MH 7550: SET THEORETIC TOPOLOGY

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1. Trees and topological characterizations of the Cantor set and irrationals

It will be convenient to think of an ordinal number as the set of its predecessors (as in done in modern set theory). So, 0 is the empty set, $1 = \{0\}$, $2 = \{0, 1\}$, $\dots$, $n = \{0, 1, 2, \dots, n-1\}$, $\omega = \{0, 1, 2, \dots\}$ and is the set of natural numbers, ω_1 is the set of countable ordinals, etc.. Note $\beta < \alpha$ iff $\beta \in \alpha$ holds for any pair of ordinals α and β.

A binary relation $\leq$ on a set X is a *partial order* if it is reflexive ($x \leq x$ for all x), antisymmetric ($x \leq y$ and $y \leq x$ implies $x = y$), and transitive ($x \leq y$ and $y \leq z$ implies $x \leq z$). It is a *linear order* if also for any $x, y \in X$, either $x < y, y < x$, or $x = y$. A linear order is a *well-order* of X if every nonempty subset of X has a least element.

A partially ordered set $(T, \leq)$ is a *tree* if for all $t \in T$, the set $P_t = \{s \in T : s < t\}$ is well-ordered by . The elements of T are called the *nodes* of T. A maximal linearly ordered subset of T is called a *branch* of T.

Recall that any well-ordered set is order-isomorphic to (the predecessors of) an ordinal. If $(T, \leq)$ is a tree, then for any ordinal α, the set $L_\alpha = \{t \in T : P_t \text{ is isomorphic to } \alpha\}$ is called the α^{th} *level* of T. The least α such that $L_\alpha = \emptyset$ is called the *height* of T.

Example: the Cantor tree. Let T be the set of all finite sequences of 0's and 1's (including the empty sequence). We can equivalently describe T as the set of all functions σ from some natural number n into 2, where here we think of n as the set $\{0, 1, 2, \dots, n-1\}$ and 2 as the set $\{0, 1\}$. If $\sigma, \tau \in T$, define $\sigma < \tau$ iff σ is an initial segment of τ. (E.g., $110 < 1100$, $01 < 0111010$, etc.) Then $(T, \leq)$ is called the *Cantor tree*.

If we let B^A denote all functions from set A into set B, then 2^n denotes the set of all functions from $n = \{0, 1, \dots, n-1\}$ into $2 = \{0, 1\}$. (Note that $2^0 = 2^\emptyset = \emptyset$.) Let $2^{<\omega} = \cup\{2^n : n < \omega\}$. Then the set $2^{<\omega}$, ordered by extension, is another way to describe the Cantor tree.

Note that the branches of the Cantor tree can be identified with 2^ω, the set of all functions $f : \omega \to 2$. Given $f \in 2^\omega$, $\{f \restriction n : n < \omega\}$ is a branch, and given a branch b, then $\cup b$ is a function from ω to 2. Note $b = \{\cup b \restriction n : n < \omega\}$.

The Cantor tree $(T, \leq)$ is completely described (up to isomorphism) as follows:

(i) The least level L_0 of T has exactly one node;
(ii) Each node σ has exactly two successors;
(iii) T has height ω.

Definition of the Cantor "middle thirds" set $\mathbb{C}$. For each finite sequence σ of 0's and 1's (including the empty sequence $\emptyset$), we define a closed subinterval I_σ of $[0, 1]$ as follows. Start by setting $I_\emptyset = [0, 1]$. Then if I_σ has been defined, let $I_{\sigma^\frown 0}$ and $I_{\sigma^\frown 1}$ be the left and right thirds, respectively, of I_σ. Thus $I_0 = [0, 1/3]$, $I_1 = [2/3, 1]$, $I_{00} = [0, 1/9]$, $I_{01} = [2/9, 1/3]$, etc. For each n, let $C_n = \cup\{I_\sigma : \sigma \text{ has length } n\}$. So, $C_0 = [0, 1]$, $C_1 = [0, 1/3] \cup [2/3, 1]$, etc. Finally, $\mathbb{C} = \bigcap_{n \in \mathbb{N}} C_n$.

Theorem 1. *The Cantor set $\mathbb{C}$ as defined above is (as a subspace of the real line $\mathbb{R}$) compact, metrizable, and has no isolated points. $\mathbb{C}$ is an uncountable closed subset of $\mathbb{R}$ with empty interior in $\mathbb{R}$. If $\mathcal{B} = \{I_\sigma \cap \mathbb{C} : \sigma \in 2^{<\omega}\}$, then $\mathcal{B}$ is a countable base of open and closed (clopen) sets in $\mathbb{C}$ such that $(\mathcal{B}, \supseteq)$ is isomorphic to the Cantor tree.*

Proof. Since $\mathbb{C} = \cap_{n\in\mathbb{N}} C_n$ and each C_n is a finite union of closed sets, then C_n is closed and thus $\mathbb{C}$ is closed. $\mathbb{C}$ is compact since it is a closed subset of the compact space I. $\mathbb{C}$ is metrizable since it is a subspace of the metrizable space $\mathbb{R}$.

Claim 1. $I_\sigma \cap \mathbb{C}$ is clopen.

Let $\mathcal{B} = \{I_\sigma \cap \mathbb{C} : \sigma \in 2^{<\omega}\}$. For a given σ, $I_\sigma \cap \mathbb{C}$ is closed since it is the intersection of two closed subsets. Also, $C_n \setminus I_\sigma$ is closed since C_n is a finite union of disjoint closed intervals one of which is I_σ, so $\mathbb{C} \setminus (\mathbb{C} \cap I_\sigma) = \mathbb{C} \cap (C_n \setminus I_\sigma)$ is closed, which gives us $\mathbb{C} \cap I_\sigma$ is open.

Claim 2. $\mathcal{B}$ is a base for $\mathbb{C}$, and $\mathbb{C}$ has empty interior in $\mathbb{R}$.

Let U be any open set in $\mathbb{C}$ and let $x \in U$. There is some $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \cap \mathbb{C} \subset U$. Pick some $n \geq 1$ with $\frac{1}{3^n} < \varepsilon$. Since $x \in C_n$, $x \in I_\sigma$ for some σ of length n. Notice $diam I_\sigma = \frac{1}{3^n}$. So $x \in \mathbb{C} \cap I_\sigma \subset \mathbb{C} \cap (x-\varepsilon, x+\varepsilon) \subset U$.

Now if U were open in $\mathbb{R}$ and contained in $\mathbb{C}$, then as above we would have $I_\sigma \subset U \subset \mathbb{C}$ for some σ. But the middle third of I_σ is disjoint from $\mathbb{C}$, contradiction. So $\mathbb{C}$ has empty interior in $\mathbb{R}$.

Claim 3. $\mathbb{C}$ is uncountable.

For each $f : \omega \to 2$, there is exactly one point x_f in the intersection of the branch coded by f, i.e., $\{x_f\} = \bigcap_{n<\omega} I_{f\upharpoonright n} \cap \mathbb{C}$. Suppose $f, g \in 2^\omega$ and $f \neq g$. Let n be minimal such that $f(n) \neq g(n)$. Then $x_f \in I_{f\upharpoonright n+1}$, $x_g \in I_{g\upharpoonright n+1}$, and $I_{f\upharpoonright n+1} \cap I_{g\upharpoonright n+1} = \emptyset$. Hence $x_f \neq x_g$. Thus the map $f \mapsto x_f$ is one-to-one, and so $\mathbb{C}$ is uncountable (in fact it has the same cardinality as the real line).

Claim 4. $\mathbb{C}$ has no isolated points.

Let $\sigma \in 2^n$. If f and g are distinct members of 2^ω such that $f \upharpoonright n = g \upharpoonright n = \sigma$, then x_f and x_g are distinct points in $I_\sigma \cap \mathbb{C}$. So every basis element has at least two points, hence $\mathbb{C}$ has no isolated points. □

Remark. A space which has a base of open and closed sets is sometimes called *zero-dimensional*, or more precisely, is said to have *small inductive dimension zero*, denoted by $ind(X) = 0$. (There are several concepts of dimension.)

Theorem 2. *A space X is homeomorphic to the Cantor set $\mathbb{C}$ iff X has a base $\mathcal{B}$ consisting of clopen sets such that $(\mathcal{B}, \supseteq)$ is isomorphic to the Cantor tree and the intersection of each branch of this tree is a single point.*

Proof. Let X be a space with a base $\mathcal{B} = \{B_\sigma : \sigma \in 2^<\omega\}$ of clopen sets isomorphic to the Cantor tree such that the intersection of any branch is a single point. For each $c \in \mathbb{C}$, there is a unique $\sigma_c \in 2^\omega$ such that $\{c\} = \cap_{n\in\omega} I_{\sigma_c\upharpoonright n}$. By assumption $\cap_{n\in\omega} B_{\sigma\upharpoonright n}$ contains a single element, say x_c. Define $h : \mathbb{C} \to X$ by $c \mapsto x_c$.

Claim 1. h is surjective. Suppose not. Then there exists $x \in X$ and $\sigma \in 2^{<\omega}$ such that $x \in \mathcal{B}_\sigma$ but $x \notin B_{\sigma^\frown 0} \cup B_{\sigma^\frown 1}$. Then $B_\sigma \setminus B_{\sigma^\frown 0} \cup B_{\sigma^\frown 1}$ is an open neighborhood of x not containing any basis elements, a contradiction. So h is surjective.

Claim 2. h is injective. Suppose not. Then there exists $\sigma, \sigma' \in 2^\omega$ and $x \in X$ such that $\sigma \neq \sigma'$ but $\cap_{n\in\omega} B_{\sigma\upharpoonright n} = \{x\} = \cap_{n\in\omega} B_{\sigma'\upharpoonright n}$. Let $k \in \omega$ such that $\sigma \upharpoonright k = \sigma' \upharpoonright k$ but $\sigma \upharpoonright (k+1) \neq \sigma' \upharpoonright (k+1)$. Put $\tau = \sigma \upharpoonright k$. Then $x \in B_{\tau^\frown 0} \cap B_{\tau^\frown 1}$, so there exists $u \in 2^{<\omega}$ such that $x \in B_u \subset B_{\tau^\frown 0} \cap B_{\tau^\frown 1}$. Thus $\tau^\frown 0$ and $\tau^\frown 1$ are initial segments of u a contradiction.

Now, let $B_\sigma \in \mathcal{B}$. Then $h^{-1}(B_\sigma) = \{c \in \mathbb{C} : \sigma \leq \sigma_c\} = I_\sigma \cap \mathbb{C}$, so h is continuous. h is bijective, so $h(I_\sigma \cap \mathbb{C}) = B_\sigma$, implying h is open. Thus h is a homeomorphism. □

Theorem 3. *A space X is homeomorphic to $\mathbb{C}$ iff X is compact Hausdorff, has no isolated points, and has a countable base of clopen sets.*

Hint. Let B_n, $n < \omega$, be the clopen base. Construct a Cantor tree of clopen sets such that every node at level n either meets B_n or is disjoint from B_n.

Proof. The forward implication is immediate. So let X be a compact Hausdorff space with no isolated points and a countable base $\mathcal{B} = \{B_n : n \geq 1\}$ of clopen sets. Put $A_\emptyset = X$. Suppose for some $n \geq 1$ that $\{A_\sigma : \sigma \in 2^{n-1}\}$ partitions X into clopen sets. Let $\sigma \in 2^{n-1}$. If $A_\sigma \cap B_n \neq \emptyset$ and $A_\sigma \setminus B_n \neq \emptyset$, put $A_{\sigma^\frown 0} = A_\sigma \cap B_n$ and $A_{\sigma^\frown 1} = A_\sigma \setminus B_n$. Otherwise, note that A_σ either misses B_n or is contained in B_n, so the members of any partition of A_σ will too. So, pick $x, y \in A_\sigma$ with $x \neq y$. There is $m \geq 1$ such that B_m contains x but not y. Put $A_{\sigma^\frown 0} = B_m$ and $A_{\sigma^\frown 1} = A_\sigma \setminus B_m$.

By induction, $\{A_\sigma : \sigma \in 2^n\}$ partitions X into clopen sets for each $n \geq 1$. Moreover, if $n \geq 1$ and $\sigma \in 2^n$, then either $A_\sigma \subset B_n$ or $A_\sigma \cap B_n = \emptyset$.

Let $\mathcal{A} = \{A_\sigma : \sigma \in 2^{<\omega}\}$. Observe that $A_\sigma \supset A_\tau \iff \sigma \leq \tau$, so $(\mathcal{A}, \supseteq)$ is isomorphic to the Cantor tree.

Let $U \subset X$ be open and $x \in U$. There exists $n \geq 1$ such that $x \in B_n \subset U$. $\{A_\sigma : \sigma \in 2^n\}$ partitions X, so $x \in A_\sigma$ for some $\sigma \in 2^n$. Since $A_\sigma \cap B_n \neq \emptyset$, $A_\sigma \subset B_n$. Thus, $x \in A_\sigma \subset B_n \subset U$ implying $\mathcal{A}$ is a base. Let $f \in 2^\omega$. Then $\cap_{n\in\omega} A_{f\restriction n} \neq \emptyset$ because $A_{f\restriction n}$ is compact, $A_{f\restriction n} \supset A_{f\restriction n+1}$, and $A_{f\restriction n}$ is closed for all $n \in \omega$.

We claim that $\cap_{n\in\omega} A_{f\restriction n}$ contains a single element. Suppose not. Then there are $x, y \in \cap_{n\in\omega} A_{f\restriction n}$ with $x \neq y$. Moreover, there is $n \geq 1$ such that $x \in B_n$ and $y \notin B_n$. Also, there exist $\sigma, \tau \in 2^n$ such that $x \in A_\sigma$ and $y \in A_\tau$. But $A_\sigma \cap B_n \neq \emptyset$, so $A_\sigma \subset B_n$, and $y \notin B_n$ implies $A_\tau \cap B_n = \emptyset$. Thus $A_\sigma \cap A_\tau = \emptyset$, so $\sigma \neq \tau$. Thus $f \restriction n = \sigma \neq \tau = f \restriction n$, a contradiction.

By Theorem 2, X is isomorphic to $\mathbb{C}$. □

Corollary 4. *The following are homeomorphic to $\mathbb{C}$: $\mathbb{C}^2$, $\mathbb{C}^\omega$, 2^ω (where 2 denotes the two-point discrete space $\{0,1\}$), and any product of the form $\Pi_{n\in\omega} F_n$, where F_n is finite discrete space with at least two points.*

Let $\leq$ be a linear order on a set X. For $a, b \in X$, let $(a,b) = \{x \in X : a < x < b\}$. Also let $(-\infty, a) = \{x \in X : x < a\}$ and $(b, \infty) = \{x \in X : b < x\}$. Let $\mathcal{B} = \{(a,b) : a, b \in X\} \cup \{(-\infty, a) : a \in X\} \cup \{(b, \infty) : b \in X\}$. Then $\mathcal{B}$ is a base for a topology τ on X, and τ is called the *order topology on X induced by* $\leq$. A topological space (X, τ) is called a *linearly ordered space* if there is a linear order on X which induces the topology τ.

Theorem 5. *Let $(T, \leq)$ be the Cantor tree. Let X be the set of all branches of T. Note that if b is a branch of T, then $\cup b \in 2^\omega$. Define a linear order $\prec$ on X as follows: if $b, b' \in X$, and n is the least integer such that $\cup b(n) \neq \cup b'(n)$, then $b \prec b'$ iff $\cup b(n) = 0$ and $\cup b'(n) = 1$. Then X with the topology induced by this order is homeomorphic to $\mathbb{C}$.*

Hint: Show that 2^ω with the topology induced by the lexicographic order is the same as the usual Tychonoff product topology.

Theorem 6. *The following are continuous images of* $\mathbb{C}$*:*

(i) *the unit interval* $[0,1]$*;*
(ii) *the Hilbert cube* $[0,1]^\omega$*;*
(iii) *any closed subset of* $\mathbb{C}$*;*
(iv) *any compact metric space.*

Proof. (i) Define a Cantor tree of closed subintervals of $[0,1]$ as follows: $J_\emptyset = [0,1]$, and if $J_\sigma = [l,r]$ has been defined for $\sigma \in 2^n$, let $J_{\sigma\frown 0}$ and $J_{\sigma\frown 1}$ be $[l,m]$ and $[m,r]$ respectively, where m is the midpoint of $[l,r]$. Note that each branch of this tree is a single point, and every point of $[0,1]$ is the intersection of some branch. The map that sends the intersection of a branch of the Cantor tree to the intersection of the corresponding branch of this tree for $[0,1]$ is easily seen to be continuous and onto. (Remark. It is not a bijection–it can't be, else it'd be a homeomorphism–but it is two-to-one on a countable set and one-to-one otherwise.)

(ii)Let f be a continuous function from $\mathbb{C}$ onto $[0,1]$. For $\vec{x} \in \mathbb{C}^\omega$, let $f(\vec{x}) = (f(x_n))_{n\in\omega} \in [0,1]^\omega$. Then f is clearly continuous and onto. Since $\mathbb{C}^\omega \cong \mathbb{C}$, we are done with (ii).

(iii) If K is a closed subset of $\mathbb{C}$, it is easily seen that $K \times \mathbb{C}$ is compact, has no isolated points, and has a countable base of clopen sets. Hence $K \times \mathbb{C}$ is homeomorphic to $\mathbb{C}$. The projection onto the first coordinate is a continuous map onto K.

(iv) Let M be any compact metric space. Then M is homeomorphic to a (compact, hence closed) subset N of the Hilbert cube. Let $f : \mathbb{C} \to I^\omega$ be continuous and onto. Let $K = f^{-1}(N)$. Then K is a closed subset of $\mathbb{C}$, so by (iii) there is a continuous surjection $g : \mathbb{C} \to K$. Then $f \circ g : \mathbb{C} \to N$ is continuous and onto. Since $N \cong M$ we are done. $\square$

For each $n \in \omega$, let ω^n denote the set of all functions from n (i.e., from the set $\{0, 1, 2, \dots n-1\}$) into ω, and let $\omega^{<\omega} = \bigcup_{n\in\omega} \omega^n$. (In other words, ω^n is the set of all n-length sequences of natural numbers, and $\omega^{<\omega}$ is the set of all finite sequences of natural numbers.) If $\sigma, \tau \in \omega^{<\omega}$, let $\sigma < \tau$ iff σ is an initial segment of τ. Then $(\omega^{<\omega}, <)$ is a tree of height ω with one node at the least level (the empty sequence) and such that every node has a countable infinite number of immediate successors.

Exercise. Let $\mathbb{P}$ denote the irrationals (as a subspace of $\mathbb{R}$). Show that $\mathbb{P}$ has a base $\mathcal{B}$ of clopen sets such that $(\mathcal{B}, \supseteq)$ is a tree isomorphic to $(\omega^{<\omega}, \leq)$, and such that the intersection of each branch is a singleton.

Proof. Let $q_1, q_2, ...$ be an enumeration of the rationals, with $q_1 = 0$. We define clopen subsets $\{B_\sigma : \sigma \in \omega^{<\omega}\}$ of $\mathbb{P}$ such that

(i) $B_\emptyset = \mathbb{P}$;
(ii) For each σ, the collection $\{B_{\sigma\frown n} : n \in \omega\}$ is a clopen partition of B_σ;
(iii) If σ has length n, then $diam(B_\sigma) \leq 1/n$.

If we do the above, the resulting B_σ's will form a base which under $\supset$ is a tree isomorphic to $(\omega^{<\omega}, \leq)$. There is one extra thing we need to be concerned about: how do we know the intersection of each branch is a single point? (It could be empty.)

To start, of course we set $B_\emptyset = \mathbb{P}$. The collection $\{(n, n+1) \cap \mathbb{P} : n \in \mathbb{Z}\}$ is a countable clopen partition of $\mathbb{P}$, so we may let $\{B_n : n \in \omega\}$ index it. As we continue, each B_σ will have the form $(a,b) \cap \mathbb{P}$ for some rationals a, b. We want to make sure that each rational is included as an endpoint at some stage. Note that since $q_1 = 0$, q_1 is taken care of. We will make sure q_n gets taken care of at or preceding the construction of the nth level of the tree.

So, suppose B_σ has been defined for all σ of length $\leq n$. Let σ have length n. We show how to construct $B_{\sigma^\frown n}$ for $n \in \omega$. We have assumed $B_\sigma = (a, b) \cap \mathbb{P}$ for some rationals a, b. Let $a_0, a_1, ...$ be a decreasing sequence of rationals in (a, b) converging to a, and $b_0, b_1, ...$ an increasing sequence of rationals converging to b. We may assume we make the choice such that $a_0 < b_0$, and that the intervals $(a_0, b_0), (a_{n+1}, a_n)$, and (b_n, b_{n+1}) for $n \in \omega$ have diameter $\leq \frac{1}{n+1}$. We may also assume that if the nth rational q_n lies between a and b, then q_n is one of the a_n's or b_n's. Note that if q_n was not already an endpoint of some chosen interval, then it will be taken care of at this stage. Finally, let $\{B_{\sigma^\frown n} : n \in \omega\}$ be a listing of $\{(a_0, b_0) \cap \mathbb{P}\} \cup \{(a_{n+1}, a_n) \cap \mathbb{P} : n \in \omega\} \cup \{(b_n, b_{n+1}) \cap \mathbb{P} : n \in \omega\}$.

Clearly the B_σ's satisfy (i)-(iii) above. We need to see that the intersection of each branch is a singleton. Note that each branch corresponds to a decreasing sequence $I_0 \cap \mathbb{P}, I_1 \cap \mathbb{P}, ...$ where the endpoints of the I_n's are rational and the diameters go to 0. In the whole real line, $\bigcap_{n\in\omega} I_n$ is a single point, say x. If x is irrational, we are done. Suppose x is rational. At some stage, say n, x becomes an endpoint of one of the intervals. So it is either an endpoint of I_n, or an endpoint of some disjoint interval. In either case, $x \notin \bigcap_{n\in\omega} I_n$, and so x must be irrational. □

A space X is said to be *nowhere-locally-compact* if no point of X has a compact neighborhood.

