal{B}(f_n, A_n, \epsilon_n) \subset \mathcal{B}(f_m, A_m, \epsilon_m)$ and $A_n \supset A_m \supset A_k \Rightarrow |f_n(x) - f_m(x)| \leq \epsilon_n \leq \frac{1}{2^n}$. Thus $\{f_n|_{A_k} \mid n \in \omega\}$ converges to some continuous $h_k : A \to \mathbb{R}$. Note that $h_k(A_k \setminus int(C_{k-1})) \subset (-\frac{1}{2^{k-1}}, \frac{1}{2^{k-1}})$ since $f_k(A_k \setminus int(C_{k-1})) \subset (-\frac{1}{2^k}, \frac{1}{2^k})$ and $|f_m(x) - f_n(x)| < \frac{1}{2^k}$ for all $x \in A_k$ and $m \geq k$. Note also that $k' \geq k \Rightarrow h_{k'}|_{A_k} = h_k$ by definition. Thus $\cup_{k\in\omega}h_k$ is a continuous function from $\cup_{k\in\omega}A_k$ to $\mathbb{R}$. Define $h : X \to \mathbb{R}$ by $h|_{\cup_{k\in\omega}A_k} = \cup_{k\in\omega}h_k$ and $h(x) = 0$ for all $x \notin \cup_{k\in\omega}A_k$.

Clearly h is continuous on $\cup_{k\in\omega}A_k$. If $x \notin \cup_{k\in\omega}A_k$ and $\epsilon > 0$, let m be such that $\frac{1}{2^m} < \epsilon$. There exists an open O containing x such that $O \cap (\cup_{n\in\omega}L_n \cup A_m = \emptyset$. For any $x' \in O$, if $x' \notin \cup_{k\in\omega}A_k$, $h(x') = 0$. If not, $x' \in A_k \setminus int(C_{k-1})$ for some $k > m$. Thus $h(x') = h_k(x') < \frac{1}{2^{k-1}} \leq \frac{1}{2^m} < \epsilon$. Thus h is continuous. Note that for any $n \in \omega$ and $i > n$, $f_i \in \mathcal{B}(f_n, A_n, \epsilon_n)$ and so for any $x \in A_n$, $|f_i(x) - f_n(x)| < \epsilon_n$. Since $h = \lim_{i\to\infty} f_i$, $|h(x)-f_n(x)| \leq \epsilon$ so $h \in \mathcal{B}(f_n, A_n, 2\epsilon_n) \subset \mathcal{B}(f_{n-1}, A_{n-1}, \epsilon_{n-1})$ for any $n \in \omega$. Thus $h \in \cap_{n\in\omega}\mathcal{B}(f_n, A_n, \epsilon_n)$, as was to be shown. $\square$

Remark. It is an unsolved problem whether or not (a) and (b) in Theorem 113 are equivalent for all completely regular spaces X (we saw earlier that (a) implies (b) always holds). In fact, we do not know of any property P of a space X such that X has P iff $C_k(X)$ is Baire (ditto for Choquet).