Theorem 7. *The following are equivalent for a space X:*

(i) *X is homeomorphic to the space of irrationals (as a subspace of the real line with the usual Euclidean topology);*
(ii) *X has a base $\mathcal{B}$ consisting of clopen sets such that $(\mathcal{B}, \supseteq)$ is isomorphic to the tree $(\omega^{<\omega}, \leq)$, and the intersection of each branch of this tree is a single point;*
(iii) *X is a nowhere-locally-compact complete separable metric space and has a countable base of clopen sets.*

Proof. Suppose X is a topolgical space. (i) $\Longrightarrow$ (ii). Exercise. (ii) $\Longrightarrow$ (iii). Suppose X has a base $\mathcal{B}$ such that $(\mathcal{B}, \supseteq) \simeq (\omega^{<\omega}, \leq)$ and the intersection of each branch of $(\mathcal{B}, \supseteq)$ is a single point. We show (1) X has a countable base of clopen sets (implying X is separable), (2) X is nowhere locally compact, and (3) X is completely metrizable.

(1) $\omega^{<\omega}$ is countable.

(2) Let $\mathcal{B} = \{B_\sigma : \sigma \in \omega^{<\omega}\}$ be an enumeration of $\mathcal{B}$ such that $B_\sigma \supseteq B_\tau$ iff τ extends σ. Note that for each $\sigma \in \omega^{<\omega}$, $\{B_{\sigma^\frown j} : j \in \omega\}$ partitions B_σ: Each $B_{\sigma^\frown j}$ is contained in B_σ since $\sigma^\frown j$ extends σ. We now show $\{B_{\sigma^\frown j} : j \in \omega\}$ covers B_σ. Suppose $x \in B_\sigma$. If $i \neq j \in \omega$ then the subsets $B_{\sigma^\frown i}$ and $B_{\sigma^\frown j}$ of B_σ are incomparable under $\supseteq$. Thus B_σ contains more than one point. Let $y \in B_\sigma$ with $y \neq x$. Since $\mathcal{B}$ is a basis, there exists $B \in \mathcal{B}$ with $x \in B$ and $y \notin B$ and there exists $B_\tau \subseteq B \cap B_\sigma$ with $x \in B_\tau$. B_τ is a proper subset of B_σ containing x, so τ properly extends σ and $x \in B_{\tau\restriction n+1} \in \{B_{\sigma^\frown j} : j \in \omega\}$. Thus $\{B_{\sigma^\frown j} : j \in \omega\}$ covers B_σ. Finally, if $i \neq j \in \omega$ we have $B_{\sigma^\frown i} \cap B_{\sigma^\frown j} = \varnothing$. Otherwise there exists $B_\tau \subseteq B_{\sigma^\frown i} \cap B_{\sigma^\frown j}$. Then τ extends both $\sigma^\frown i$ and $\sigma^\frown j$, which is impossible.

Now we are ready to show X is nowhere locally compact. Suppose U is a neighborhood. Then U has nonempty interior. There exists $B_\sigma \subseteq \text{int}U \subseteq U$. If U is compact then since B_σ is closed, we have B_σ is compact. But $\{B_{\sigma^\frown j} : j \in \omega\}$ is an open cover of B_σ with no finite subcover.

(3) For $x, y \in X$ let $L_{x,y} = \{i \in \omega : (\exists \sigma \in \omega^i)(x, y \in B_\sigma)\}$. Define $d : X \times X \to [0, \infty)$ by

$$d(x,y) = \begin{cases} 1/\text{max}L_{x,y} & \text{if } L_{x,y} \text{ is bounded and } \max L_{x,y} > 0 \\ 1 & \text{if } L_{x,y} \text{ is bounded and } \max L_{x,y} = 0 \\ 0 & \text{if } L_{x,y} \text{ is unbounded} \end{cases}.$$

d is a metric: (a) $d(x, y) = 0$ iff $x = y$: If $x = y$ then x and y may not be separated by sets in $\mathcal{B}$, so $L_{x,y}$ is unbounded, so $d(x, y) = 0$. Conversely, if $x \neq y$ then there exist $\sigma, \tau \in \omega^{<\omega}$ with $x \in B_\sigma$, $y \in B_\tau$, and $B_\sigma \cap B_\tau = \varnothing$. Then x and y are split at level $\max(\text{dom}(\sigma), \text{dom}(\tau))$. So $L_{x,y}$ is bounded. So $d(x, y) \neq 0$. (b) $d(x, y) = d(y, x)$: Trivial. (c) $d(x, y) \leq d(x, z)+d(y, z)$: This is trivial if $d(x, y) = 0$, so suppose $d(x, y) = 1/n$. Then x and z or y and z are contained in disjoint basis sets at level $n + 1$ (say x and z). So there is a max level $m \leq n$ at which x and z are together. Thus $d(x, y) = 1/n \leq 1/m = d(x, z) \leq d(x, z) + d(y, z)$.

Now we show the topology induced by d coincides with the topology induced by $\mathcal{B}$ (the original topology on X). It is easy to see that $B(x, 1/n) = B_\tau$ where $\tau \in \omega^{n+1}$ is such that $x \in B_\tau$. So metric ball contains a set in $\mathcal{B}$. Conversely, if $\varnothing \neq \sigma \in \omega^{<\omega}$ and $x \in B_\sigma$, then $B(x, 1/\text{dom}(\sigma)) = B_{\sigma^\frown i} \subseteq B_\sigma$ (i such that $x \in B_{\sigma^\frown i}$). If $\sigma = \varnothing$ then $B_\sigma = X$ so any metric ball is conained in it.

d is complete: Suppose $(x_n) \in X^\omega$ is a Cauchy sequence w.r.t. d. There exists an increasing sequence $(m_n) \in \omega^\omega$ such that for each n, $k \geq m_n$ implies $d(x_{m_n}, x_k) < 1/2^{n+1}$. The balls $B(x_{m_n}, 1/2^n)$, $n \in \omega$, are nested (decreasing). Indeed, if $n \in \omega$ and $x \in B(x_{m_{n+1}}, 1/2^{n+1})$ then we have $d(x, x_{m_n}) \leq d(x, x_{m_{n+1}})+d(x_{m_n}, x_{m_{n+1}}) < 1/2^{n+1} + 1/2^{n+1} = 1/2^n$, whence $x \in B(x_{m_n}, 1/2^n)$. Thus the balls $B(x_{m_n}, 1/2^n)$ correspond to a branch in the tree $(\mathcal{B}, \supseteq)$, which intersects to a point x. Claim x is the limit of (x_n). Well, let $\epsilon > 0$. Let $n \in \mathbb{N}$ such that $2/2^n < \epsilon$. For $k \geq m_n$ we have $d(x, x_k) \leq d(x_{m_n}, x) + d(x_{m_n}, x_k) < 1/2^n + 1/2^{n+1} < 2/2^n < \epsilon$.

(iii) $\Longrightarrow$ (i). Suppose X is nowhere locally compact, completely metrizable, and has a countable base $\mathcal{B}$ of clopen sets.

Claim: For every nonempty open $U \subseteq X$ and $n \in \mathbb{N}$ there exists a partition of U into ω-many nonempty clopen sets, each with diameter less than $1/n$. Let $U \subseteq X$ nonempty open and $n \in \mathbb{N}$. Using the fact that X is nowhere locally compact, $\mathcal{U}_n = \{B \in \mathcal{B} : \text{diam}(B) < 1/n \text{ and } B \subseteq U\}$ is an open cover of U with no finite subcover. Enumerate $\mathcal{U}_n = \{B_0, B_1, ...\}$. After throwing away the empty elements in $\mathcal{U}_n^* = \{B_n \setminus \bigcup_{i<n} B_i : n \in \omega\}$, we have the desired partition.

Using the claim, recursively construct a tree of clopen sets isomorphic to $(\omega^{<\omega}, \leq)$. Each branch of this tree intersects to at most one point since its nodes have diameters shrinking to 0. The intersection of each branch is nonempty: create a Cauchy sequence by selecting a sequence up the branch. It converges by completeness, to a point which must be in the intersection of the branch. Construct a homeomorphism between X and the space of irrationals by mapping branches to branches, as in the proof of Theorem 2. □

Theorem 8. *Let $\mathbb{P}$ denote the space of irrationals. Then $\mathbb{P}$ is homeomorphic to $\mathbb{P}^2$, $\mathbb{P}^\omega$, and ω^ω (where ω is given the discrete topology).*

Remark. A subset A of the real line $\mathbb{R}$ is said to be *analytic* if there is a continuous surjection $f : \mathbb{P} \to A$. It is known that every Borel set[1] is analytic, every analytic set is Lebesgue measurable, and that there are analytic sets that are not Borel (a highly nontrivial result!) and measurable sets that are not analytic. Analytic sets play a major role in the field of "descriptive set theory".

2. Topological characterizations of $[0, 1]$ and the rationals

Theorem 9. *Suppose $(X, <)$ and $(Y, \prec)$ are countable linearly ordered sets with no first or last point, and both are densely ordered (i.e., between any two points there is another point). Then there is an order-preserving bijection $f : X \to Y$.*

Proof. Let $(X, <) = \{x_n\}_{n\in\omega}$ and $(Y, \prec) = \{y_n\}_{n\in\omega}$ be densely and linearly ordered sets with no first or last point. Put $X_0 = \{x_0\}$ and $Y_0 = \{y_0\}$. Let $f_0 : X_0 \to Y_0$. Suppose for some $n \in \omega$ that $f_n : X_n \to Y_n$ is an order-preserving bijection with $\{x_0, \dots, x_n\} \subset X_n$, $\{y_0, \dots, y_n\} \subset Y_n$, and both X_n and Y_n finite.

If $x_{n+1} \in X_n$, put $X'_{n+1} = X_n$, $Y'_{n+1} = Y_n$, and $f'_{n+1} = f_n$. Otherwise, let $A = \{x \in X_n : x < x_{n+1}\}$ and $B = \{x \in X_n : x_{n+1} < x\}$. If $A = X_n$, pick $y \in Y \setminus Y_n$ such that $y' \prec y$ for all $y' \in Y_n$. If $B = X_n$, pick $y \in Y \setminus Y_n$ such that $y \prec y'$ for all $y' \in Y_n$. Otherwise, let $a = \max A$ and $b = \min B$ and pick $y \in (f_n(a), f_n(b))$. Set $X'_{n+1} = X_n \cup \{x_{n+1}\}$ and $Y'_{n+1} = Y_n \cup \{y\}$, and define $f'_{n+1} : X'_{n+1} \to Y'_{n+1}$ so that $f'_{n+1} \restriction X_n = f_n$ and $f'_{n+1}(x_{n+1}) = y$.

If $y_{n+1} \in Y'_{n+1}$, put $X_{n+1} = X'_{n+1}$, $Y_{n+1} = Y'_{n+1}$, and $f_{n+1} = f'_{n+1}$. Otherwise, let $A = \{y \in Y'_{n+1} : y \prec y_{n+1}\}$ and $B = \{y \in Y'_{n+1} : y_{n+1} \prec y\}$. If $A = Y'_{n+1}$, pick $x \in X \setminus X'_{n+1}$ such that $x' < x$ for all $x' \in X'_{n+1}$. If $B = Y'_{n+1}$, pick $x \in X \setminus X'_{n+1}$ such that $x < x'$ for all $x' \in X'_{n+1}$. Otherwise, let $a = \max A$ and $b = \min B$ and pick $x \in \big((f'_{n+1})^{-1}(a), (f'_{n+1})^{-1}(b)\big)$.

Finally, set $X_{n+1} = X'_{n+1} \cup \{x\}$ and $Y'_{n+1} = Y_n \cup \{y_{n+1}\}$, and define $f_{n+1} : X_{n+1} \to Y_{n+1}$ so that $f_{n+1} \restriction X'_{n+1} = f'_{n+1}$ and $f_{n+1}(x) = y_{n+1}$. Then $f_{n+1} : X_{n+1} \to Y_{n+1}$ is an order-preserving bijection with $\{x_0, \dots, x_{n+1}\} \subset X_{n+1}$, $\{y_0, \dots, y_{n+1}\} \subset Y_{n+1}$, and both X_{n+1} and Y_{n+1} finite.

Let $f = \cup_{n\in\omega} f_n$. Then $f : X \to Y$ is an order-preserving bijection. □

Theorem 10. *A linearly ordered space X is connected iff the following hold:*

(i) *X is densely ordered;*
(ii) *Every bounded subset of X has a least upper bound.*

Proof. Let X be a connected linearly ordered space. Let $x, y \in X$ with $x \neq y$. Without loss of generality we may assume that $x < y$. If $(x, y) = \emptyset$, then $X = (-\infty, y) \cup (x, \infty)$, implying X is not connected. It follows that $(x, y) \neq \emptyset$, whence X is densely ordered.

Now, let A be a bounded set and suppose on the contrary that A has no least upper bound. Then for all $x \in A$ there exists some $y \in A$ with $x < y$. Also, for each upper bound b of A there exists another upper bound b' such that $b' < b$. Take $U = \cup\{(-\infty, y) : y \in A\}$ and $V = \cup\{(b, \infty) : b \text{ is an upper bound of } A\}$. Then

[1]The collection of Borel sets the smallest collection $\mathcal{B}$ of subsets of $\mathbb{R}$ containing all open sets, and closed under the taking of complements, countable intersections, and countable unions.

$X = U \cup V$ with U, V open and $U \cap V = \emptyset$, a contradiction. It follows that every bounded subest of X has a least upper bound.

For the other direction, let X be a linearly ordered non-connected space in which every bounded subset has a least upper bound. Let $U, V \subset X$ be disjoint, nonempty, and open such that $X = U \cup V$. Without loss of generality we may assume there exists $x \in U$ and $y \in V$ such that $x < y$. Put $U' = (-\infty, y) \cap U$ and let b be a least upper bound of U'. Note that either $b \in U'$ or $b \in V$.

If $b \in U'$, then there exists $(a, c) \subset X$ with $b \in (a, c) \subset U'$. Since $(b, c) \cap V = \emptyset$ and $(b, c) \cap U = \emptyset$, it must be the case that $(b, c) = \emptyset$. Similarly, if $b \in V$ there exists $(a, c) \subset X$ with $b \in (a, c) \subset V$. Now $(a, b) \cap U' = \emptyset$ and $(a, b) \cap V = \emptyset$, since b is a least upper bound, so $(a, b) = \emptyset$. It follows that X is not densely ordered. □

Theorem 11. *Suppose X is a connected linearly ordered space with no first or last point, and is separable. Then X is order-isormorphic to the real line $\mathbb{R}$ with the usual order.*

Proof. Let X be a separable, connected, and linearly ordered space with no first or last point. Let D be a countable dense subset of X. Observe that D is densely ordered and has no first or last point. By Theorem 9, there exists an order-preserving bijection $g : D \to \mathbb{Q}$.

For each $x \in X$ put $L_x = \{d \in D : d < x\}$. Define $f : X \to \mathbb{R}$ by $f(x) = \sup g(L_x)$. Since $g(L_x)$ is bounded above, $\sup g(L_x)$ exists, so f is well-defined.

Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Without loss of generality, we may assume that $x_1 < x_2$. Since X is densely ordered and D is dense, there exist $d_1, d_2 \in D$ such that $x_1 < d_1 < d_2 < x_2$. Then $\sup g(L_{x_1}) \leq g(d_1) < g(d_2) \leq \sup g(L_{x_2})$, so $f(x_1) < f(x_2)$. Thus f is one-to-one and order-preserving.

Let $y \in \mathbb{R}$ and put $L = \{q \in \mathbb{Q} : q < y\}$. Then $g^{-1}(L) \subset D$ is bounded above, so $x = \sup g^{-1}(L) \in X$ exists. But $L_x = g^{-1}(L)$, so $f(x) = y$. It follows that f is onto. □

Corollary 12. *Suppose X is a separable compact connected linearly ordered space. Then X is homeomorphic to the unit interval* $[0, 1]$.

Proof. Suppose X is a separable compact connected linearly ordered space.

First we show X has a first and last point. Well, if X has neither then $X \simeq \mathbb{R}$ by Theorem 11, so that X is not compact, a contradiction. Without loss of generality, suppose x_0 is a first point of X. If X has no last point, then the collection $\{[x_0, x) : x \in X\}$ is an open cover of X, so it has a finite subcover. So there exists $x \in X$ such that $[x_0, x) = X$. But $x \in X \setminus [x_0, x)$, a contradiction. So X must have a last point x_1.

Now $X \setminus \{x_0, x_1\}$ has no first or last point (for instance, if it has a first point x_2 then $(x_0, x_2) = \varnothing$, contradicting Theorem 10). So by Theorem 11, $(x_0, x_1) \simeq \mathbb{R} \simeq (0, 1)$. So $X \simeq [0, 1]$. □

Theorem 13. *If X is completely regular and $|X| < |\mathbb{R}|$, then X has a base of clopen sets.*

Proof. Since X is completely regular T_1, it may be embedded into some product $\prod_{i \in I} \mathbb{R}_i$ of real lines (with the product topology). For instance, the the map $X \to \mathbb{R}^{C(X)}$ given by $x \mapsto (f(x))_{f \in C(X)}$ is an embedding. We therefore may (and do) consider X as a subspace of $\prod_{i\in I}\mathbb{R}_i$. Let

$$\mathcal{B}=\{\bigcap_{i\in F}\pi_i^{-1}[(c_i,d_i)]\cap X : F \text{ is a finite subset of } I \text{ and } c_i<d_i\in\mathbb{R}_i\setminus\pi_i[X] \text{ for each } i\in F\}.$$

If $F\subseteq I$ and $c_i<d_i\in\mathbb{R}_i\setminus\pi_i[X]$ for each $i\in F$, we have $\bigcap_{i\in F}\pi_i^{-1}[(c_i,d_i)]\cap X=\bigcap_{i\in F}\pi_i^{-1}[[c_i,d_i]]\cap X$, so that the sets of $\mathcal{B}$ are clopen in X. We now show $\mathcal{B}$ is a basis for X. Suppose U is an open set in X and $(x_i)\in U$. We find a clopen subset of X containing (x_i), contained in U. There exists an open $U'\subseteq\prod_{i\in I}\mathbb{R}_i$ such that $U'\cap X=U$. There exists a basic product open set $\bigcap_{i\in F}\pi_i^{-1}[U_i']$ with

$$(x_i)\in\bigcap_{i\in F}\pi_i^{-1}[U_i']\subseteq U$$

(F some finite subset of I, U_i open in $\mathbb{R}_i$). For each $i\in F$ there exists $a_i,b_i\in\mathbb{R}_i$ such that $x_i\in(a_i,b_i)\subseteq U_i'$.

Since $|X|<|\mathbb{R}|$, we have: $\pi_i(X)$ does not contain an interval of reals, for any $i\in I$. So for any $i\in I$ and $a<b\in\mathbb{R}_i$ there exists $r\in(a,b)\cap\mathbb{R}_i\setminus\pi_i(X)$. In particular, for each $i\in F$ there exist $c_i<d_i\in(a_i,b_i)\cap\mathbb{R}_i\setminus\pi_i(X)$ with $x_i\in(c_i,d_i)$. We have

$$(x_i)\in\bigcap_{i\in F}\pi_i^{-1}[(c_i,d_i)]\cap X\subseteq\bigcap_{i\in F}\pi_i^{-1}[(a_i,b_i)]\cap X\subseteq\bigcap_{i\in F}\pi_i^{-1}[U_i']\cap X\subseteq U'\cap X=U.$$

□

Theorem 14. *Suppose κ is an infinite cardinal. If κ is the least cardinal of a base for a space X, then for every base $\mathcal{B}$, there is a base $\mathcal{C}\subset\mathcal{B}$ such that $|\mathcal{C}|=\kappa$.*

Hint: For every infinite cardinal κ, the set of all finite subsets of κ also has cardinality κ. Or, use Theorem 50(i) which says that the union of $\leq\kappa$-many sets each of cardinality $\leq\kappa$ has cardinality $\leq\kappa$.

Proof. Let X be a space and let $\mathcal{A}$ and $\mathcal{B}$ be bases thereof with $|\mathcal{A}|=\kappa$.

Fix $A\in\mathcal{A}$. Let $\mathcal{B}(A)=\{B\in\mathcal{B}:B\subset A\}$ and let $\mathcal{A}(A)=\{A'\in\mathcal{A}:\exists B\in\mathcal{B}(A) \text{ with } A'\subset B)\}$. Since A is open and $\mathcal{B}$ is a base, we must have $\cup\mathcal{B}(A)=A$. Similarly, $\cup\mathcal{A}(A)=A$. Finally, for each $A'\in\mathcal{A}(A)$, let $B(A')\in\mathcal{B}(A)$ such that $A'\subset B(A')$. Put $\mathcal{C}(A)=\{B(A'):A'\in\mathcal{A}(A)\}$. Since $\cup\mathcal{A}(A)\subset\cup\mathcal{C}(A)\subset\cup\mathcal{B}(A)$, we have $\cup\mathcal{C}(A)=A$. Moreover, $|\mathcal{C}(A)|\leq\kappa$.

Let $\mathcal{C}=\cup_{A\in\mathcal{A}}\mathcal{C}(A)$. Certainly $\mathcal{C}\subset\mathcal{B}$ is a base. By Theorem 50(i), $|\mathcal{C}|\leq\kappa$. Since κ is the least cardinal of a base for X, $|\mathcal{C}|=\kappa$. □

Remark. The least cardinal of a base for a space X is called the *weight* of X and is denoted by $w(X)$.

Theorem 15. *Suppose X is countable, regular, first-countable, and has no isolated points. Then X is homeomorphic to the rationals $\mathbb{Q}$.*

Proof. Let X be a countable, regular and first-countable space with no isolated points. Observe that X is second-countable, hence metrizable and completely regular. By Lemmas 13 and 14, X has a countable base $\mathcal{B}=\{B_n:n\in\omega\}$ of clopen sets.

Claim 1: X embeds into 2^ω.

For each $n \in \omega$, define $f_n : X \to 2$ so that $f_n(B_n) = \{0\}$ and $f_n(X \setminus B_n) = \{1\}$. Certainly f_n is continuous for each $n \in \omega$. Put $\mathcal{F} = \{f_n : n \in \omega\}$ and define $e_{\mathcal{F}} : X \to 2^\omega$ by $e_{\mathcal{F}}(x) = \langle f_n(x)\rangle_{n\in\omega}$. Note that $\pi_n \circ e_{\mathcal{F}} = f_n$ for each $n \in \omega$, so $e_{\mathcal{F}}$ is continuous.

Let $x \in X$ and $H \subset X$ closed not containing x. Then there exists $n \in \omega$ such that $x \in B_n \subset X \setminus H$. It follows that $f_n(x) = 0$ and $\overline{f_n(H)} = \{1\}$, so $\mathcal{F}$ separates points from closed sets. Since X is T_1, $\mathcal{F}$ separates points so $e_{\mathcal{F}}$ is one-to-one.

Finally, we show $e_{\mathcal{F}}$ is closed. Let $H \subset X$ be closed and suppose on the contrary that $e_{\mathcal{F}}(H)$ is not closed. Let $y \in \overline{e_{\mathcal{F}}(H)} \setminus e_{\mathcal{F}}(H)$ and $x \in X \setminus H$ such that $e_{\mathcal{F}}(x) = y$. $\mathcal{F}$ separates points from closed sets, so there exists $n \in \omega$ such that $f_n(x) = 0$ and $\overline{f_n(H)} = \{1\}$. It follows that $y \in \prod_{i=1}^{n-1} 2 \times \{0\} \times \prod_{i=n+1}^{\infty} 2$ and $\overline{e_{\mathcal{F}}(H)} \subset \prod_{i=1}^{n-1} 2 \times \{1\} \times \prod_{i=n+1}^{\infty} 2$. But this implies $y \notin \overline{e_{\mathcal{F}}(H)}$, a contradiction. Thus $e_{\mathcal{F}}$ is closed, whence $X \cong e_{\mathcal{F}}(X) \subset 2^\omega$.

Remark. By Theorem 95 from 1st year topology class, if X is T_1 and $\mathcal{F}$ separates points from closed sets, then $e_{\mathcal{F}}$ is a homeomorphic embedding (into I^ω, but clearly the range of this $e_{\mathcal{F}}$ is 2^ω).

Claim 2: X is homeomorphic to a dense subset of $\mathbb{C}$.

Since $2^\omega \cong \mathbb{C}$, there exists an embedding $f : X \to \mathbb{C}$ by Claim 1. Observe that $\overline{f(X)} \subset \mathbb{C}$ is compact Hausdorff and has a countable base of clopen sets. Since $f(X)$ has no isolated points, $\overline{f(X)}$ does not either. By Theorem 3, there exists a homeomorphism $h : \overline{f(X)} \to \mathbb{C}$. Since $f(X)$ is dense in $\overline{f(X)}$, $h(f(X))$ is dense in $\mathbb{C}$.

Claim 3: Let $f \in 2^\omega$. Then $h : 2^\omega \to 2^\omega$ defined by $h(g) = f + g$ is a homeomorphism.

Certainly h is a bijection. Let $U \subset 2^\omega$ be a basic open set. Then

$$U = \prod_{i\in F_0} \{0\} \times \prod_{i\in F_1} \{1\} \times \prod_{i\notin F_0\cup F_1} 2,$$

where $F_0, F_1 \subset \omega$ are finite and $F_0 \cap F_1 = \emptyset$. Thus

$$h(U) = \prod_{i\in F_0'} \{0\} \times \prod_{i\in F_0''} \{1\} \times \prod_{i\in F_1'} \{1\} \times \prod_{i\in F_1''} \{0\} \times \prod_{i\notin F_0\cup F_1} 2,$$

where $F_j' = \{i \in F_j : f(i) = 0\}$ and $F_j'' = \{i \in F_j : f(i) = 1\}$ for $j = 0, 1$. It follows that h is open. By the same argument h^{-1} is open, whence h is a homeomorphism.

Claim 4: Let $E \subset \mathbb{C}$ be the collection of endpoints of all I_σ, where $\sigma \in 2^{<\omega}$ (see the definition of $\mathbb{C}$ prior to Theorem 1). Then X is homeomorphic to a subset of $\mathbb{C}$ disjoint from E.

Let X be (densely) embedded in $\mathbb{C}$ and let h witness $\mathbb{C} \cong 2^\omega$. Pick $a \in 2^\omega \setminus \{f+g : f \in h(X), g \in h(E)\}$. Put $X_a = \{f + a : f \in h(X)\}$. Then $h(X) \cong X_a$ and $h(E) \cap X_a = \emptyset$. It follows that $X \cong h^{-1}(X_a)$ and $E \cap h^{-1}(X_a) = \emptyset$.

Claim 5: Let $D \subset \mathbb{C}$ be dense and disjoint from E. Then D is densely ordered and has no first or last point.

Suppose on the contrary that D has a first point, say d. Since $D \cap E = \emptyset$, we must have $d > 0$. Thus $(-\infty, d)$ is a nonempty open set of $\mathbb{C}$ missing D, a contradiction. By a similar argument, D has no last point.

Now, let $d_1, d_2 \in D$ with $d_1 < d_2$. Then there exists $\sigma \in 2^{<\omega}$ such that $d_1 \in I_\sigma$ but $d_2 \notin I_\sigma$. It follows that (d_1, d_2) contains the right endpoint of I_σ. Since (d_1, d_2) is nonempty, $(d_1, d_2) \cap D \neq \emptyset$, whence D is densely ordered.

By the above claims, let X be densely embedded in $\mathbb{C}$ such that $X \cap E = \emptyset$. By Theorem 9 there exists an order-preserving bijection $f : X \to \mathbb{Q}$, hence $X \cong \mathbb{Q}$. □

Corollary 16. *The following are homeomorphic to the space $\mathbb{Q}$ of rationals:*
(a) *$\mathbb{Q}^n$ for every positive integer n;*
(b) *Any countable dense subset of $\mathbb{R}^n$ for any $1 \leq n \leq \omega$, or of the Cantor set $\mathbb{C}$;*
(c) *$\mathbb{Q}$ with the right half-open interval topology.*

3. Some metrization theorems

Theorem 17. *Let X be a compact Hausdorff space, and suppose there is a countable collection $\mathcal{U}$ of open sets such that, whenever $x \neq y \in X$, there is some $U \in \mathcal{U}$ with $x \in U$ and $y \notin \overline{U}$. Then X is metrizable.*

Hint. Let $\mathcal{B} = \{X \setminus \overline{\cup \mathcal{V}} : \mathcal{V}$ is a finite subset of $\mathcal{U}\}$. Show that $\mathcal{B}$ is a countable base for X.

Proof. Since X is compact Hausdorff, X is regular. Let $\mathcal{B} = \{X \setminus \overline{\cup \mathcal{V}} : \mathcal{V} \subset \mathcal{U}$ is finite$\}$. Clearly $\mathcal{B}$ is countable.

Let $U \subset X$ be open and let $x \in U$. For all $y \in X \setminus U$, there exists $U_y \in \mathcal{U}$ with $y \in U_y$ and $x \notin \overline{U_y}$. Observe that the U_y cover $X \setminus U$, so there exist finitely many, say $U_1, \ldots, U_n$, which cover $X \setminus U$. Then $x \in \cap_{i=1}^n (X \setminus \overline{U_i}) \subset U$. But $\cap_{i=1}^n (X \setminus \overline{U_i}) = X \setminus \overline{\cap_{i=1}^n U_i} \in \mathcal{B}$, so $\mathcal{B}$ is a base. It follows that X is second-countable, hence separable metrizable. □

Definition. A space X is said to have a G_δ*-diagonal* if the diagonal $\Delta = \{(x, x) \in X^2 : x \in X\}$ is a G_δ-set in X^2 (i.e., there are open subsets U_n, $n \in \omega$, of X^2 such that $\Delta = \bigcap_{n \in \omega} U_n$).

Theorem 18. *A compact Hausdorff space X is metrizable iff X has a G_δ-diagonal.*

Proof. Suppose X is compact Hausdorff. ($\Rightarrow$). Suppose X is metrizable. As a closed subset of a metrizable space $X \times X$, Δ is G_δ in $X \times X$.

($\Leftarrow$). Assume X has a G_δ diagonal. There exist open subsets U_n, $n \in \omega$, of $X \times X$, such that$\Delta = \bigcap_{n \in \omega} U_n$. We show X is metrizable via Lemma 17. Because X is compact Hausdorff, X is regular. So for each $n \in \omega$ and $x \in X$ there exists an open $B_n(x) \subseteq X$ containing x, with $\overline{B_n(x)} \times \overline{B_n(x)} \subseteq U_n$. For each $n \in \omega$ let $\mathcal{V}_n$ be a finite subcover of $\{B_n(x) : x \in X\}$. Let $\mathcal{B} = \bigcup_{n \in \omega} \mathcal{V}_n$. Clearly $\mathcal{B}$ is countable. And if $x \neq y \in X$ then there exists $n \in \omega$ such that $(x, y) \notin U_n$. So there exists $B \in \mathcal{V}_n$ such that $(x, x) \subseteq \overline{B} \times \overline{B} \subseteq U_n$ and so $y \notin \overline{B}$. □

Definition. A collection $\mathcal{N}$ of subsets of a space X is a *network* for X if whenever $x \in U$, where U is open in X, then there is some $N \in \mathcal{N}$ with $x \in N \subset U$.

Theorem 19. *If X has a countable network, then so does every continuous image of X.*

Proof. Suppose X has a countable network $\{N_i : i \in \omega\}$ and $f : X \to Y$ is a continuous surjection. Claim $\{f[N_i] : i \in \omega\}$ is a (countable) network for Y. Well, suppose V is open in Y and $y \in V$. We find $i \in \omega$ such that $y \in f[N_i] \subseteq V$. Since f is surjective, there exists $x \in f^{-1}(\{y\})$. Since $\{N_i : i \in \omega\}$ is a network for X and $f^{-1}(V)$ is an open subset of X containing x, there exists $i \in \omega$ such that $x \in N_i \subseteq f^{-1}(V)$. Then $y = f(x) \in f[N_i] \subseteq f[f^{-1}(V)] \subseteq V$. □

Theorem 20. *The following are equivalent for a T_1-space X:*

(a) *X has a countable network;*
(b) *X is the continuous image of a separable metric space.*

Hint for (a)⇒(b): If $\mathcal{N}$ is a countable network for X, show that the topology on X obtained by taking $\mathcal{N} \cup \{X \setminus N : N \in \mathcal{N}\}$ as a subbase is a separable metrizable topology finer (i.e., it has more open sets) than the original topology on X. (Recall that $\mathcal{B}$ is a *subbase* for a space X if the collection of all intersections of finite subsets of $\mathcal{B}$ is a base for X.)

Proof. Let $(X, \mathcal{T})$ be a T_1-space with a countable network $\mathcal{N}$. Put $\mathcal{S} = \mathcal{N} \cup \{X \setminus N : N \in \mathcal{N}\}$ and $\mathcal{B} = \{\cap \mathcal{F} : \mathcal{F} \subset \mathcal{S} \text{ is finite}\}$. Let $\mathcal{T}_{\mathcal{N}}$ be the topology on X generated by $\mathcal{B}$.

Certainly $(X, \mathcal{T}_{\mathcal{N}})$ is second-countable. Let $x \in X$ and $H \subset X$ closed in $(X, \mathcal{T}_{\mathcal{N}})$ not containing x. Pick $U \in \mathcal{B}$ such that $x \in U \subset X \setminus H$. Recall that $U = \cap \mathcal{F}$ for some finite $\mathcal{F} \subset \mathcal{S}$. Since each $F \in \mathcal{F}$ is clopen, U is clopen. It follows that U and $X \setminus U$ are disjoint open sets containing x and H, respectively. Thus $(X, \mathcal{T}_{\mathcal{N}})$ is regular, hence separable metrizable.

Let $U \in \mathcal{T}$. Then $U = \cup \mathcal{N}'$ for some $\mathcal{N}' \subset \mathcal{N}$. Since $\mathcal{N}' \subset \mathcal{B}$, $U \in \mathcal{T}_{\mathcal{N}}$. Therefore, $f : (X, \mathcal{T}_{\mathcal{N}}) \to (X, \mathcal{T})$ defined by $x \mapsto x$ is continuous.

To see the other direction, recall that separable metric spaces are second-countable. Since any countable base is necessarily a countable network, X has a countable network by Theorem 19. □

Remark. Because of Theorem 20, spaces having a countable network are sometimes called *cosmic* spaces.

Theorem 21. *If X is regular and has a countable network, then X has a G_δ-diagonal.*

Proof. Suppose X is regular and has a countable network $\mathcal{N}$. We show $X \times X$ has a G_δ diagonal Δ by showing $(X \times X) \setminus \Delta$ is the countable union of closed sets.

Clearly $\mathcal{L} = \{\overline{N} \times \overline{M} : N, M \in \mathcal{N} \text{ and } \overline{N} \times \overline{M} \subseteq (X \times X) \setminus \Delta\}$ is countable collection of closed subsets of $X \times X$. We claim $(X \times X) \setminus \Delta = \bigcup \mathcal{L}$. Well, suppose $(x, y) \in (X \times X) \setminus \Delta$. Since $X \times X$ is regular there exists a basic open set $B_1 \times B_2$ with $(x, y) \in \overline{B_1} \times \overline{B_2} \subseteq (X \times X) \setminus \Delta$. There exists $N, M \in \mathcal{N}$ such that $x \in N \subseteq B_1$ and $y \in M \subseteq B_2$, so that $(x, y) \in \overline{N} \times \overline{M} \subseteq \overline{B_1} \times \overline{B_2} \subseteq (X \times X) \setminus \Delta$. □

Theorem 22. *If X is a compact metrizable space, then so is every Hausdorff continuous image of X.*

Proof. Suppose X is compact metrizable and Y is a Hausdorff continuous image of X. Every compact metrizable space has a countable basis. In particular, X has a countable network. By Theorem 19, Y has a countable network. As the continuous image of a compact space, Y is compact. Since Y is also Hausdorff by assumption, we have Y is regular. By Theorem 21, Y has a G_δ diagonal. This, together with the fact that Y is compact Hausdorff, implies Y is metrizable (Theorem 18). □

Recall that a mapping $f : X \to Y$ is *closed* if $f(H)$ is closed in Y whenever H is closed in X.

Theorem 23. *Let $f : X \to Y$ be a continuous surjection. Then the following are equivalent:*

(a) *f is closed;*
(b) *Whenever $y \in Y$ and U is an open set in X containing $f^{-1}(y)$, there is an open set V in Y containing y with $f^{-1}(V) \subset U$;*
(c) *For each open set U in X, the set $f^*(U) = \{y \in Y : f^{-1}(y) \subset U\}$ is open in Y.*

Proof. Let $f : X \to Y$ be a continuous surjection.

$(a) \Rightarrow (b)$. Suppose $y \in Y$. Let $U \subseteq X$ be an open set containing $f^{-1}(y)$. Then $V = Y \setminus f[X \setminus U]$ is open in Y containing y. Since $V \cap f[X \setminus U] = \varnothing$, we have $f^{-1}[V] \subseteq U$.

$(b) \Rightarrow (c)$. Let $U \subseteq X$ be open and let $y \in f^*(U)$. There exists an open $V \subseteq Y$ such that $y \in V$ and $f^{-1}[V] \subseteq U$. So $f^{-1}(y') \subseteq U$ for all $y' \in V$. So $V \subseteq f^*(U)$. So $f^*(U)$ is open.

$(c) \Rightarrow (a)$. Suppose $H \subseteq X$ is closed and y is a limit point of $f[H]$. By (c), $f^*(X \setminus H)$ is open; it clearly misses $f[H]$, so $y \notin f^*(X \setminus H)$. Hence exists $x \in H$ with $f(x) = y$. So $y \in f[H]$. This proves $f[H]$ is closed. □

Definition. A continuous surjection $f : X \to Y$ is said to be *perfect* if f is closed, and $f^{-1}(y)$ is compact for each $y \in Y$.

Theorem 24. *The perfect image of a separable metrizable space is separable metrizable.*

Proof. Let $f : X \to Y$ be a perfect surjection, where X is separable metrizable. Then X has a countable base $\mathcal{B}$. W.l.o.g., we may assume $\mathcal{B}$ is closed under finite unions. Now let $\mathcal{C} = \{f^*(B) : B \in \mathcal{B}\}$, where f^* is as defined in Theorem 23(c). So $\mathcal{C}$ is a countable collection of open sets.

We prove that $\mathcal{C}$ is a base for Y. Let $y \in U$, where U is open in Y. Then $f^{-1}(y)$ is a compact subset of the open set $f^{-1}(U)$. Since $\mathcal{B}$ is closed under finite unions, there is some $B \in \mathcal{B}$ with $f^{-1}(y) \subset B \subset f^{-1}(U)$. Then $y \in f^*(B) \subset U$. Hence $\mathcal{C}$ is a countable base for Y.

Since a regular space with a countable base is separable and metrizable, it remains to prove that Y is regular. Each point of X is closed, and f is a closed map, so it follows that each point of Y is closed; so Y is T_1. Now suppose $y \in Y$ and H is a closed subset of Y not containing y. Since X is normal, there are disjoint open sets U and V containing $f^{-1}(y)$ and $f^{-1}(H)$, respectively. Then $f^*(U)$ and $f^*(V)$ are disjoint open sets containing y and H. Thus Y is regular. □

Remark. By a more complicated argument, the perfect image of any metrizable space, separable or not, is metrizable.

4. *ccc* vs. separable

Definition. A space X is said to have the *countable chain condition (ccc)* if every pairwise-disjoint collection of open subsets of X is countable.

The following is an easy observation:

Theorem 25. *Every separable space has the ccc.*

Let $I = [0,1]$.

Theorem 26. *I^I is separable.*

Proof. Let D be the set of all functions $f : I \to I$ of the form

$$f(t) = \begin{cases} q_1 & \text{if } t \in [0, p_1) \\ q_2 & \text{if } t \in [p_1, p_2) \\ \vdots & \\ q_n & \text{if } t \in [p_{n-1}, 1] \end{cases},$$

where $p_i, q_i \in \mathbb{Q} \cap I$, $p_1 < p_2 < \cdots < p_{n-1}$, and $n \in \omega$. Certainly this is a countable dense subset of I^I. □

Remark. Let $\mathfrak{c} = |I|$. By a similar argument, any product of $\mathfrak{c}$ (or fewer) separable spaces is separable.

Let $\mathfrak{c}^+$ denote the least cardinal greater than $\mathfrak{c}$.

Theorem 27. *$I^{\mathfrak{c}^+}$ is not separable.*

Proof. Suppose on the contrary that $I^{\mathfrak{c}^+}$ is separable and let D be a countable dense subset thereof. For each $\alpha \in \mathfrak{c}^+$, put $D_\alpha = \pi_\alpha^{-1}\left([0, \frac{1}{2})\right) \cap D$. If $\alpha \neq \beta \in \mathfrak{c}^+$, then $\pi_\alpha^{-1}\left((\frac{1}{2}, 1]\right) \cap D_\beta \neq \emptyset$, so $D_\alpha \neq D_\beta$. It follows that D has at least $\mathfrak{c}^+$ distinct subsets, a contradiction. □

Theorem 28. *(Δ-system lemma) Let $\mathcal{F}$ be an uncountable collection of finite sets. Then there is an uncountable subcollection $\mathcal{G}$ of $\mathcal{F}$ and a set R such that $G_1 \cap G_2 = R$ for any two distinct $G_1, G_2 \in \mathcal{G}$.*

Hint. W.l.o.g., every member of $\mathcal{F}$ has the same cardinality k. Induct on k.
Remark. A collection $\mathcal{G}$ satisfying the conclusion of Theorem 28 is called a Δ-*system* and the set R is called the *root* of the Δ-system.

Proof. Suppose $\mathcal{S}$ is an uncountable collection of finite sets. Since $\mathcal{S} = \bigcup_{n\in\omega}\{A \in \mathcal{S} : |A| = n\}$, there exists $n \in \mathbb{N}$ and an uncountable $\mathcal{S}^{'} \subseteq \mathcal{S}$ such that $|A| = n$ for all $A \in \mathcal{S}^{'}$. We prove the following via induction on n: (*) If $\mathcal{S}'$ is an uncountable collection of sets and $n \in \mathbb{N}$ such that $|A| = n$ for all $A \in \mathcal{S}'$, then there exists an uncountable $\mathcal{S}''' \subseteq \mathcal{S}'$ and a set r such that $A \cap B = r$ for all $A \neq B \in \mathcal{S}'''$.

Well, if $n = 1$ then all members of $\mathcal{S}'$ are distinct, so let $\mathcal{S}''' = \mathcal{S}'$ and $r = \varnothing$.

Supose $k \in \mathbb{N}$ and (*) holds whenever $n = k$. We show (*) holds when $n = k+1$. Suppose $\mathcal{S}'$ is uncountable and $|A| = k+1$ for all $A \in \mathcal{S}'$. For each $A \in \mathcal{S}'$, enumerate $A = \{A(1), ..., A(k+1)\}$ and let $A_{\leq k} = \{A(1), ..., A(k)\}$.

Case 1: $\{A_{\leq k} : A \in \mathcal{S}'\}$ is countable. Then there exists an uncountable $\mathcal{S}''' \subseteq \mathcal{S}'$ such that $A_{\leq k} = B_{\leq k}$ and $A(k+1) \neq B(k+1)$ for all $A \neq B \in \mathcal{S}'''$. Let $r = A_{\leq k}$ for some (any) $A \in \mathcal{S}'''$.

Case 2: $\{A_{\leq k} : A \in \mathcal{S}'\}$ is uncountable. Then by the induction hypothesis there exists an uncountable $\mathcal{S}'' \subseteq \mathcal{S}'$ and a set r such that $A_{\leq k} \cap B_{\leq k} = r$ for all $A_{\leq k} \neq B_{\leq k} \in \{A_{\leq k} : A \in \mathcal{S}''\}$.

If $\{A(k+1) : A \in \mathcal{S}''\}$ is countable, then there is an uncountable $\mathcal{S}''' \subseteq \mathcal{S}''$ and x such that $A(k+1) = x$ for all $A \in \mathcal{S}'''$; then $\mathcal{S}'''$ and $r \cup \{x\}$ work. If $\{A(k+1) : A \in \mathcal{S}''\}$ is uncountable, then there is an uncountable $\mathcal{S}''' \subseteq \mathcal{S}''$ such that the $A(k+1)$'s are distinct as A ranges over $\mathcal{S}'''$. Note that $A(k+1) \notin r$ for any $A \in \mathcal{S}'''$. Hence $A \cap B = r$ for any $A \neq B \in \mathcal{S}'''$. □

Theorem 29. *Let $\{X_\alpha : \alpha \in \kappa\}$ be a collection of ccc spaces. Then $\Pi_{\alpha\in\kappa} X_\alpha$ has the ccc iff every finite subproduct has the ccc. (A finite subproduct is a product of the form $\Pi_{\alpha\in F} X_\alpha$ for some finite subset F of κ.)*

Proof. Let $X = \prod_{\alpha\in\kappa} X_\alpha$. Suppose for some finite $F \subset \kappa$ that $X_F = \prod_{\alpha\in F} X_\alpha$ does not have the *ccc*. Then there exists an uncountable collection $\{U_i : i \in \Lambda\}$ of pairwise-disjoint basic open subsets of X_F. If for each $i \in \Lambda$, $U_i = \prod_{\alpha\in F} U^i_\alpha$ with U^i_α open in X_α, then $\{\prod_{\alpha\in F} U^i_\alpha \times \prod_{\alpha\in\kappa\setminus F} X_\alpha : i \in \Lambda\}$ is an uncountable collection of pairwise-disjoint open subsets X. Thus, X does not have the *ccc*.

Now, suppose X does not have the *ccc*. Then there exists an uncountable collection $\{U_i : i \in \Lambda\}$ of pairwise-disjoint basic open subsets of X. For each $i \in \Lambda$, let $F_i \subset \kappa$ be the support of U_i. By Theorem 28, there exists an uncountable subcollection Λ' of Λ and $R \subset \kappa$ such that $F_i \cap F_j = R$ for all distinct $i, j \in \Lambda'$. Observe that $R \neq \emptyset$, else the U_i would not be pairwise-disjoint. If $i \neq j \in \Lambda'$, there exists $\alpha \in R$ such that $\pi_\alpha(U_i) \cap \pi_\alpha(U_j) = \emptyset$. It follows that $\{\prod_{\alpha\in R} \pi_\alpha(U_i) : i \in \Lambda'\}$ is an uncountable collection of pairwise-disjoint open subsets of $\prod_{\alpha\in R} X_\alpha$, whence not every finite subproduct has the *ccc*. □

Corollary 30. *$I^{\mathfrak{c}^+}$ has the ccc but is not separable.*

5. Collectionwise normal

Definition. A collection $\mathcal{H}$ of subsets of a space X is said to be *discrete* in X if every point of X has a nbhd meeting at most one member of $\mathcal{H}$. A T_1-space X is said to be *collectionwise normal(CWN)* if, given any discrete collection $\mathcal{H}$ of closed sets, there is a pairwise-disjoint collection $\{U_H : H \in \mathcal{H}\}$ of open sets with $H \subset U_H$ for every $H \in \mathcal{H}$.

Example. Let S be the Sorgenfrey line. Let

$$\mathcal{H} = \{\{p\} : p \in S^2 \text{ is on the line } y = -x\}.$$

Then $\mathcal{H}$ is a discrete collection of closed subsets of S^2. Note that there is no pairwise-disjoint collection of open sets separating the member of $\mathcal{H}$. So S^2 is not collectionwise normal. (We already know it's not even normal, but not collectionwise normal is easier to see.)

Lemma 31. *If $\mathcal{H}$ is a discrete collection of closed sets, then $\mathcal{H}$ is pairwise-disjoint, and $\cup\mathcal{H}'$ is closed for every subcollection $\mathcal{H}'$ of $\mathcal{H}$.*

Proof. Suppose $\mathcal{H}$ is a discrete collection of closed sets. If two members of $\mathcal{H}$ intersect, then obviously $\mathcal{H}$ can't be discrete. Let $\mathcal{H}' \subseteq \mathcal{H}$, and suppose $p \notin \cup\mathcal{H}'$. Let U be an open nbhd of p meeting at most one member, say H_0, of $\mathcal{H}$. If $p \in H_0$, then $H_0 \notin \mathcal{H}'$, so U misses $\cup\mathcal{H}'$. If $H_0 \in \mathcal{H}'$, then $p \notin H_0$, and then $V = U \setminus H_0$ is an open nbhd of p missing $\cup\mathcal{H}'$. Thus p is not a limit point of $\cup\mathcal{H}'$, and it follows that $\cup\mathcal{H}'$ is closed. □

Theorem 32. *Every paracompact T_2-space is collectionwise normal.*

Proof. Suppose X is paracompact Haudorff. Let $\mathcal{H}$ be a discrete collection of closed subsets of X. For each $x \in X$, let U_x be a neighborhood of x intersecting at most one member of $\mathcal{H}$. Let $U_x^{'}$ be an open set containing x whose closure misses

- $\cup\{H^{'} \in \mathcal{H} : H^{'} \neq H\}$ if $H \in \mathcal{H}$ such that $U_x \cap H \neq \emptyset$
- $\cup\mathcal{H}$ if no such H.

Let $\mathcal{V}$ be a locally finite refinement of $\{U_x \cap U_x^{'} : x \in X\}$. Note that the closure of each member of $\mathcal{V}$ meets at most one member of $\mathcal{H}$.

Define, for $H \in \mathcal{H}$, $W_H = X \setminus cl(\cup\{V \in \mathcal{V} : cl(V) \cap H = \emptyset\} = X \setminus \cup\{cl(V) : V \in \mathcal{V}, cl(V) \cap H = \emptyset\}$. W_H is an open set containing H. Suppose $H^{'} \neq H$ and $x \in W_H \cap W_{H'}$. Let $V \in \mathcal{V}$ contain x; then $cl(V)$ meets at most one member of $\mathcal{H}$. Therefore, either $cl(V) \cap H = \emptyset$ or $cl(V) \cap H^{'} = \emptyset$. Therefore, either $x \notin W_H$ or $x \notin W_{H'}$, a contradiction. So $W_H \cap W_{H'} = \emptyset$. □

Corollary 33. *Every metrizable space is collectionwise normal.*

Example. (Bing's G) *Let A be an uncountable set, and let $\mathcal{P}(A)$ be the set of all subsets of A. For each $\alpha \in A$, define $e_\alpha : \mathcal{P}(A) \to \{0,1\}$ by $e_\alpha(B) = 1$ if $\alpha \in B$ and $e_\alpha(B) = 0$ if $\alpha \notin B$. Let $E = \{e_\alpha : \alpha \in A\}$. Note that E can be considered to be a subset of $2^{\mathcal{P}(A)}$. Let X be the set $2^{\mathcal{P}(A)}$ with the topology defined by declaring every point of $X \setminus E$ to be isolated, while each e_α has its usual product nbhds in $2^{\mathcal{P}(A)}$.*

Then X is normal but not collectionwise normal.

Hint: For non-collectionwise normal, show that $\{\{e_\alpha\} : \alpha \in A\}$ is a discrete collection of singleton sets and use the fact that the usual product topology on $2^{\mathcal{P}(A)}$ is ccc.

Proof. We first prove that X is not collectionwise normal. Note that $\{\{e_\alpha\} : \alpha \in A\}$ is a discrete collection of singleton sets iff E is a closed set, and its subspace topology is discrete. If $x \notin E$, then $\{x\}$ is an open set missing E; thus E is closed. Now fix $e_\alpha \in E$. Let $U_\alpha = \{f \in X : f(\{\alpha\}) = 1\}$. U_α is an open set containing $\{e_\alpha\}$ and missing $\{e_\beta\}$ for each $\beta \neq \alpha$. Thus E is discrete.

Now suppose $e_\alpha \in V_\alpha$ for all $\alpha \in A$, where V_α is open in X. There is an open set O_α in the product topology of $2^{\mathcal{P}(A)}$ such that $e_\alpha \in O_\alpha \subset V_\alpha$. The product topology is ccc, so the collection $\{O_\alpha : \alpha \in A\}$ is not pairwise disjoing, hence $\{V_\alpha : \alpha \in A\}$ is not pairwise disjoint. So X is not collectionwise normal.

We now show that X is normal. Since $X \setminus E$ consists of isolated points, it is not difficult to see that if any two disjoint subsets of E can be separated by disjoint open sets, then any two disjoint closed subsets of X can be separated by disjoint open sets. So let H and K be disjoint subsets of E. Let $A_H = \{\alpha \in A : e_\alpha \in H\}$. Let $U = \{f \in X : f(A_H) = 1\}$ and $V = \{f \in X : f(A_H) = 0\}$. Then $e_\alpha \in H \Rightarrow \alpha \in A_H \Rightarrow e_\alpha(A_H) = 1 \Rightarrow e_\alpha \in U$, and $e_\alpha \in K \Rightarrow \alpha \notin A_H \Rightarrow e_\alpha(A_H) = 0 \Rightarrow e_\alpha \in V$. So U and V are disjoint open sets containing H and K. □

6. Monotonically normal

Definition. A space X is said to be *monotonically normal* if to each pair (H, K) of disjoint closed sets, one can assign an open set $U(H, K)$ such that

(i) $H \subset U(H, K) \subset \overline{U(H, K)} \subset X \setminus K$;
(ii) If $H \subset H'$ and $K \supset K'$, then $U(H, K) \subset U(H', K')$.

An operator $U(H,K)$ satisfying conditions (i) and (ii) is called a *monotone normality operator* for X.

Theorem 34. *Metrizable spaces are monotonically normal.*

Proof. Let (X,d) be a metric space. For each $x \in X$ and $K \subset X$ closed not containing x, put $d(x,K) = inf\{d(x,y) : y \in K\}$. Some neighborhood of x misses K, so $d(x,K) > 0$. For each pair H,K of disjoint closed subsets of X, let $U(H,K) = \cup_{x\in H} B(x, d(x,K)/2)$. Since $U(H,K) \cap U(K,H) = \emptyset$, $H \subset U(H,K) \subset cl(U(H,K)) \subset X \setminus K$. If $H \subset H'$ and $K' \subset K$, with H', K' disjoint closed subsets of X, observe $\cup_{x\in H} B(x, d(x,K')/2) \subset U(H',K')$. But $d(x,K') \geq d(x,K)$, so $U(H,K) \subset \cup_{x\in H} B(x, d(x,K')/2) \subset U(H',K')$. □

Theorem 35. *If X is monotonically normal, then there is a monotone normality operator $U(H,K)$ for X satisfying $U(H,K) \cap U(K,H) = \emptyset$ for any pair H,K of disjoint closed sets.*

Proof. Assume X is monotonically normal. Let H,K be disjoint closed subsets of X. By assumption, there exists a monotone normality operator $U(\cdot,\cdot)$ for X. Define $U'(H,K) = U(H,K) \setminus cl(U(K,H))$. Clearly, $U'(\cdot,\cdot)$ inherits the properties of a monotone normality operator from $U(\cdot,\cdot)$. Furthermore, $U'(H,K) \cap U'(K,H) = [U(H,K) \setminus cl(U(K,H))] \cap [U(K,H) \setminus cl(U(H,K))] = \emptyset$. □

Theorem 36. *Monotonically normal spaces are collectionwise normal.*

Proof. Let X be a monotonically normal space and let $U(\cdot,\cdot)$ be the monotone normality operator guaranteed by lemma 35. Let $\mathcal{H}$ be a discrete collection of closed subsets of X. Note by lemma 31 that $\mathcal{H}$ is pairwise-disjoint and $\cup\mathcal{H}'$ is closed for any subcollection $\mathcal{H}'$ of $\mathcal{H}$.

We want to find a collection $\mathcal{U}$ of pairwise-disjoint open sets U_H such that $U_H \supset H$ for each $H \in \mathcal{H}$. Put $U_H = U(H, \cup(\mathcal{H} \setminus \{H\}))$ for each $H \in \mathcal{H}$.

Let $K \neq H \in \mathcal{H}$. Then

$$\begin{aligned} U_H \cap U_K &= U(H, \cup(\mathcal{H} \setminus \{H\})) \cap U(K, \cup(\mathcal{H} \setminus \{K\})) \\ &\subset U(H,K) \cap U(K,H) \\ &= \emptyset, \end{aligned}$$

so $\mathcal{U} = \{U_H : H \in \mathcal{H}\}$ is the desired collection. □

Theorem 37. *TFAE for a T_1-space X:*

(a) *X is monotonically normal*
(b) *To each $x \in X$ and open nbhd U of x, one can assign an open nbhd U_x of x satisfying:*
$$U_x \cap V_y \neq \emptyset \Rightarrow x \in V \text{ or } y \in U;$$
(c) *Same as (b), but with the nbhds U restricted to members of a given base $\mathcal{B}$.*

Proof. Suppose X is T_1.

$(a) \Rightarrow (b)$. Singletons in X are closed since X is T_1. Let U be a monotone normality operator satisfying the condition of Theorem 35. For each $x \in X$ and open neighborhood W of x, let $W_x = U(\{x\}, X \setminus W)$. Then W_x is an open neighborhood of x. Suppose $x,y \in X$, W and V are open neighborhoods of x and y, respectively, $x \notin V$ and $y \notin W$. Then $X \setminus W \supseteq \{y\}$ and $\{x\} \subseteq X \setminus V$. So $W_x \cap V_y = U(\{x\}, X \setminus W) \cap U(\{y\}, X \setminus V) \subseteq U(\{x\},\{y\}) \cap U(\{y\},\{x\}) = \emptyset$.

$(b) \Rightarrow (c)$. Trivial.

$(c) \Rightarrow (a)$. For disjoint closed H and K, define $U(H,K) = \bigcup\{U_x : x \in H \text{ and } U \cap K = \varnothing\}$. For each $x \in H$ there exists a (basic) open neighborhood of x which misses K, so that $H \subseteq U(H,K) \subseteq \overline{U(H,K)}$. We show $\overline{U(H,K)} \subseteq X \setminus K$. Suppose $p \in K$. Let V be an open set containing p which misses H. We may assume $V_p \subseteq V$. If $x \in H$ and $U \cap K = \varnothing$, then we can have neither $x \in V_p$ nor $y \in U_x$, so $V_p \cap U_x = \varnothing$. So $V_p \cap U(H,K) = \varnothing$, so $p \notin \overline{U(H,K)}$. □

Theorem 38. *Every subspace of a monotonically normal space is monotonically normal.*

Proof. Let X be a monotonically normal T_1-space. For each $x \in X$ and each open neighborhood U' of x, let U'_x be defined as in Theorem 37b. Let $Y \subset X$. For each open $U \subset Y$, there exists U' open in X such that $U = U' \cap Y$. For each $y \in Y$ and each open neighborhood U of y, put $U_y = U'_y \cap Y$.

Let $x, y \in Y$. If $U_x \cap V_y \neq \emptyset$, then $Y \cap (U'_x \cap V'_y) \neq \emptyset$, so $U'_x \cap V'_y \neq \emptyset$. It follows that $x \in V'$ or $y \in U'$. But $x \in V'$ implies $x \in V$ and $y \in U'$ implies $y \in U$, so $U_x \cap V_y \neq \emptyset \implies x \in V$ or $y \in U$. By the previous theorem, Y is monotonically normal. □

Theorem 39. *Every linearly ordered space is monotonically normal.*

Hint: Let $\prec$ be any well-ordering of the linearly ordered space X. For $a < x < b$, define a_x to be a if $(a,x) = \emptyset$, else let a_x be the $\prec$-least element of (a,x). Define b_x analogously, and let $(a,b)_x = (a_x, b_x)$.

Proof. Let X be linearly ordered by $<$. We will show X is monotonically normal by way of Theorem 37. Let $\mathcal{B}$ be the base of all open intervals (a,b) with $a, b \in X \cup \{-\infty, \infty\}$. Let $\prec$ be any well-ordering of X.

For $a < x < b$, define a_x as a if $(a,x) = \emptyset$, otherwise let a_x be the $\prec$-least element of (a,x). Similarly, define b_x to be x if $(x,b) = \emptyset$, and the $\prec$-least element of (x,b) otherwise. Now, let $x \in (a,b) \in \mathcal{B}$ and $y \in (c,d) \in \mathcal{B}$. Put $(a,b)_x = (a_x, b_x)$ and $(c,d)_y = (c_y, d_y)$.

Assume $(a,b)_x \cap (c,d)_y \neq \emptyset$. W.l.o.g., $d \geq c$. Suppose that $x \notin (c,d)$ and $y \notin (a,b)$. Then $x \leq c$ and $y \geq b$. Also, $(a,b)_x \cap (c,d)_y = (a_x, b_x) \cap c_y, d_y) \neq \emptyset$ implies $c_y < b_x$. So we have $x \leq c < c_y < b_x < b \leq y$. But b_x is the $\prec$-least element of (x,b), so $b_x \prec c_y$, and c_y is the $\prec$-least element of (c,y), so $c_y \prec b_x$, contradiction. Thus $y \in (a,b)$ or $x \in (c,d)$, so X is monotonically normal. □

Remark. M.E. Rudin proved in 2001 that the class of compact Hausdorff monotonically normal spaces is exactly the class of continuous images of compact linearly ordered spaces. The reverse direction is relatively easy: it's not hard to show that the closed image of a monotonically normal space is monotonically normal. But the forward direction is a deep and very difficult result.

7. Stationary and closed unbounded sets, regular and singular cardinals

We now discuss paracompactness of ordered spaces, with the eventual goal of the characterization theorem of section 9. First we consider ordinal spaces. The following result is a corollary of a couple of results from first year topology:[2] It is also a special case of Theorem 46.

Theorem 40. *The space ω_1 of countable ordinals is not paracompact.*

Theorem 41. (i) *Let C and D be closed (in the order topology) and unbounded subsets of ω_1. Then $C \cap D \neq \emptyset$;*
(ii) *Let C_n, $n \in \omega$, be closed unbounded subsets of ω_1. Then $\bigcap_{n\in\omega} C_n \neq \emptyset$.*

Proof. (i) Since C and D are unbounded, there exists an increasing sequence $x_0 < x_1 < x_2, ...$ with $x_{2i} \in C$ and $x_{2i+1} \in D$ for each $i \in \omega$. Then $x = \sup_{i\in\omega} x_i < \omega_1$ is a limit point of both C and D. Since C and D are closed, we have $x \in C \cap D$. (ii) Using the unboundedness of the sets C_n, we construct a strictly increasing sequence $(x_i)_{i\in\omega}$ of points in ω_1 as follows. Let $x_0 \in C_0$ and $x_1 \in C_1$. Let $x_2 \in C_0$, $x_3 \in C_1$, and $x_4 \in C_2$. Let $x_5 \in C_0$, $x_6 \in C_1$,$x_7 \in C_2$, and $x_8 \in C_3$. Continue in this manner. Let $x = \sup_{i\in\omega} x_i < \omega_1$. Then for each $n \in \omega$, $(x_i)_{i\in\omega}$ has a subsequence in C_n which limits to x. By closedness of the sets C_n, we have $x \in \bigcap_{n\in\omega} C_n$. □

Remark. By the same argument, the conclusion of both parts of the previous theorem can be strengthened from "$\neq \emptyset$" to "unbounded". I.e, the intersection of countably many closed unbounded sets is itself closed unbounded.

Theorem 42. *ω_1 is not perfectly normal.*

Hint. Show that the set H of all limit ordinals is a closed set, and any open superset of H contains all but countably many points of the space.

Proof. Let $H \subseteq \omega_1$ be the set of limit ordinals. If $\alpha \in \omega_1 \setminus H$ then $\alpha = 0$ or $\alpha = \beta+1$ for some $\beta \in \omega_1$. So α is isolated ($\{\alpha\} = [0, 1)$ or $\{\alpha\} = (\beta, \alpha + 1)$). Claim any open superset of H contains all but ω-many points of ω_1. Well, let $O \subseteq \omega_1$ be open containing H. Then $H \cap \omega_1 \setminus O = \varnothing$ and $\omega_1 \setminus O$ is closed, so it can't be unbounded and miss H. Thus $\omega_1 \setminus O$ is countable. Let $\{O_n : n \in \omega\}$ be a collection of open sets, each containing H. For each $n \in \omega$ there exists $\alpha_n \in \omega_1$ such that $O_n \supseteq [\alpha_n, \omega_1)$. Then $\bigcap_{n\in\omega} O_n \supseteq [\alpha, \omega_1)$ where $\alpha = \sup_{n\in\omega} \alpha_n$. So $\bigcap_{n\in\omega} O_n \neq H$. So H is not G_δ and hence ω_1 is not perfectly normal. □

Theorem 43. *Let C be closed unbounded in ω_1. Then C is homeomorphic to ω_1.*

Proof. Let C be a closed unbounded set in ω_1. Because C is well-ordered, there is a least $c_0 \in C$. Send c_0 to 0. Similarly, there is a least $c_1 \in C \setminus \{c_0\}$; send c_1 to 1. Now, if c_β has been defined for all $\beta < \alpha$, where $\alpha \in \omega_1$, put $c_\alpha = \min C \setminus \{c_\beta : \beta < \alpha\}$ and send c_α to α.

Let f be the map just constructed. Certainly f is bijective. Let $(\beta, \alpha] \subset \omega_1$ be basic open. We claim that $f^{-1}((\beta, \alpha]) = (c_\beta, c_\alpha] \cap C$ is basic open in C. Note that if α is a limit ordinal, then $c_\alpha = \min C \setminus \{c_\beta : \beta < \alpha\} = \sup\{c_\beta : \beta < \alpha\}$ is a limit of $C \cap (0, c_\alpha)$. If α is not a limit, $(c_{\alpha-1}, c_\alpha] \cap C = \{c_\alpha\} = f^{-1}(\{\alpha\})$. The claim follows. □

Theorem 44. *Suppose $f : \omega_1 \to \omega_1$ (not necessarily continuous). Let $C = \{\alpha : \forall\beta < \alpha(f(\beta) < \alpha)\}$. Then C is closed unbounded.*

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[2] If you had first year topology with me. If not: one can prove that ω_1 is countably compact but not compact, and a countably compact paracompact space must be compact.

Proof. (C closed?) Suppose α is a limit of C not in C. Suppose $\beta<\alpha$. Then $(\beta,\alpha]\cap C\neq\emptyset$. There is an $\alpha^{'}\in(\beta,\alpha)\cap C$. Then $f(\beta)<\alpha^{'}<\alpha$. Therefore $\alpha\in C$, a contradiction. So C is closed.

(C unbounded?) Let $\gamma\in\omega_1$. We need to show there is an $\alpha\in C$ such that $\alpha>\gamma$. Pick $\alpha_0>\gamma$. Pick $\alpha_1>sup\{f(\beta):\beta<\alpha_0\}$. Pick $\alpha_2>sup\{f(\beta):\beta<\alpha_1\}$. Having already picked α_n, pick $\alpha_{n+1}=\{f(\beta):\beta<\alpha_n\}$. Let $\alpha=sup\{\alpha_n:n\in\omega\}$. Suppose $\beta<\alpha$. Then $\beta<\alpha_n$ for some n, hence $f(\beta)<\alpha_{n+1}<\alpha$. So $\alpha\in C$. □

A subset S of ω_1 is said to be *stationary* in ω_1 if $S\cap C\neq\emptyset$ for every closed unbounded set C.

Theorem 45. *(Pressing Down Lemma) Let S be a stationary set in ω_1. If for each ordinal α in S, $\alpha>0$, we choose an ordinal $\beta_\alpha<\alpha$, then there is some $\beta<\omega_1$ such that $\beta=\beta_\alpha$ for uncountably many $\alpha\in S$.*

Proof. Suppose $f:S\to\omega_1$ such that $f(\alpha)<\alpha$, and $S\subseteq\omega_1$ is stationary. Suppose $f^{-1}\{\alpha\}$ is countable, for each $\alpha<\omega_1$. For each $\alpha<\omega$, let C_α be the set of all ordinals greater than $\sup f^{-1}\{\alpha\}$. Let $C=\{\gamma<\omega_1:\gamma\in\cap_{\alpha<\gamma}C_\alpha\}$.

Claim: C is club.

C is closed: Suppose η is a limit of C. Let $\alpha<\eta$. There exists $\gamma\in C$ such that $\alpha<\gamma<\eta$. So $\gamma\in C_\alpha$, implying $\eta\in C_\alpha$.

C is unbounded: Note that $\cap_{\alpha<\gamma}C_\alpha$ is unbounded for any $\gamma<\omega_1$. Let $\eta<\omega_1$. Let $\gamma_0=\eta$, $\gamma_0<\gamma_1\in\cap_{\alpha<\gamma_0}C_\alpha$, $\gamma_1<\gamma_2\in\cap_{\alpha<\gamma_1}C_\alpha$, and so on. We have $\eta<\sup_{i\in\omega}\gamma_i$.

$\sup_{i\in\omega}\gamma_i\in C$: Let $\alpha<\sup_{i\in\omega}\gamma_i$. There exists $i\in\omega$ such taht $\alpha<\gamma_i<\gamma_{i+1}<\sup_{i\in\omega}\gamma_i$. Then $\gamma_{i+1}\in C_\alpha$. So $\sup_{i\in\omega}\gamma_i\in C_\alpha$. So C is club. There exists $\gamma\in S\cap C$. Since $\gamma\in S$, $f(\gamma)<\gamma$. Since $\gamma\in C$ and $f(\gamma)<\gamma$, $\gamma\in C_{f(\gamma)}$, a contradiction.

(Alternative Approach)

Let f be pressing down. Suppose $|f^{-1}(\alpha)|\leq\omega$ for all α. Define $g:\omega_1\to\omega_1$ such that for all α, $g(\alpha)>\sup(\cup_{\beta\leq\alpha}f^{-1}(\beta))$. Let C be as in theorem 44: $C=\{\alpha:\forall\beta<\alpha,g(\beta)<\alpha\}$. There is an $\alpha\in C\cap S$. Let $\beta=f(\alpha)$. Then $\beta<\alpha$, but $g(\beta)>\alpha$, a contradiction to $\alpha\in C$. □

Remark. A slightly souped up version of the first proof shows that there is $\beta<\omega_1$ such that $f^{-1}(\beta)$ is not just uncountable but stationary. It is possible to use this stronger version of the pressing down lemma to show that there is a collection of ω_1-many disjoint stationary sets. The little simpler argument below shows there is a pair of disjoint stationary sets.

Proposition. *There are disjoint stationary sets.*

Proof. Suppose not. For every $S\subset\omega_1$, either S or $\omega_1\setminus S$ is non stationary, implying either S or $\omega_1\setminus S$ contains a club. We will show why this is not possible.

Identify ω_1 with a subset of $[0,1]$. Either $[0,\frac{1}{2}]$ or $[\frac{1}{2},1]$ contains a club. Wlog, $[\frac{1}{2},1]$ contains a club. Consider $[\frac{1}{2},\frac{3}{4}]$ and $[\frac{3}{4},1]$. One of them must contain a club: Say $[\frac{3}{4},1]$ is nonstationary. Then $[0,\frac{1}{2}]\cup[\frac{3}{4},1]$ is nonstationary, and $[\frac{1}{2},\frac{3}{4}]$ contains a club. Similarly, $[\frac{1}{2},\frac{5}{8}]$ or $[\frac{5}{8},\frac{3}{4}]$ contains a club. Inductively, get $I_1\supseteq I_2\supseteq I_3\supseteq\cdots$ such that for all n, I_n contains a club. $diam(I_n)=\frac{1}{2^{n-1}}$. Therefore $\cap I_n$ contains a club, but $|\cap I_n|=1$. □

Theorem 46. *Let S be a stationary subset of ω_1. Then S is not paracompact.*

Proof. Let $\mathcal{U} = \{[0, \alpha] \cap S : \alpha \in \omega_1\}$ be an open cover of S and let $\mathcal{V}$ be an open refinement thereof. Let C be the set of all points α in ω_1 that are limit points of S. Then C is closed unbounded. Let $S' = C \cap S$; it is easily checked that S' is stationary.

For each $\alpha \in S'$, there exists $V \in \mathcal{V}$ and $f(\alpha) \in S$ such that $f(\alpha) < \alpha$ and $[f(\alpha), \alpha] \cap S \subset V$. By the Pressing Down Lemma, there exists $\beta \in S$ such that $f(\alpha) = \beta$ for uncountably many $\alpha \in S'$.

It suffices to show that β is a member of more than finitely many members of $\mathcal{V}$. To that end, suppose on the contrary that β is a member of only finitely many members of $\mathcal{V}$, say $V_{\alpha_1}, \ldots, V_{\alpha_n}$. Let $\gamma \in \omega_1$ such that $\gamma > \sup \cup_{i=1}^{n} V_{\alpha_i}$. Then there exists $\delta \in S'$ such that $\delta > \gamma$ and $f(\delta) = \beta$. It follows that $[\beta, \delta] \cap S \not\subset V_{\alpha_i}$ for $i = 1, \ldots, n$. Thus $\beta \in V_\delta$ with $V_\delta \neq V_{\alpha_i}$ for all $1 \leq i \leq n$, a contradiction.

Since β is contained in more than finitely many members of $\mathcal{V}$, $\mathcal{V}$ is not point-finite. It follows that $\mathcal{U}$ has no locally-finite open refinite, whence S is not paracompact. □

Remark. The same proof shows that there is no point-finite or even point-countable open refinement of $\mathcal{U}$, and hence a stationary set S is not even metacompact or metalindelöf.

Theorem 47. *Let C be a closed subset of a linearly ordered space X. Then $X \setminus C$ is the union of a disjoint collection of convex open sets.*

Theorem 48. *A subspace S of ω_1 is metrizable iff S is paracompact iff S is nonstationary.*

Proof. Let $S \subset \omega_1$. Metrizability of S implies its paracompactness, and by Theorem 46 paracompactness of S implies S is nonstationary. It remains to prove that if S is nonstationary, then it is metrizable. If S is nonstationary, there exists a club C such that $C \cap S = \varnothing$. By Theorem 47, $\omega_1 \setminus C = \bigcup_{\alpha \in I} M_\alpha$ with the M_α convex open disjoint. As a countable subspace of a regular space first countable space, each M_α is regular and second countable. So each M_α is metrizable by a metric d_α with max 1. It is easily checked that the function d given by $d(x, y) = d_\alpha(x, y)$ if $x, y \in M_\alpha$, $d(x, y) = 2$ if $x \in M_\alpha$ and $y \in M_\beta$ with $\alpha \neq \beta$, is a metric on $\omega_1 \setminus C$ which induces the subspace topology on $\omega_1 \setminus C$. □

Example 49. *The space $\omega_1 \times (\omega_1 + 1)$ is not normal. Hence, the product of a normal space and a compact Hausdorff space need not be normal.*

Hint. Let $H = \{(\alpha, \alpha) : \alpha < \omega_1\}$ and $K = [0, \omega_1) \times \{\omega_1\}$. Show H and K are disjoint closed sets which can't be separated.

It will be convenient to define an ordinal κ to be a *cardinal* if there is no function from an ordinal $\alpha < \kappa$ onto κ. With this notation, ω and ω_1 denote the least infinite cardinal and least uncountable cardinal, resp. (as well as the least ordinals with infinitely many and uncountably many predecessors, resp.). A cardinal κ is called a *successor cardinal* if there is a cardinal $\lambda < \kappa$ such that κ is the least cardinal greater than λ. In this case, κ is often denoted by λ^+. A cardinal κ which is not a successor cardinal is called a *limit cardinal.*

E.g., $\omega_1, \omega_2, \omega_3, \ldots$ are successor cardinals, while ω and $\omega_\omega = sup\{\omega_n : n < \omega\}$ are limit cardinals.

Let λ be a limit ordinal. We define the *cofinality* of λ, denoted $cf(\lambda)$, to be the least cardinal κ such that there is a function $f : \kappa \to \lambda$ such that $\{f(\alpha) : \alpha < \kappa\}$ is unbounded in λ.

Example. If λ is any countable limit ordinal, then $cf(\lambda) = \omega$. Also, $cf(\omega_\omega) = \omega$.

An infinite cardinal κ is said to be *regular* if $cf(\kappa) = \kappa$. Otherwise, κ is called *singular*.

Clearly, ω_ω is singular. It is not difficult to see from the fact that a countable union of countable sets is countable that $cf(\omega_1) = \omega_1$, hence ω_1 is a regular cardinal.

Theorem 50. (i) *If κ is any infinite cardinal, then the union of $\leq \kappa$-many sets, each of cardinality $\leq \kappa$, has cardinality $\leq \kappa$;*
(ii) *If κ is a successor cardinal, then κ is regular;*
(iii) *For any limit ordinal λ, $cf(\lambda)$ is a regular cardinal.*

Proof of (i). This is essentially equivalent to $\kappa = |\kappa \times \kappa|$. Suppose it is true for all cardinals less than κ. Define a well-order $\prec$ on $\kappa \times \kappa$ as follows. Let (α, β) and (γ, δ) be distinct points in $\kappa \times \kappa$. Define $(\alpha, \beta) \prec (\gamma, \delta)$ iff $max\{\alpha, \beta\} < max\{\gamma, \delta\}$, or $max\{\alpha, \beta\} = max\{\gamma, \delta\}$ and (α, β) is less than (γ, δ) in the lexicographic order. It is not difficult to check that this is a well-ordering. Let $(\gamma, \delta) \in \kappa \times \kappa$ and suppose $max\{\gamma, \delta\} = \mu$. The set of predecessors of (γ, δ) has cardinality at most $|\mu \times \mu| = |\mu|$ by the induction hypothesis, which is less than κ. But κ is the only ordinal such that the cardinality of every predecessor is less than κ while the cardinality of the whole set is not. Hence $|\kappa \times \kappa| = \kappa$.

Now to complete the proof of (i), for each $\alpha < \kappa$ let $|A_\alpha| \leq \kappa$. Let $f_\alpha : \kappa \to A_\alpha$ be onto. Then $F : \kappa \times \kappa \to \bigcup_{\alpha<\kappa} A_\alpha$ by $F(\alpha, \beta) = f_\alpha(\beta)$ is onto. Hence $|\bigcup_{\alpha<\kappa} A_\alpha| \leq |\kappa \times \kappa| \leq \kappa$.

Proof of (ii). Supppose $\kappa = \lambda^+$, but that $\mu = cf(\kappa) < \kappa$. Note that $\mu \leq \lambda$. Let $f : \mu \to \kappa$ be unbounded in κ. Then $\{f(\alpha) : \alpha < \mu\}$ is a collection of $\leq \lambda$-many sets each of cardinality $\leq \lambda$, so its union has cardinality $\leq \lambda$. But its union is κ, contradiction.

Proof of (iii). Suppose $\mu = cf(\lambda)$ is not regular. Then $\nu = cf(\mu) < \mu$. Let $f : \nu \to \mu$ be cofinal and nondecreasing, and $g : \mu \to \lambda$ cofinal and nondecreasing. It is easy to check that $g \circ f : \nu \to \lambda$ is cofinal, so $cf(\lambda) \leq \nu$, contradiction. □

Theorem 51. *The following are equivalent for an infinite cardinal κ:*
(i) *κ is regular;*
(ii) *For any $A \subset \kappa$, if $|A| < \kappa$, then $sup(A) < \kappa$;*
(iii) *The union of $< \kappa$-many sets, each of cardinality $< \kappa$, has cardinality $< \kappa$.*

Proof. Let κ be an infinite cardinal. To see (i) implies (ii), suppose $A \subset \kappa$ with $|A| < \kappa$. Let $\lambda = |A|$ and enumerate $A = \{\alpha_\gamma : \gamma < \lambda\}$. Define $f : \lambda \to \kappa$ by $\gamma \mapsto \alpha_\gamma$. Then $\sup A = \sup_{\gamma<\lambda} f(\gamma) < \kappa$.

For (ii) implies (iii), suppose $|A_\gamma| < \kappa$ for $\gamma < \lambda < \kappa$. Without loss of generality, we may assume $A_\gamma \subset \kappa$ for each $\gamma < \lambda$ and that the A_γ are pairwise-disjoint. By (ii), $\sup A_\gamma < \kappa$ for each $\gamma < \lambda$. Then $\sup_{\gamma<\lambda}(\sup A_\gamma) < \kappa$, so

$$|\cup_{\gamma<\lambda} A_\gamma| \leq |\sup_{\gamma<\lambda}(\sup A_\gamma)| \leq \sup_{\gamma<\lambda}(\sup A_\gamma) < \kappa.$$

To see (iii) implies (i), suppose $\lambda < \kappa$ and let $f : \lambda \to \kappa$. For $\gamma < \lambda$, define $A_\gamma = f(\gamma)$ (i.e., $A_\gamma = \{\alpha : \alpha < f(\gamma)\}$). Then $| \cup_{\gamma<\lambda} A_\gamma| < \kappa$, implying f is bounded in κ. □

Closed unbounded and stationary sets are defined for any uncountable regular cardinal κ in the same way they were defined for ω_1, and the analogues of Theorems 41, 44, and the Pressing Down Lemma hold by similar arguments.

Theorem 52. *Let κ be an uncountable regular cardinal.*

(i) *If $\mathcal{C}$ is a collection of $< \kappa$-many closed unbounded subsets of κ, then $\cap\mathcal{C}$ is closed unbounded;*
(ii) *If $f : \kappa \to \kappa$ is a function, then the set $C = \{\alpha < \kappa : \forall\beta < \alpha(f(\beta) < \alpha)\}$ is closed unbounded;*
(iii) *If $S \subset \kappa$ is stationary, and to each $\alpha \in S$ with $\alpha > 0$ we choose $\beta_\alpha < \alpha$, then there is some $\beta < \kappa$ such that $\beta_\alpha = \beta$ for κ-many $\alpha \in S$.*

Theorem 53. *Let S be a stationary subset of a regular cardinal κ. Then S is not paracompact.*

Proof. Suppose κ is regular and $S \subseteq \kappa$ is stationary. Let $\mathcal{U} = \{[0, \alpha] \cap S : \alpha \in S\}$, and suppose $\mathcal{V}$ is a locally finite open refinement of $\mathcal{U}$. For each $\alpha \in S$ there exists $\beta_\alpha < \alpha$ such that $[\beta_\alpha, \alpha] \cap S \subseteq V_\alpha \in \mathcal{V}$. By Theorem 52(iii) there exists β with $\beta_\alpha = \beta$ for κ-many $\alpha \in S$. On the other hand, only finitely many members of $\mathcal{V}$ contain β. There exists $\alpha \in S$ such that $\beta_\alpha = \beta$ and $\alpha > \sup\bigcup\{V \in \mathcal{V} : \beta \in V\}$. But there exists $V' \in \mathcal{V}$ such that $[\beta, \alpha] \subseteq V'$, contradicting $\alpha > \sup\bigcup\{V \in \mathcal{V} : \beta \in V\}$. □

8. Characterization of compact and paracompact linearly ordered spaces

Theorem 54. *A linearly ordered space X is compact iff every subset of X has a least upper bound and a greatest lower bound.*

Proof. First, let X be a compact linearly ordered space. Suppose on the contrary that there exists some $Y \subset X$ without a least upper bound. Put $\mathcal{U} = \{(-\infty, y) : y \in Y\}$ and $\mathcal{V} = \{(x, \infty) : x \in X, \forall y \in Y(y < x)\}$. Certainly $\mathcal{U} \cup \mathcal{V}$ is an open cover of X, so there exist finite subcollections $\mathcal{U}'$ and $\mathcal{V}'$ of $\mathcal{U}$ and $\mathcal{V}$, respectively, such that $\mathcal{U}' \cup \mathcal{V}'$ covers X.

If $Y \neq \emptyset$, $\mathcal{U}' \neq \emptyset$ so we may assume $\mathcal{U}' = \{(-\infty, y_1), \ldots, (-\infty, y_n)\}$ with $y_1 < \cdots < y_n$. It follows that $Y \subset (-\infty, y_n)$ implying $y_n \notin Y$, a contradiction. On the other hand, if $Y = \emptyset$ we may assume $\mathcal{V}' = \{(x_1, \infty), \ldots, (x_m, \infty)\}$ with $x_1 < \cdots < x_m$. Then $X \subset (x_1, \infty)$ so $x_1 \notin X$, a contradiction. It follows that every $Y \subset X$ must have a least upper bound and, by a similar argument, a greatest lower bound.

Now, let X be a linearly ordered space in which every subset has a least upper bound and a greatest lower bound. Observe that X has a least element, say 0, and a greatest element, say 1.

Let $\mathcal{U}$ be an open cover of X and let A be the collection of all $x \in X$ such that $[0, x]$ can be covered by finitely many members of $\mathcal{U}$. Put $a = \sup A$. We claim that $a \in A$ and $a = 1$.

If $a \notin A$, there exists $U \in \mathcal{U}$ such that $(x, a] \subset U$ and $x \in A$. But this means $[0, a]$ can be covered by finitely many members of $\mathcal{U}$, so we must have $a \in A$.

Suppose on the contrary that $a < 1$. If a has no immediate successor, there exists $U \in \mathcal{U}$ such that $(a, y] \subset U$ for some $y > a$. But this means $[0, y]$ can be covered by finitely many members of $\mathcal{U}$, so a must have an immediate successor, say b. But some member of $\mathcal{U}$ must contain b, so $[0, b]$ can be covered by finitely many members of $\mathcal{U}$, contradicting the definition of a. It follows that $a = 1$, whence X is compact. □

Theorem 55. *Every linearly ordered space X is a dense subset of a compact linearly ordered space $\hat{X}$.*

Hint. Given X, call a subset A of X *left-closed* if A is closed, and $a \in A$ and $b < a$ implies $b \in A$. For example, for each $x \in X$, the set $\{a \in X : a \leq x\}$ is left-closed. (Other left-closed sets are often called "gaps".) Let $\hat{X}$ be all left-closed sets, ordered by $\subseteq$. Include the empty set in $\hat{X}$ iff X has no least element.

Proof. Let $\hat{X} = (\{A \subseteq X : \text{A is left-closed}\}, \subseteq)$. By theorem 54, it suffices to show every subet of $\hat{X}$ has a lub and glb. Let $\mathcal{A} \subset \hat{X}$.

Claim: $cl(\cup\mathcal{A})$ is the lub of $\mathcal{A}$.

Clearly $cl(\cup\mathcal{A})$ is an upper bound of $\mathcal{A}$. $cl(\cup\mathcal{A}) \in \hat{X}$. $b < a \in cl(\cup\mathcal{A}) \implies b < a^{'} \in A$ for some $A \in \mathcal{A} \implies b \in A \implies b \in cl(\cup\mathcal{A})$. So $cl(\cup\mathcal{A})$ is left-closed.

Let $B \subsetneq cl(\cup\mathcal{A})$ be left-closed. We show that B is not an upper bound of $\mathcal{A}$. Suppose it were. Then $B \supset A$ for every $A \in \mathcal{A}$. But B is closed, so then $B \supset cl(\cup\mathcal{A})$, contradiction.

Claim: $\cap\mathcal{A}$ is glb of $\mathcal{A}$.

$\cap\mathcal{A}$ is closed. If $b < a \in \cap\mathcal{A}$ then $b < a \in A$ for all $A \in \mathcal{A}$, so $b \in A$ for all $A \in \mathcal{A}$ and $b \in \cap\mathcal{A}$. So $\mathcal{A} \in \hat{X}$. Clearly $\cap\mathcal{A}$ is a lb for $\mathcal{A}$. If $\cap\mathcal{A}$ is a proper subset of a left-closed set B, then there exists $b \in B$ with $\cap A < b$. If B were a lower bound for $\mathcal{A}$, then $b \in B \subset A$ for every $A \in \mathcal{A}$, whence $b \in \cap\mathcal{A}$, contradiction.

Claim: X is dense in $\hat{X}$.

Let $(A, B) \subseteq \hat{X}$ be nonempty. There is $C \in \hat{X}$ such that $A \subset C \subset B$. There exists $x \in C$ with $A < x$ such that $A \subset \{a \in X : a \leq x\} \subset C \subset B$. □

Let $\mathcal{U}$ be a collection of sets, and let $U, V \in \mathcal{U}$. A *finite linked chain in $\mathcal{U}$ from U to V* is a sequence $U_1, U_2, \ldots, U_n$ of members of $\mathcal{U}$ such that $U_1 = U$, $U_n = V$, and $U_i \cap U_{i+1} \neq \emptyset$ for any $i = 1, 2, \ldots, n-1$. $\mathcal{U}$ is said to be *connected* if there is a finite linked chain in $\mathcal{U}$ between any two members of $\mathcal{U}$.

The next result has nothing to do with ordered spaces, but is good to know.

Theorem 56. *A space X is connected iff every cover of X by nonempty open sets is connected.*

Proof. ($\Rightarrow$). Suppose X is connected and $\mathcal{U}$ is an open cover of X. Let $U \in \mathcal{U}$ and let

$$\mathcal{U}(U) = \{V \in \mathcal{U} : \text{ there is a finite linked chain between } U \text{and } V\}.$$

Claim that $O = \bigcup\mathcal{U}(U)$ is connected. Clearly O is open. We show it is also closed. Suppose x is a limit point of O. Since $\mathcal{U}$ is an open cover, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. Then $U_x \cap O \neq \varnothing$, so there exists $V \in \mathcal{U}(U)$ such that $U_x \cap V \neq \varnothing$. Thus we may extend a finite linked chain between U and V to a finite linked chain between U and U_x, so that $x \in U_x \subseteq O$. This completes the proof that O is closed. Since X is connected and O is clopen, we must have $O = X$. Let $U_1, U_2 \in \mathcal{U}$. There exist $V_1, V_2 \in \mathcal{U}(U)$ (note that $\mathcal{U}(U)$ covers X) such that $V_1 \cap U_1 \neq \varnothing \neq V_2 \cap U_2$. Extend chains between V_1 and U, and V_2 and U, to a chain between U_1 and U_2.

($\Leftarrow$). If U and V are nonempty disjoint clopen sets with $U \cup V = X$, then $\mathcal{U} = \{U, V\}$ is an open cover of X which is not connected. □

Theorem 57. *A linearly ordered space X is paracompact iff X does not contain a closed subspace homeomorphic to a stationary subset of a regular uncountable cardinal.*

Hint: For the "if" direction, first show that it suffices to construct a locally finite open refinement of $\mathcal{U}$ on $\cup\mathcal{U}$, where $\mathcal{U}$ is a connected collection of open intervals.

9. Suslin lines, Suslin trees, and Aronszajn trees

Lemma 58. *The following are equivalent for a space X:*

(a) *X is hereditarily Lindelöf;*
(b) *Every open subspace of X is Lindelöf;*
(c) *For any collection $\mathcal{U}$ of open subsets of X, there is a countable $\mathcal{V} \subset \mathcal{U}$ such that $\cup\mathcal{V} = \cup\mathcal{U}$;*
(d) *There is no subset $\{x_\alpha : \alpha < \omega_1\}$ of X with the property that, for each $\alpha < \omega_1$, $x_\alpha \notin \overline{\{x_\beta : \beta > \alpha\}}$.*

Proof. Clearly a $\implies$ b $\implies$ c $\implies$ a.

To see $\neg$d $\implies$ $\neg$c, let $\{x_a : \alpha < \omega_1\} \subset X$ such that $x_\alpha \notin \overline{\{x_\beta : \beta > \alpha\}}$ for all $\alpha \in \omega_1$. Then for each $\alpha \in \omega_1$ there exists an open neighborhood U_α of x_α such that $U_\alpha \cap \{x_\beta : \beta > \alpha\} = \emptyset$. Take $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\}$. If $\mathcal{V} \subset \mathcal{U}$ is countable, $\cup\mathcal{V}$ may contain only countably many members of $\{x_\alpha : \alpha < \omega_1\}$, so $\cup\mathcal{V} \neq \cup\mathcal{U}$.

To see $\neg$c $\implies$ $\neg$d, let $\mathcal{U}$ be a collection of open subsets of X such that $\cup\mathcal{V} \neq \cup\mathcal{U}$ for all countable $\mathcal{V} \subset \mathcal{U}$. Pick $U_0 \in \mathcal{U}$ and $x_0 \in U_0$. Let $0 < \alpha < \omega_1$ and suppose $U_\beta \in \mathcal{U}$ and $x_\beta \in U_\beta$ have been defined for all $\beta < \alpha$. Since $\cup\{U_\beta : \beta < \alpha\}$ is a proper subset of $\cup\mathcal{U}$, there exists $U_\alpha \in \mathcal{U}$ such that $U_\alpha \setminus \cup\{U_\beta : \beta < \alpha\} \neq \emptyset$. Pick $x_\alpha \in U_\alpha \setminus \cup\{U_\beta : \beta < \alpha\}$ and observe that for each $\beta < \alpha$, U_β is a neighborhood of x_β not containing x_α. It follows that $\{x_\alpha : \alpha < \omega_1\}$ is a subset of X such that $x_\alpha \notin \overline{\{x_\beta : \beta > \alpha\}}$ for all $\alpha \in \omega_1$. □

Remark. A set $\{x_\alpha : \alpha < \omega_1\}$ satisfying the conditions of Lemma 58(d) is said to be *right-separated in type* ω_1. *Left-separated in type* ω_1 is defined analogously, and the analogous result is that a space is hereditarily separable iff it does not contain a subspace which is left-separated in type ω_1.

Theorem 59. *Suppose X is a ccc linearly ordered space. Then X is hereditarily Lindelöf.*

Hint: Since open subspaces of *ccc* linearly ordered spaces are also *ccc* linearly ordered, it suffices to show X is Lindelöf. Let $\mathcal{U}$ be an open cover of X. Define $x \sim y$ iff $[x, y]$ is covered by some countable subcollection of $\mathcal{U}$. Show that each equivalence class E is open and is covered by some countable subcollection of $\mathcal{U}$.

A linearly ordered space X is a *Suslin line* if X is *ccc*, connected, has no first or last point, and is not homeomorphic to the real line. By Theorem 11, a Suslin line cannot be separable.

Theorem 60. *If there is a ccc nonseparable linearly ordered space, then:*

(i) *there is one which is densely ordered and such that no nonempty open interval is separable;*
(ii) *there is a Suslin line.*

Proof of (i). Let X be a *ccc* nonseparable linearly ordered space. Define $x \sim y$ iff the interval from x to y is separable. Note that equivalence classes are convex. We claim that the set $S = \{[x] : x \in X\}$ of equivalence classes with the natural order, denoted by $\prec$, satisfies the desired conditions.

It follows easily that S is *ccc* because X is. To see that S is densely ordered, suppose $([x],[y])$ is empty. Then (x,y) is separable and $[x]=[y]$, contradiction.

To complete the proof, we need to show that S has no separable open intervals. First we show that every equivalence class $[x]$ is separable. It follows from the *ccc* of X that $cf([x])$ is countable, and similarly so is the coinitiality of $[x]$. Let us assume $[x]$ has a greatest point but no least point; other cases can be handled similarly. Let $y_0, y_1, ...$ be a decreasing sequence of points of $[x]$ which is coinitial in $[x]$. Then $[x] = \bigcup_{n\in\omega}(y_n, z]$, where z is the greatest point of $[x]$. Since each (y_n, z) is separable, so is $[x]$. Now suppose $([x],[y])$ is separable, say with dense set $D = \{[z_i] : i \in \omega\}$. For each i, let E_i be a dense (in X) subset of $[z_i]$. It follows that $(x,y) \cap [\bigcup_{i\in\omega} E_i]$ is dense in (x,y), hence $[x]=[y]$, contradiction.

Proof of (ii). Let X be a *ccc* nonseparable linearly ordered space. By (i), we may also assume X is densely ordered and that no nonempty open interval in X is separable. Let $\hat{X}$ be the compactification of X as in Theorem 55.

We claim that $\hat{X}$ is densely ordered. Suppose not. Let $a, b \in \hat{X}$ with $(a,b) = \emptyset$. If $a \in \hat{X} \setminus X$ and $b \in X$, then $(-\infty, a) \cap X$ is a left-closed subset of X. Note that in X, every neighborhood of b contains an interval $(c, b]$, where $c < b$. But then $c \in (-\infty, a)$ and since X is densely ordered, $(c,b) \cap (-\infty, a) \neq \emptyset$. So b is a limit point of $(-\infty, a)$, a contradiction to it being left closed.

If $a, b \in \hat{X} \setminus X$, then $(-\infty, a) \cap X = (-\infty, b) \cap X$, so these subsets are identified with the same element of $\hat{X}$, forcing $a = b$. If $a \in X$ and $b \in \hat{X} \setminus X$, then $(-\infty, a] \cap X = (-\infty, b) \cap X$, giving $a = b$ as above. So $\hat{X}$ is densely ordered.

By Theorem 54, every subset of $\hat{X}$ has a least upper bound. Because $\hat{X}$ is densely ordered and every bounded subset of $\hat{X}$ has a least upper bound, $\hat{X}$ is connected by Theorem 10. Throw out the first and last points of $\hat{X}$ to get a space X' that is connected, *ccc*, and has a subspace that is not separable. It follows that X' is not homeomorphic to $\mathbb{R}$ and is thus a Suslin line. □

Remark. Sometimes a Suslin line is defined to be a nonseparable *ccc* linearly ordered space. By Theorem 60, this is equivalent to our definition (in the sense that one exists iff the other exists).

Theorem 61. *If T is a tree of height ω, and every level of T is finite, then T has an infinite branch.*

Proof. Let T be a tree of height ω such that L_α is finite for each $\alpha < \omega$. For each $t \in T$, put $S_t = \{s \in T : t < s\}$. Since T has height ω, there must be infinitely many nodes in T.

Pick $b_0 \in L_0$ so that S_{b_0} is infinite. This is possible since L_0 is finite. Suppose for some $n \in \omega$ that $b_0 < \cdots < b_n$ have been defined so that $b_i \in L_i$ and S_{b_n} is infinite. Pick $b_{n+1} \in S_{b_n} \cap L_{n+1}$ such that $S_{b_{n+1}}$ is infinite. This is possible since $S_{b_n} \cap L_{n+1}$ is finite.

Certainly $b = \{b_n : n \in \omega\}$ is an infinite branch of T. □

An *Aronszajn tree* is a tree of height ω_1 such that every branch and every level is countable. A *Suslin tree* is a tree of height ω_1 such that every branch and every antichain is countable. Note that every Suslin tree is Aronszajn.

Theorem 62. *There is an Aronszajn tree.*

Proof. We construct the tree T by induction, along with a function $\theta : T \to \mathbb{Q}$ which is increasing in the sense that $x < y \in T \Rightarrow \theta(x) < \theta(y)$. This function will guarantee that there are no branches all the way through the tree, and will also tell us some things about the structure of T (see the remark after this proof).

For each $\alpha < \omega_1$, we let T_α denote the set of nodes of T of height $< \alpha$. To start, let $T_\omega = \omega^{<\omega}$. Let $\theta(\emptyset) = 0$. The first level L_1 of T_ω, i.e., the set of immediate successors of $\emptyset$, is countably infinite, so we can let $\theta \restriction L_1$ be a bijection from L_1 to the positive rationals. Given $t \in L_1$ with $\theta(t) = q_t$, let θ send the immediate successors of t one-to-one and onto the set of rationals $> q_t$. And so on.

From here on, we denote $\theta(t)$ by q_t.

Next we define $T_{\omega+\omega}$ as follows. For each $s \in T_\omega$ and $q \in \mathbb{Q}$ with $q > q_s$, choose a branch $b(s,q)$ of T_ω with $s \in b(s,q)$ and $q = sup\{q_t : t \in b(s,q)\}$. This is possible by the construction of θ on T_ω. Now let $\{T(s,q)) : s \in T_\omega, q > q_s\}$ be a collection of copies of $\omega^{<\omega}$ disjoint from each other and from T_ω; we'll put $T(s,q)$ above the nodes of $b(s,q)$. Let $T_{\omega+\omega} = T_\omega \cup \bigcup_{s\in T_\omega, q>q_s} T(s,q))$. Give the copies of $\omega^{<\omega}$ their usual order, and if $x \in T_\omega$ and $y \in T(s,q)$, define $x < y \iff x \in b(s,q)$. To help see the picture, note that if $\emptyset_{s,q}$ is the copy in $T(s,q)$ of the empty node of $\omega^{<\omega}$, then the set of predecessors of $\emptyset_{s,q}$ in $T_{\omega+\omega}$ is precisely $b(s,q)$. So $\emptyset_{s,q}$ stands at the top of $b_{s,q}$ at level ω, and $T(s,q)$ extends above it. Define $\theta(\emptyset_{s,q}) = q$, and extend θ to $T(s,q)$ level by level similar to the way it was defined on T_ω.

Now suppose α is a limit ordinal $> \omega + \omega$, and T_β and $\theta \restriction T_\beta$ has been defined for all limit ordinals $\beta < \alpha$ such that the following holds:

$(*)$ If $\gamma < \delta < \beta$, s is in the γ^{th} level of T_β, and $q \in \mathbb{Q}$ is such that $q > q_s$, then there is a successor t of s in the δ^{th} level of T_β with $q_t = q$.

It is easily checked that $(*)$ holds for T_ω and $T_{\omega+\omega}$.

If α is a limit of limit ordinals, we simply let $T_\alpha = \bigcup_{\beta<\alpha} T_\beta$ and $\theta \restriction T_\alpha = \bigcup_{\beta<\alpha} \theta \restriction T_\beta$. It is easy to check that $(*)$ holds with $\beta = \alpha$.

It remains to construct T_α for the limit ordinal α when there is a maximal limit ordinal $\beta < \alpha$, so $\alpha = \beta + \omega$. The construction in this case is similar to the construction of $T_{\omega+\omega}$ from T_ω, First we show that for each $s \in T_\beta$ and $q > q_s$, there is a branch $b(s,q)$ of T_β which meets every level of T_β below β and such that $q = sup\{q_t : t \in b(s,q)\}$. Let $\gamma_0, \gamma_1, ...$ be an increasing sequence of ordinals with supremum β, such that γ_0 is the level of $s_0 = s$. Let $q_0 < q_1 < ...$ be a sequence of rationals with $q_0 = q_s$ and $q = sup\{q_n : n \in \omega\}$. By $(*)$, there is a successor s_1 of s_0 at level γ_1 with $q_{s_1} = q_1$, then a successor s_2 of s_1 at level γ_2 with $q_{s_2} = q_2$, etc.. Let $b(s,q)$ be the branch determined by the sequence $s_0, s_1, ...$ Let $\{T(s,q)) : s \in T_\beta, q > q_s\}$ be a collection of copies of $\omega^{<\omega}$ disjoint from each other and from T_β and let $T_\alpha = T_\beta \cup \bigcup_{s\in T_\beta, q>q_s} T(s,q)$. Define the order on T_α and extend θ to T_α in the same way this was done for $T_{\omega+\omega}$ vis-à-vis T_ω.

This defines T_α for all $\alpha < \omega_1$. Let $T = \bigcup_{\alpha<\omega_1} T_\alpha$ and for each $t \in T$ let $\theta(t) = q_t \in \mathbb{Q}$ be as defined in the construction. Then it is immediate from the construction that T has height ω_1, every level of T is countable, and as noted earlier, since $s < t \Rightarrow q_s < q_t$ there are no uncountable branches. □

Remark. Let $\theta(t) = q_t$ be as in the above proof, and for each $q \in \mathbb{Q}$ let $A_q = \{t \in T : q_t = q\}$. Note that each A_q is an antichain of T, so $T = \bigcup_{q \in \mathbb{Q}} A_q$ is the union of countably many antichains, at least one of which must be uncountable. So this tree is definitely not Suslin. Indeed, any Aronszajn tree which is the union of countably many antichains is called a *special* Aronszajn tree. It is consistent with the axioms of set theory not only that there are no Suslin trees, but that *every* Aronszajn tree is special (e.g., this holds under the axiom $MA(\omega_1)$ discussed in the next section).

Theorem 62 shows that the natural analogue of Theorem 61 for trees of height ω_1 is false: every level can be countable yet there is no branch going all the way to the top of the tree. (If, however, you require every level to be finite, then there is such a branch.)

Theorem 63. *If there is a Suslin line, then there is a Suslin tree.*

Proof. Let X be a Suslin line. Define separable closed subsets C_α, $\alpha < \omega_1$, of X as follows. Let $C_0 = \emptyset$. If α is a limit ordinal and C_β has been defined for all $\beta < \alpha$, let $C_\alpha = \overline{\bigcup_{\beta<\alpha} C_\beta}$. If $\alpha = \gamma + 1$ and C_γ has been defined, let L_γ be the collection of convex components of $X \setminus C_\gamma$. For each $I \in L_\gamma$, choose a countable sequence of points converging to each endpoint of I. Let C'_γ be the collection of these chosen points for all $I \in L_\gamma$. Then let $C_\alpha = \overline{C_\gamma \cup C'_\gamma}$. Since the the closure of the union of countably many separable sets is separable, and C'_γ is countable, we see that each C_α is separable.

Now let L_α be the set of all convex components of $X \setminus C_\alpha$, and let $T = \bigcup_{\alpha<\omega_1} L_\alpha$.

We claim that T ordered by $\supseteq$ is a Suslin tree. Let $I \in T$. Then there is a unique α such that I is a convex component of $X \setminus C_\alpha$. Since the C_α's get bigger with α, if $I \subset J \in T$, then J is a convex component of $X \setminus C_\beta$ for some $\beta < \alpha$. Furthermore, for each $\beta < \alpha$ there is a unique such J, call it J_β. Then $\{J_\beta : \beta < \alpha\}$ is the set of predecessors of I and has order type α, so I is a member of level α of T. Since each C_α is separable, $X \setminus C_\alpha \neq \emptyset$, so T has a node at every level $\alpha < \omega_1$, and hence T has height ω_1.

Let L_α denote the α^{th} level of T. Suppose $I \in L_\alpha$, $J \in L_\beta$, $\alpha \leq \beta$, and I and J are incomparable. There is a unique $J' \in L_\alpha$ with $J \subseteq J'$. Since I and J are incomparable, $I \neq J'$, but this means that $I \cap J' = \emptyset$, and so $I \cap J = \emptyset$. Thus an antichain in T corresponds to a pairwise-disjoint collection of open sets, so every antichain of T is countable.

Note that by construction, each I in T has at least two immediate successors. We need to show that T has no uncountable chain. We show something more general: if T is a tree with no uncountable antichains, and each node of T has at least two immediate successors, then T has no uncountable chain. Suppose it did. Then there is a chain $\{t_\alpha : \alpha < \omega_1\}$ with $t_\alpha \in L_\alpha$. Let s_α be an immediate successor of t_α which is not equal to $t_{\alpha+1}$. It is easy to check that $\{s_\alpha : \alpha < \omega_1\}$ is an uncountable antichain, contradiction. □

Theorem 64. *If there is a Suslin tree, then there is a Suslin line.*

Hint. Let T be a Suslin tree, and let $\prec$ be an arbitrary linear order on T. Let X be the set of all branches of T. For any branch b, let $b(\alpha)$ be the member of b in level α of the tree. Given $b_1, b_2 \in X$, let α be minimal such that $b_1(\alpha) \neq b_2(\alpha)$, and then define $b_1 < b_2$ iff $b_1(\alpha) \prec b_2(\alpha)$. Show that X with this order is *ccc* and nonseparable.

Proof. Let T be a Suslin tree and let $\prec$ be some linear order on T. Let X be the set of all branches of T. For each $b \in X$ let $b(\alpha)$ be the member of b at level α of the tree. Given $b_1 \neq b_2 \in X$ let α be least such that $b_1(\alpha) \neq b_2(\alpha)$ and define $b_1 < b_2$ iff $b_1(\alpha) \prec b_2(\alpha)$. Then clearly $<$ linearly orders X.

Claim 1. *X is ccc.* Let $\mathcal{U} = \{(a_i, b_i) : i \in I\}$ be a collection of nonempty pairwise disjoint open intervals in X. For each $i \in I$ let $c_i \in (a_i, b_i)$, then let α be minimal such that $c_i(\alpha) \neq a_i(\alpha), b_i(\alpha)$ and let $t_i = c_i(\alpha + 1)$. Then $\{t_i : i \in I\}$ is an antichain in T: if $i \neq j \in I$ and c is branch containing t_i and t_j then $c \in (a_i, b_i) \cap (a_j, b_j)$, a contradiction. So I is countable by the ccc in T.

Claim 2. *X is nonseparable.* Let $C \subseteq X$ be countable. Let $\kappa = \sup\{\text{height}(b) : b \in C\} + 1$. Then there exists $t \in T$ at level κ and three branches $a < b < c \in X$, each containing t. Then $a(\alpha) = b(\alpha)$ for each $\alpha < \kappa$ so any $d \in C$ lies on the same side of a and b, hence $d \notin (a, b)$. So (a, b) is a nonempty open set missing C, so C is not dense in X. □

Theorem 65. *If S is a Suslin line, then S^2 does not have the ccc.*

Hint. Let $T = \bigcup_{\alpha<\omega_1} L_\alpha$ be as in the hint for Theorem 63. For each α, choose $I_\alpha \in L_\alpha$. There are disjoint $I^0_\alpha, I^1_\alpha \in L_{\alpha+1}$ contained in I_α. Show that $\{I^0_\alpha \times I^1_\alpha : \alpha < \omega_1\}$ is pairwise-disjoint.

Proof. Let S be a Suslin line and let $T = \cup_{\alpha<\omega_1} L_\alpha$ be as in the hint for Theorem 63. For each α, pick $I_\alpha \in L_\alpha$ and let $I^0_\alpha, I^1_\alpha \in L_{\alpha+1}$ be disjoint subsets of I_α.

Let $\alpha, \beta \in \omega_1$ with $\alpha < \beta$. Certainly I_β has to miss either I^0_α or I^1_α. Thus either I^0_β misses I^0_α or I^1_β misses I^1_α, so $(I^0_\alpha \times I^1_\alpha) \cap (I^0_\beta \times I^1_\beta) = \emptyset$. It follows that $\{I^0_\alpha \times I^1_\alpha : \alpha < \omega_1\}$ is a pairwise-disjoint collection. □

Theorem 66. *If there is a Suslin line, there is one such that no nondegenerate interval is separable. Such a Suslin line is the union of ω_1-many nowhere-dense sets.*

Hint for the second part: study the hint for Theorem 63.

Proof. If there is a Suslin line, then by Theorem 60(i), there is a densely ordered *ccc* linearly ordered space X such that no nonempty open interval is separable. In the proof of 60(ii), a Suslin line Y is constructed from such a linearly ordered space such that X is dense in in Y. It follows that Y has no nonempty separable open intervals either. Now let $\mathcal{C} = \{C_\alpha : \alpha < \omega_1\}$ be the strictly increasing sequence of separable closed subsets of Y as constructed in the proof of Theorem 63. For each $\alpha < \omega_1$, C_α is closed so if it failed to be nowhere dense, it would contain an open interval of Y, and this interval would be separable because C_α is. This is a contradiction, so C_α is nowhere dense.

We claim that $Y = \bigcup_{\alpha<\omega_1} C_\alpha$. Suppose otherwise, and let $p \in Y \setminus \bigcup_{\alpha<\omega_1} C_\alpha$. Then for each α, $p \notin C_\alpha$, so there is a convex component I_α of $Y \setminus C_\alpha$ containing p. In the proof of Theorem 63, it was shown that any pair I, J of convex components of a pair of C_α's is either disjoint or comparable. So $\{I_\alpha : \alpha < \omega_1\}$ is an uncountable chain in the Suslin tree constructed there, which is a contradiction. Thus Y is a Suslin line which is the union of ω_1-many nowhere dense sets. □

10. Martin's Axiom

Like the Continuum Hypothesis (CH) , Martin's Axiom (MA) is an axiom of set theory that is known to be consistent with and independent of the usual Zermelo-Frankel axioms together with the Axiom of Choice (abbreviated ZFC). Martin's Axiom has had many applications in certain parts of general topology and real analysis. It's most powerful when conjuncted with the *negation* of the Continuum Hypothesis. Roughly speaking, assuming MA+¬CH, one can bump up to ω_1 or higher some results that are true in ZFC for ω. For example, in ZFC you know how to prove that the real line is not the union of countably many nowhere dense sets. Assuming MA+¬CH, $\mathbb{R}$ is not the union of ω_1-many nowhere dense sets. We'll also see that MA+¬CH kills Suslin lines.

Understanding and using MA takes some practise, but with a little experience you will get the idea.

Let $(P, \leq)$ be a partially ordered set. We say $D \subset P$ is *dense* in P if for any $p \in P$, there is $q \leq p$ with $q \in D$. A subset G of P is called a *filter* in P if

(i) For each $p, q \in G$, there is $r \in G$ with $r \leq p$ and $r \leq q$;
(ii) For each $p \in G$, if $p \leq q$ then $q \in G$.

For example, let X be any set, and let $\mathcal{P}(X) \setminus \{\emptyset\}$ be the collection of all nonempty subsets of X. For $p, q \in \mathcal{P}(X) \setminus \{\emptyset\}$, define $p \leq q$ iff $p \subset q$. Then G is a filter on $\mathcal{P}(X) \setminus \{\emptyset\}$ iff G is a filter of subsets of X in the usual sense ((i) and (ii) tell you G is closed under finite intersections and under supersets).

We say that two elements p, q of P are *comparable* if $p \leq q$ or $q \leq p$, *compatible*(abbreviated $p \not\perp q$) if there is $r \in P$ with $r \leq p$ and $r \leq q$, and *incompatible* (abbreviated $p \perp q$) if they are not compatible.

In the above example of $\mathcal{P}(X) \setminus \{\emptyset\}$, p and q are comparable iff one is a subset of the other, compatible iff they have nonempty intersection, and thus incompatible iff they are disjoint.

An *antichain* in P is a subset A of P such that every two elements of A are incompatible. We say that $(P, \leq)$ has the *ccc* if every antichain is countable, or equivalently, every uncountable subset of P has a pair of compatible elements.

Note that $\mathcal{P}(X) \setminus \{\emptyset\}$ will not have the *ccc* if X is uncountable. But now let X be a topological space, and let $\mathcal{O}(X)$ be the collection of all nonempty open sets, and define $p \leq q$ iff $p \subset q$. Then the partial order $\mathcal{O}(X)$ has the *ccc* iff the space X has the *ccc*. Note that a subset $\mathcal{D}$ of $\mathcal{O}(X)$ is dense in this partial order iff $\mathcal{D}$ is a collection of nonempty open sets such that every nonempty open set contains a member of $\mathcal{D}$. (Such a collection $\mathcal{D}$ is sometimes called a π-*base* for the space X.)

Finally, we now define Martin's Axiom(MA): Let κ be a cardinal. $MA(\kappa)$ is the following statement: Whenever $(P, \leq)$ is a *ccc* partial order, and $\mathcal{D}$ is a family of $\leq \kappa$-many dense sets, then there is a filter G in P such that $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.

MA is the statement that $MA(\kappa)$ holds for every $\kappa < 2^\omega$.

Theorem 67. *$MA(\omega)$ is true, and hence the Continuum Hypothesis (CH) implies MA.*

Proof. Let $D_0, D_1, \ldots$ be countably many dense subsets of a partial order P. Choose $d_0 \in D_0$. Since D_1 is dense, there is $d_1 \in D_1$ with $d_1 \leq d_0$. Similarly, there is $d_2 \in D_2$ with $d_2 \leq d_1$, and so on. Let $F = \{p \in P : \exists n \in \omega \text{ with } d_n \leq p\}$. It is easy to check that F is a filter in P. By construction, $F \cap D_n \neq \emptyset$ for all n. □

Theorem 68. *Assume $MA(\kappa)$. Let X be a compact Hausdorff ccc space. If $\mathcal{U}$ is a collection of $\leq \kappa$-many dense open sets, then $\cap\mathcal{U} \neq \emptyset$.*

Hint: Use the partial order $\mathcal{O}(X)$ defined above.

Proof. Let $\mathcal{U}$ be a collection of $\leq \kappa$-many dense open sets in the compact Hausdorff *ccc*-space X. Let $\mathcal{O}(X)$ be the collection of nonempty open subsets of X, ordered by $\subseteq$. Note that incompatible elements of $\mathcal{O}(X)$ are disjoint, so since X has the *ccc*, so does $\mathcal{O}(X)$.

For each $U \in \mathcal{U}$, let $D_U = \{V \in \mathcal{O}(X) : \overline{V} \subset U\}$. Since X is regular, it is easy to check that each D_U is dense in the partial order $\mathcal{O}(X)$. By $MA(\kappa)$, there is a filter G in $\mathcal{O}(X)$ which meets each D_U. Let $g_U \in G \cap D_U$. Since G is a filter, whenever $U_1, ..., U_n$ are in $\mathcal{U}$, there is $h \in \mathcal{O}(X)$ with $h \subseteq g_{U_i}$ for each $i = 1, 2, ..., n$. It follows that $\{g_U : U \in \mathcal{U}\}$ has the f.i.p., and so does $\{\overline{g}_U : U \in \mathcal{U}\}$. Since X is compact, $\bigcap_{U\in\mathcal{U}} \overline{g}_U \neq \emptyset$. Since $g_U \in D_U$, $\overline{g}_U \subset U$, and so $\cap\mathcal{U} \neq \emptyset$. □

The topological statement in Theorem 68 is actually equivalent to $MA(\kappa)$. So Martin's Axiom is equivalent to the statement that no compact Hausdorff *ccc* space is the union of fewer than 2^ω-many nowhere-dense sets. In particular, MA implies that the real line is not the union of fewer than 2^ω-many nowhere-dense sets. One can also show that MA implies the real line is not the union of fewer than 2^ω-many Lebesgue measure zero sets.

Corollary 69. *$MA(2^\omega)$ is false.*

Proof. Let $X = [0, 1]$, and for each $x \in [0, 1]$, let $U_x = X \setminus \{x\}$. Then $\mathcal{U} = \{U_x : x \in [0, 1]$ is a collection of 2^ω-many dense open subsets of the interval, and of course $\cap\mathcal{U} = \emptyset$. □

Corollary 70. *Assume $MA(\omega_1)$. Then there are no Suslin lines.*

Proof. By Theorem 66, if there is a Suslin line, then there is a Suslin line X which is a union of ω_1-many nowhere dense sets, say $\{A_\alpha : \alpha < \omega_1\}$. Then $\{X \setminus \overline{A}_\alpha : \alpha < \omega_1$ is s collection of ω_1-many dense open sets with empty intersection. □

We will soon see other ways (e.g., via Theorem 73) to show that Corollary 70 is true.

Lemma 71. *Assume $MA(\omega_1)$. Suppose X is ccc and $\{U_\alpha : \alpha < \omega_1\}$ is a collection of nonempty open subsets of X. Then there is an uncountable subset A of ω_1 such that $\{U_\alpha : \alpha \in A\}$ has the f.i.p..*

Hint. First show that there is $\alpha_0 < \omega_1$ such that $\forall \alpha > \alpha_0$

$$\overline{\bigcup_{\beta>\alpha} U_\beta} = \overline{\bigcup_{\beta>\alpha_0} U_\beta}.$$

Then apply MA with $P = \{O \in \mathcal{O}(X) : O \subset \bigcup_{\beta>\alpha} U_{\alpha_0}\}$.

Proof. Suppose MA(ω_1), X is a ccc space, and $\{U_\alpha : \alpha < \omega_1\}$ is an uncountable collection of (distinct) nonempty open subsets of X.

We claim that there exists $\alpha_0 < \omega_1$ such that $\bigcup_{\beta>\alpha_0} U_\beta \subseteq \overline{\bigcup_{\beta>\alpha} U_\beta}$ for each $\alpha > \alpha_0$. If not, there exists $\alpha_0 > 0$ such that $\bigcup_{\beta>0} U_\beta \setminus \overline{\bigcup_{\beta>\alpha_0} U_\beta} \neq \varnothing$. If $\delta < \omega_1$ and α_γ has been defined for all $\gamma < \delta$, pick $\alpha_\delta = \sup_{\gamma<\delta} \alpha_\gamma$ if δ is a limit, and pick $\alpha_\delta > \alpha_{\delta-1}$ such that $\bigcup_{\beta>\alpha_{\delta-1}} U_\beta \setminus \overline{\bigcup_{\beta>\alpha_\delta} U_\beta} \neq \varnothing$ if δ is a successor. Then $\{\bigcup_{\beta>\alpha_\delta} U_\beta \setminus \overline{\bigcup_{\beta>\alpha_{\delta+1}} U_\beta} : \delta < \omega_1\}$ violates the ccc in X.

Let $\mathbb{P} = \{O \in \mathcal{O}(X) : O \subseteq \bigcup_{\beta>\alpha_0} U_\beta\}$. For each $\alpha > \alpha_0$ let $D_\alpha = \{O \in \mathbb{P} : (\exists\beta > \alpha)(O \subseteq U_\beta)\}$. Then D_α is dense in $\mathbb{P}$: Let $O \in \mathbb{P}$. Then $O \subseteq \bigcup_{\beta>\alpha_0} U_\beta \subseteq \overline{\bigcup_{\beta>\alpha} U_\beta}$ so there exists $\beta > \alpha$ such that $O \cap U_\beta \neq \varnothing$. Then $O \cap U_\beta \in D_\alpha$ and $O \cap U_\beta \subseteq O$ as desired.

Note that X is ccc implies $\mathcal{O}(X)$ is ccc implies $\mathbb{P}$ is ccc. By MA(ω_1) there exists a filter G on $\mathbb{P}$ such that $G \cap D_\alpha \neq \varnothing$ for all $\alpha > \alpha_0$. Since G is closed upward, for each $\alpha > \alpha_0$ there exists $\beta > \alpha$ such that $U_\beta \in G$. Let $\gamma_0 > \alpha_0 + 1$ such that $U_{\gamma_0} \in G$. If $\delta < \omega_1$ and an increasing sequence $(\gamma_\beta)_{\beta<\delta}$ has been defined so that $U_{\gamma_\beta} \in G$ for each $\beta < \delta$, let $\gamma_\alpha > \sup_{\beta<\delta} \gamma_\beta$ such that $U_{\gamma_\delta} \in G$. Then $\{U_{\gamma_\delta} : \delta < \omega_1\}$ is an uncountable subcollection of $\{U_\alpha : \alpha < \omega_1\}$ with the finite intersection property. □

A space X is said to have *property K* if every uncountable collection $\mathcal{U}$ of nonempty open sets contains an uncountable subcollection $\mathcal{V}$ such that $V_1 \cap V_2 \neq \emptyset$ for every $V_1, V_2 \in \mathcal{V}$.

Theorem 72. *Assume $MA(\omega_1)$. Then every ccc space has property K.*

Proof. Immediate from Lemma 71. □

Theorem 73. *Assume $MA(\omega_1)$. If X and Y are ccc, so is $X \times Y$. Hence any product of ccc spaces is ccc.*

Proof. Suppose X, Y have the ccc and $\{U_\alpha : \alpha \in \Lambda\}$ is an uncountable collection of pairwise disjoint nonempty open subsets of $X \times Y$. Assume the U_α's are basic; $U_\alpha = V_\alpha \times W_\alpha$. By Theorem 72 since X has the ccc, X has property K. There exists an uncountable $A \subseteq \Lambda$ such that $\{V_\alpha : \alpha \in A\}$ is linked. Then $\{W_\alpha : \alpha \in A\}$ is uncountable so by Theorem 72 there exists an uncountable $B \subseteq A$ such that $\{W_\beta : \beta \in B\}$ is linked. Let $\beta \neq \gamma \in B$. Then $U_\beta \cap U_\gamma = (V_\beta \times W_\beta) \cap (V_\gamma \times W_\gamma) \neq \varnothing$ since $V_\beta \cap V_\gamma \neq \varnothing$ and $W_\beta \cap W_\gamma \neq \varnothing$. Contradiction. The theorem now follows from the Δ-system Lemma (see Theorem 29). □

Lemma 74. *(a)A regular Lindelöf space X is hereditarily Lindelöf iff X is perfectly normal.*

(b) If X is compact Hausdorff, then X is first-countable iff every point of X is a G_δ-set. Consequently, compact Hausdorff perfectly normal spaces are first-countable.

Proof of (a). Let $(X, \mathcal{T})$ be a regular Lindelöf space and suppose X is hereditarily Lindelöf. Certainly X is normal. Let $H \subset X$ be closed and for each $x \in X \setminus H$ let $U_x \in \mathcal{T}$ such that $x \in U_x \subset \overline{U}_x \subset X \setminus H$. Since $X \setminus H$ is Lindelöf, there exist $x_0, x_1, \ldots \in X$ such that $\{U_{x_i} : i \in \omega\}$ covers $X \setminus H$. It follows that $H = \cap_{i\in\omega} X \setminus \overline{U}_{x_i}$.

Now suppose X is perfectly normal. Let $\mathcal{U} \subset \mathcal{T}$ be uncountable and put $H = X \setminus (\cup\mathcal{U})$. Let $\{V_i : i \in \omega\} \subset \mathcal{T}$ such that $H = \cap_{i\in\omega} V_i$. Observe that each $X \setminus V_i$ is closed, hence Lindelöf. For each $i \in \omega$, let $\mathcal{V}_i \subset \mathcal{U}$ be a countable cover of $X \setminus V_i$. Put $\mathcal{V} = \cup_{i\in\omega} \mathcal{V}_i$. Then $\mathcal{V} \subset \mathcal{U}$ is countable with $\cup\mathcal{V} = \cup\mathcal{U}$, whence X is hereditarily Lindelöf. □

Proof of (b). Let $(X, \mathcal{T})$ be a compact Hausdorff space and suppose X is first-countable. Let $x \in X$ and let $\{U_i : i \in \omega\}$ be a local base at x. If $y \in X \setminus \{x\}$ there must exist some $i \in \omega$ such that $y \notin U_i$, so $\{x\} = \cap_{i \in \omega} U_i$.

Now suppose every point of X is G_δ. Let $x \in X$ and $\{V_i : i \in \omega\} \subset \mathcal{T}$ with $\{x\} = \cap_{i\in\omega} V_i$. Put $U_0 = V_0$. If $n \in \omega$ and U_n has been defined, let $U_{n+1} \in \mathcal{T}$ such that $x \in U_{n+1} \subset \overline{U}_{n+1} \subset U_n \cap V_{n+1}$. Then $U_0 \supset \overline{U}_1 \supset U_1 \supset \dots$ and $\{x\} = \cap_{i\in\omega} U_i = \cap_{i\in\omega} \overline{U}_i$. Let U be an open neighborhood of x. Since $X \setminus U$ is compact and covered by $\{X \setminus \overline{U}_i : i \in \omega\}$, there exists $k \in \omega$ such that $X \setminus U \subset X \setminus \overline{U}_k$. It follows that $U_k \subset U$, whence $\{U_i : i \in \omega\}$ is a local base at x. □

Theorem 75. *Assume $MA(\omega_1)$. If X is a compact Hausdorff hereditarily Lindelöf space, then X is hereditarily separable.*

Hint. Suppose not. Then there are points x_α, $\alpha < \omega_1$, in X such that $x_\alpha \notin \overline{\{x_\beta : \beta < \alpha\}}$. Let $Y = \overline{\{x_\alpha : \alpha < \omega_1\}}$. It follows from first-countability that every point of Y is in $\overline{\{x_\beta : \beta < \alpha\}}$ for some $\alpha < \omega_1$. Use compactness of Y and Lemma 71 applied to Y to get a contradiction.

Proof. Assume MA(ω_1). Suppose X is compact Hausdorff hereditarily Lindelof but *not* hereditarily separable. Then there exists a subspace S of X which is not separable. Then $\overline{S}$ is not separable: Suppose $\{x_n : n \in \omega\}$ is dense in $\overline{S}$. By Theorem 74 parts (a) and (b) X is first countable. For each $n \in \omega$ let $\{U_i^n : i \in \omega\}$ be a local base at x_n. For each $n, i \in \omega$ there exists $s_i^n \in S \cap U_i^n$. Then $\{s_i^n : n, i \in \omega\}$ is countable and dense in S.

(i) There are points $x_\alpha \in \overline{S}$, $\alpha < \omega_1$, such that $x_\alpha \notin \mathrm{cl}_{\overline{S}}\{x_\beta : \beta < \alpha\} = \overline{\{x_\beta : \beta < \alpha\}}$: Let $x_0 \in \overline{S}$. Assuming $\alpha < \omega_1$ and x_β, $\beta < \alpha$, have been chosen appropriately, $\{x_\beta : \beta < \alpha\}$ is countable and thus there exists $x_\alpha \in \overline{S} \setminus \mathrm{cl}_{\overline{S}}\{x_\beta : \beta < \alpha\}$.

Let $Y = \overline{\{x_\alpha : \alpha < \omega_1\}}$. (ii) For each $y \in Y$ there exists $\alpha < \omega_1$ such that $y \in \overline{\{x_\beta : \beta < \alpha\}}$: If $y \in Y$ then, letting $\{U_n : n \in \omega\}$ be a local base at y and $x_{\alpha_n} \in U_n$ for each n, we have $y \in \overline{\{x_\beta : \beta < \sup_{n\in\omega} \alpha_n\}}$.

Y is compact Hausdorff and therefore Y is regular, so by (i) for each $\alpha < \omega_1$ there exists an open $U_\alpha \subseteq Y$ such that $x_\alpha \in U_\alpha \subseteq \overline{U_\alpha} \subseteq Y \setminus \overline{\{x_\beta : \beta < \alpha\}}$. Note that Y is ccc since it is hereditarily Lindelof. By Lemma 71 there exists a cofinal subset A of ω_1 such that $\{U_\alpha : \alpha \in A\}$ has the finite intersection property. Then $\{\overline{U_\alpha} : \alpha \in A\}$ has the finite intersection property. Since Y is compact we have $\bigcap_{\alpha \in A} \overline{U_\alpha} \neq \varnothing$. This contradicts (ii), as a point in this intersection cannot be in $\overline{\{x_\beta : \beta < \alpha\}}$ for any $\alpha < \omega_1$. □

The next lemma, and a couple later results, require a poset consisting of ordered pairs. We include the proof of this one, since it is a prototype of many Martin's Axiom arguments.

Theorem 76. *Assume $MA(\kappa)$. Let $\{U_\alpha : \alpha < \kappa\}$ be a collection of dense open subsets of the real line $\mathbb{R}$. Then there is a dense G_δ-set G such that $G \subset \bigcap_{\alpha<\kappa} U_\alpha$.*

Proof. Let $\mathcal{B}$ be a countable base for $\mathbb{R}$. Let P be all pairs of the form $(\vec{C}, F)$, where $\vec{C}$ is a finite sequence $\langle C_0, C_1, \dots, C_n\rangle$ of members of $\mathcal{B}$, and F is a finite subset of κ. For $(\vec{C}', F'), (\vec{C}, F) \in P$, define $(\vec{C}', F') \le (\vec{C}, F)$ iff $\vec{C}'$ extends $\vec{C}$, $F' \supseteq F$, and for each $i \in dom(\vec{C}') \setminus dom(\vec{C})$, we have

$$C_i \subset \bigcap_{\alpha \in F} U_\alpha$$

.

Claim 1. $(P, \leq)$ is a partially ordered set.

We need to prove transitivity. Suppose $(\vec{C}'', F'') \leq (\vec{C}', F') \leq (\vec{C}, F)$. Then $\vec{C}''$ extends $\vec{C}'$ extends $\vec{C}$ and $F'' \supseteq F' \supseteq F$, so $\vec{C}''$ extends $\vec{C}$ and $F'' \supset F$. If $i \in dom(\vec{C})'' \setminus dom(\vec{C})$, then either $i \in dom(\vec{C}'') \setminus dom(\vec{C}')$ in which case $C_i \subset \bigcap_{\alpha \in F'} U_\alpha \subset \bigcap_{\alpha \in F} U_\alpha$, or $i \in dom(\vec{C}') \setminus dom(\vec{C})$ in which case $C_i \subset \bigcap_{\alpha \in F} U_\alpha$. Thus $(\vec{C}'', F'') \leq (\vec{C}, F)$.

Claim 2. $(P, \leq)$ has the ccc.

Suppose $(\vec{C}_\alpha, F_\alpha)$ is in P for each $\alpha < \omega_1$. We need to show that there is $\alpha \neq \beta$ such that $(\vec{C}_\alpha, F_\alpha)$ and $(\vec{C}_\beta, F_\beta)$ are compatible. Since $\mathcal{B}$ is countable, so is the collection of finite sequences from $\mathcal{B}$, so there are $\alpha \neq \beta$ such that $\vec{C}_\alpha = \vec{C}_\beta = \vec{C}$. Then it is easy to see that $(\vec{C}, F_\alpha \cup F_\beta)$ is less than or equal to both $(\vec{C}_\alpha, F_\alpha)$ and $(\vec{C}_\beta, F_\beta)$, and hence they are compatible.

Now we define some dense sets. For each $B \in \mathcal{B}$ and $k \in \omega$, let

$$D_{B,k} = \{(\vec{C}, F) \in P : \exists i > k (C_i \subset B)\}.$$

Remark: The $D_{B,k}$'s serve two purposes. One, they will make sure that the generic filter G contains elements whose first coördinate is a sequence of arbitrarily long length, and thus G will determine an infinite sequence $< C_0^G, C_1^G, \cdots >$ of members of $\mathcal{B}$. Two, they make sure that $\bigcup_{i>n} C_i^G$ is dense in $\mathbb{R}$ for each n. (We'll argue this later.)

Also, for each $\alpha \in \kappa$, let

$$E_\alpha = \{(\vec{C}, F) \in P : \alpha \in F\}.$$

The E_α's will make sure that, for each α, there is some n so that $\bigcup_{i>n} C_i \subset U_\alpha$.

Let us show that the $D_{B,k}$'s and the E_α's are indeed dense in P. Let $(\vec{C}, F) \in P$. Then $(\vec{C}, F \cup \{\alpha\}) \leq (\vec{C}, F)$, so E_α is dense. Now fix $B \in \mathcal{B}$ and $k \in \omega$. Suppose $\vec{C} =< C_0, C_1, ..., C_n >$. Let $m > max\{n, k\}$ and choose C_i for $i = n+1, ..., m$ such that $C_i \subset \bigcap_{\alpha \in F} U_\alpha$, and also $C_m \subset B \cap \bigcap_{\alpha \in F} U_\alpha$. This is possible since $\bigcap_{\alpha \in F} U_\alpha$ is dense open. Let $\vec{C}' =< C_0, C_1, \ldots, C_m >$. Then $(\vec{C}', F) \in D_{B,k}$ and $(\vec{C}', F) \leq (\vec{C}, F)$. So $D_{B,k}$ is dense.

Let G be a filter in P meeting all E_α's, $\alpha < \kappa$, and all $D_{B,k}$'s, $B \in \mathcal{B}$ and $k \in \omega$.

Claim 3. If $(\vec{C}, F)$ and $(\vec{C}', F')$ are in G, then the sequence $\vec{C}$ extends $\vec{C}'$ or vice-versa; i.e., if $i \in dom(\vec{C}) \cap dom(\vec{C}')$, then $C_i = C_i'$. Well, if $(\vec{C}, F)$ and $(\vec{C}', F')$ are in G, then there is some $(\vec{C}'', F'')$ in G such that $(\vec{C}'', F'')$ is less than or equal to both $(\vec{C}, F)$ and $(\vec{C}', F')$, and hence the sequence $\vec{C}''$ extends both $\vec{C}$ and $\vec{C}'$. The claim follows.

Now, since G meets all $D_{B,k}$'s, there are arbitrarily long sequences appearing as first coördinates of members of G, so we can define an infinite sequence $\vec{C}_G =< C_0^G, C_1^G, \cdots >$ of members of $\mathcal{B}$ by defining C_i^G to be the i^{th} term of $\vec{C}$ for some

$(\vec{C}, F) \in G$ such that $i \in dom(\vec{C})$. It follows from Claim 3 that it doesn't matter which member of G we use to define C_i^G. Note that

$$\vec{C}_G = \bigcup\{\vec{C} : \exists F((\vec{C}, F) \in G)\},$$

where $\vec{C}$ is viewed as a set of ordered pairs.

Claim 4. For each $n \in \omega$, $\bigcup_{i>n} C_i^G$ is dense in $\mathbb{R}$. If $(\vec{C}, F) \in G \cap D_{B,n}$, then we have $C_i \subset B$ for some $i > n$. It follows that every basic open set $B \in \mathcal{B}$ contains C_i^G for some $i > n$, from which it easily follows that $\bigcup_{i>n} C_i^G$ is dense in $\mathbb{R}$.

Claim 5. For each $\alpha < \kappa$, there is some $n \in \omega$ such that $\bigcup_{i>n} C_i^G \subset U_\alpha$. Fix $\alpha < \kappa$, and let $(\vec{C}, F) \in G \cap E_\alpha$. Then $\alpha \in F$. Let $\vec{C} =< C_0, C_1, \ldots, C_n >$. Suppose $i > n$, and let $(\vec{C}', F') \in G$ such that $i \in dom(\vec{C}')$. There is some $(\vec{C}'', F'') \in G$ with $(\vec{C}'', F'')$ less than or equal to both $(\vec{C}', F')$ and $(\vec{C}, F)$. Then $C_i^G = C_i''$ and $C_i'' \subset U_\alpha$ because $(\vec{C}'', F'') \leq (\vec{C}, F)$ and $\alpha \in F$.

Finally, from Claims 4 and 5, it follows that $\bigcap_{n\in\omega}(\bigcup_{i>n} C_i^G)$ is a dense G_δ subset of $\mathbb{R}$ which is contained in $\bigcap_{\alpha<\kappa} U_\alpha$. □

A subset X of $\mathbb{R}$ is said to be *first category* in $\mathbb{R}$ if X is contained in the union of countably many nowhere-dense subsets of $\mathbb{R}$. Since the closure of a nowhere-dense set is nowhere-dense, it is equivalent to say that X is first category iff the complement of X contains a dense G_δ-set. (Some texts call sets of first category *meager*, and their complements *comeager.*)

Theorem 77. *Assume MA. Then the union of $< 2^\omega$-many first category subsets of $\mathbb{R}$ is first category. In particular, any subset of $\mathbb{R}$ of cardinality $< 2^\omega$ is first category.*

Proof. Assume MA, and let $\mathcal{F}$ be a collection of $< 2^\omega$-many first category subsets of $\mathbb{R}$. Let $\kappa = |\mathcal{F}|$. For each $F \in \mathcal{F}$, let $\mathcal{N}(\mathcal{F})$ be a countable collection of nowhere-dense sets covering F. Since the closure of a nowhere-dense set is nowhere-dense, we may assume each member of $\mathcal{N}(\mathcal{F})$ is closed. Let

$$\mathcal{U} = \{\mathbb{R} \setminus N : N \in \mathcal{N}(\mathcal{F}) \text{ for some } F \in \mathcal{F}\}.$$

Then $\mathcal{U}$ is a collection of κ-many dense open sets. By Lemma 76, there is a dense G_δ-set $G \subset \cap\mathcal{U}$. Then $\mathbb{R} \setminus G$ is the union of countably many nowhere dense sets and contains $\cup\mathcal{F}$. Hence $\cup\mathcal{F}$ is first category. □

Remark. It is also true that assuming MA, the union of $< 2^\omega$-many Lebesgue measure zero subsets of $\mathbb{R}$ has measure zero, and, in particular, any subset of $\mathbb{R}$ of cardinality $< 2^\omega$ has measure zero.

11. Q-sets, and normal vs. collectionwise normal

Example. Let A be a subset of the real line $\mathbb{R}$. Let $X(A)$ be the space whose set is

$$(A \times \{0\}) \cup \{(q, r) : q, r \in \mathbb{Q}, r > 0\}.$$

That is, $X(A)$ consists of the points on the x-axis corresponding to $a \in A$, together with all points in the upper half-plane with rational coördinates.

Let the points in the upper half-plane be isolated, and for each $a \in A$ and $n > 0$, let a basic neighborhood of $(a, 0)$ be

$$D(a, n) = \{(a, 0)\} \cup \{(q, r) : q, r \in \mathbb{Q} \text{ and } \sqrt{(q-a)^2 + (r - 1/n)^2} < 1/n\}.$$

That is, $D(a, n)$ consists of the point $(a, 0)$ together with all points with rational coördinates in the interior of a disk of radius $1/n$ tangent to the x-axis at $(a, 0)$.

Note that $A \times \{0\}$ is closed in $X(A)$, and discrete as a subspace. It follows that the collection of singletons $\{\{(a, 0)\} : a \in A\}$ is a discrete collection of closed sets, and hence the union of any subcollection, i.e., $B \times \{0\}$ for any $B \subset A$, is closed in $X(A)$.

Theorem 78. *Let $A \subset \mathbb{R}$, and let $X(A)$ be as defined above. Then:*

(a) *If $A = \mathbb{R}$, then $X(A)$ is not normal;*
(b) *If A is uncountable, then $X(A)$ is not collectionwise normal;*
(c) *If every subset of A is a G_δ-set in A, then $X(A)$ is normal.*

Hint for (c): Recall when working with Bing's G, we noted that to prove normality for a space consisting of a closed discrete set plus isolated points, we need only show that any subset of the closed discrete set, and its complement in the closed discrete set, can be put into disjoint open sets.

Proof. (a) Let $H = \mathbb{Q} \times \{0\}$ and $K = \mathbb{P} \times \{0\}$. Then H and K are closed and disjoint. Suppose U and V are open and disjoint with $H \subseteq U$ and $K \subseteq V$. For each $p \in \mathbb{P}$ there exists $n_p \in \mathbb{N}$ such that $D(p, n_p) \subseteq V$. For each $n \in \mathbb{N}$ let $A_n = \{p \in \mathbb{P} : n_p = p\}$. Then $\bigcup_{n \in \mathbb{N}} A_n = \mathbb{P}$. Since $\mathbb{P}$ is a Baire space there exists $n \in \mathbb{N}$ such that A_n is dense in some interval (a, b) of $\mathbb{R}$. Let $q \in (a, b) \cap \mathbb{Q}$ and (p_i) an increasing sequence of irrationals in $(a, b) \cap A_n$ converging to q. Then any basic neighborhood of $(q, 0)$ must meet a basic open neighborhood $D(p_i, n)$ for some i. So $U \cap V \neq \varnothing$.

(b) $\mathcal{H} = \{\{(a, 0)\} : a \in A\}$ is a discrete collection of closed sets. Suppose $\mathcal{U} = \{U_H : H \in \mathcal{H}\}$ is a disjoint collection of open sets separating $\mathcal{H}$. Then we may identify each U_H with a unique $(q_H, r_H) \in U_H$ where $q_H, r_H \in \mathbb{Q}$. Contradiction.

(c) Let $A \subset \mathbb{R}$ such that every subset of A is G_δ in A. Let $H \subset A$ and put $K = A \setminus H$. Let $\{H_n : n \in \omega\}$ witness H is F_σ. For each $n \in \omega$, set $U_n = \cup\{D(x, 1) : x \in H_n\}$.

Certainly $H \times \{0\} \subset \cup_{n \in \omega} U_n$. Fix $n \in \omega$ and let $y \in K$. Let $(a, b) \subset \mathbb{R}$ be an interval containing y and missing H_n and pick $k \in \mathbb{N}$ such that $D(y, k) \cap (D(a, 1) \cup D(b, 1)) = \emptyset$. If $x \in H_n$, then $x \leq a$ or $x \geq b$, so $D(y, k) \cap D(x, 1) = \emptyset$ as well. It follows that $(y, 0) \notin \overline{U}_n$, so $\overline{U}_n \cap (K \times \{0\}) = \emptyset$ for each $n \in \omega$.

Similarly, there is a collection $\{V_n : n \in \omega\}$ of open sets such that $K \times \{0\} \subset \cup_{n \in \omega} V_n$ and $\overline{V}_n \cap (H \times \{0\}) = \emptyset$ for all $n \in \omega$. By Lemma 7500.45, $H \times \{0\}$ and $K \times \{0\}$ may be separated, so $X(A)$ is normal. □

Remark. Lemma 7500.45 says: Suppose H and K are disjoint subsets of a space X. If H can be covered by countably many open sets whose closures miss K, and K can be covered by countably many open sets whose closures miss H, then H and K can be separated by disjoint open sets.

An uncountable subset A of $\mathbb{R}$ whose every subset is G_δ in A is called a *Q-set.*

Theorem 79. *If $2^\omega < 2^{\omega_1}$, then there are no Q-sets.*

Hint: How many G_δ-sets can there be in a space with a countable base?

Proof. Suppose A is an uncountable subspace of reals and $2^\omega < 2^{\omega_1}$. Then A is has a countable basis $\{B_n : n \in \omega\}$. Define $f : \tau_A \to 2^\omega$ by

$$f(U)(n) = \begin{cases} 1 & \text{if } B_n \subseteq U \\ 0 & \text{else} \end{cases}.$$

Suppose $U \neq V \in \tau_A$. Then $\{n \in \omega : B_n \subseteq U\} \neq \{n \in \omega : B_n \subseteq V\}$. Wlog assume $n \in \{n \in \omega : B_n \subseteq U\} \setminus \{n \in \omega : B_n \subseteq V\}$. Then $f(U)(n) = 1 \neq 0 = f(V)(n)$, so $f(U) \neq f(V)$. So f is an injection.

Let $\mathcal{G}_\delta$ be the family of G_δ-subsets of A. For each $G \in \mathcal{G}_\delta$ choose a sequence of open sets $(U_n^G) \in \tau_A^\omega$ such that $G = \bigcap_{n\in\omega} U_n^G$. Clearly the mapping $G \mapsto (f(U_n^G))$ is an injection from $\mathcal{G}_\delta$ into $(2^\omega)^\omega$.

We have $|\mathcal{G}_\delta| \leq (2^\omega)^\omega = 2^\omega < 2^{\omega_1} \leq |\mathcal{P}(A)|$, so not every subset of A is G_δ. □

Theorem 80. *Let κ be an uncountable cardinal, and assume $MA(\kappa)$. Then every subset of $\mathbb{R}$ of cardinality κ is a Q-set.*

Proof. Let $A \subset \mathbb{R}$ have cardinality κ, where $\kappa < \mathfrak{c}$, and let $B \subset A$. Let $\mathcal{C}$ be a countable base for A. The plan is to use $MA(\kappa)$ to show that there is a sequence $\langle C_0, C_1, \dots\rangle$ of members of $\mathcal{C}$ such that every $x \in B$ is in infinitely many C_i's, while every $y \in A \setminus B$ is in only finitely many C_i's. If we do this, then note that $B = \bigcap_{n\in\omega}(\bigcup_{i>n} C_i)$, and hence B is a G_δ-set in the space A.

To this end, let P be all pairs of the form $(\vec{C}, F)$, where $\vec{C}$ is a finite sequence $\langle C_0, C_1, \dots, C_n\rangle$ of members of $\mathcal{C}$, and F is a finite subset of $A \setminus B$. For $(\vec{C}', F'), (\vec{C}, F) \in P$, define $(\vec{C}', F') \leq (\vec{C}, F)$ iff $\vec{C}'$ extends $\vec{C}$, $F' \supseteq F$, and for each $i \in dom(\vec{C}') \setminus dom(\vec{C})$, we have $C_i \cap F = \emptyset$.

That $(P, \leq)$ is a partially ordered set, and has the *ccc*, follows in the same way as in the proof of Theorem 76.

Now we define some dense sets. For each $b \in B$ and $k \in \omega$, let

$$D_{b,k} = \{(\vec{C}, F) \in P : \exists i > k(b \in C_i)\}.$$

Also, for each $a \in A \setminus B$, let

$$E_a = \{(\vec{C}, F) \in P : a \in F\}.$$

That these sets are dense in P follows easily as in the proof of Theorem 76.

Let G be a filter in P meeting all E_a's, $a \in A \setminus B$, and all $D_{b,k}$'s, $b \in B$ and $k \in \omega$. Since G meets all $D_{b,k}$'s, there are arbitrarily long sequences appearing as first coordinates of members of G, so we can define an infinite sequence $\vec{C}_G =< C_0^G, C_1^G, \cdots >$ of members of $\mathcal{C}$ by defining C_i^G to be the i^{th} term of $\vec{C}$ for some $(\vec{C}, F) \in G$ such that $i \in dom(\vec{C})$. Note that

$$\vec{C}_G = \bigcup\{\vec{C} : \exists F((\vec{C}, F) \in G)\},$$

where $\vec{C}$ is viewed as a set of ordered pairs.

Fix $b \in B$. Since G meets $D_{b,k}$ for each $k \in \omega$, it follows that $b \in C_i^G$ for infinitely many i. Now fix $a \in A \setminus B$. There is some $(\vec{C}, F) \in G \cap E_a$. Then $a \in F$. Let $\vec{C} =< C_0, C_1, \dots, C_n >$. Suppose $i > n$, and let $(\vec{C}', F') \in G$ such that $i \in dom(\vec{C}')$. There is some $(\vec{C}'', F'') \in G$ with $(\vec{C}'', F'')$ less than or equal to both $(\vec{C}', F')$ and $(\vec{C}, F)$. Then $C_i^G = C_i''$ and $a \notin C_i''$ because $(\vec{C}'', F'') \leq (\vec{C}, F)$ and

$a \in F$. It follows that $a \notin \bigcup_{i>n} C_i^G$, and hence a is in only finitely many terms of $\vec{C}_G$. Now the result follows as indicated in the first paragraph. □

Corollary 81. *Assume $MA(\omega_1)$. Then there is a Q-set, and hence there is a normal first-countable separable non-collectionwise normal space.*

Corollary 82. *Assume MA. Then $2^\kappa = 2^\omega$ for every infinite cardinal $\kappa < 2^\omega$.*

Proof. Assume MA, and let $\kappa < 2^\omega$ be an infinite cardinal. There is a subset A of the real line of cardinality κ. By MA and the previous theorem, every subset of A is G_δ in A. There is a countable base for A and every open subset of A is a union of members of this base, so there are no more than 2^ω-many open sets in A. Each G_δ-set is by definition a countable intersection of open sets, so there are no more than $(2^\omega)^\omega = 2^{\omega\times\omega} = 2^\omega$ many G_δ-sets in A. So the cardinality of the power set of A, which is 2^κ, is no more than 2^ω. And of course it is at least that, so $2^\kappa = 2^\omega$. □

Remark. Bob Heath showed that if there is a normal first-countable separable non-collectionwise normal space, then there is a Q-set. So, by 78(c), the existence of such a space is equivalent to the existence of a Q-set.

Lemma 83. *Let X be a separable space, and let D be a closed discrete subset of X. If $2^{|D|} > 2^\omega$, then X is not normal.*

Proof. Let X be a normal space, let $D \subset X$ be closed discrete, and let $G \subset X$ be dense. For $H \subset D$, let U_H and V_H be disjoint open sets containing H and $D \setminus H$, respectively. Define $f : \mathcal{P}(D) \to \mathcal{P}(G)$ by $f(H) = U_H \cap G$.

Let $H, K \subset D$ with $H \neq K$. Without loss of generality, we may assume $H \setminus K$ is nonempty. Then $G \cap (U_H \cap V_K)$ is a nonempty subset of $f(H)$ missing $f(K)$, so $f(H) \neq f(K)$. Since f is one-to-one, $2^{|D|} \leq 2^{|G|}$.

This proves a more general version of the lemma. Apply this with G a countable dense set to obtain the lemma as stated. □

Remark. Since $2^{2^\omega} > 2^\omega$, Lemma 83 gives another way to show Theorem 78(a), the square of the Sorgenfrey line is not normal, and the like.

Corollary 84. *If $2^\omega < 2^{\omega_1}$, then every normal separable space is collectionwise normal.*

Hint. In a normal space, any *countable* discrete collection of closed sets can be separated by disjoint open sets.

Proof. Suppose $2^\omega < 2^{\omega_1}$ and let X be a normal, separable space. Let $\mathcal{H}$ be a discrete collection of closed sets in X. For each $H \in \mathcal{H}$, choose $x_H \in H$ and put $D = \{x_H : H \in \mathcal{H}\}$. Then D is closed discrete, so $2^{|D|} \leq 2^\omega$, implying $|D| \leq \omega$ and $|\mathcal{H}| \leq \omega$.

Enumerate $\mathcal{H} = \{H_i : i \in \omega\}$. For each $i \in \omega$, observe that H_i and $(\cup\mathcal{H}) \setminus H_i$ are disjoint closed sets. Since X is normal, for each $i \in \omega$ there exist disjoint open sets U_i and V_i containing H_i and $(\cup\mathcal{H}) \setminus H_i$, respectively. For each $i \in \omega$, put $W_i = U_i \cap (\cap_{j<i} V_j)$. Certainly $H_i \subset W_i$ for each $i \in \omega$. Moreover, if $i < j \in \omega$ then $W_i \cap W_j = \emptyset$ since $W_i \subset U_i$ and $W_j \subset V_i$. □

12. Almost disjoint families

A collection $\mathcal{A}$ of infinite subsets of ω is said to be *almost-disjoint* if the intersection of any two distinct members of $\mathcal{A}$ is finite. By a standard Zorn's Lemma argument, every almost-disjoint family $\mathcal{A}$ is contained in a maximal almost-disjoint family $\mathcal{A}'$.

Theorem 85. *There is an almost-disjoint family of subsets of ω of cardinality 2^ω.*

Proof. Clearly it suffices to show that there is an almost-disjoint family of subsets of some countable set, such as $\mathbb{Q}$, of cardinality 2^ω. For each $x \in \mathbb{R}$, let S_x be the terms of a sequence of rationals converging to x. Then $x \neq y \Rightarrow S_x \cap S_y$ is finite , so $\{S_x : x \in \mathbb{R}\}$ is an almost-disjoint family of subsets of $\mathbb{Q}$ of cardinality 2^ω. $\square$

It is easy to check that any finite partition of ω into infinite sets is a maximal almost-disjoint family, but ...

Lemma 86. *No countably infinite almost-disjoint family of subsets of ω is maximal.*

Proof. Suppose $\{A_n : n \in \omega\}$ is an almost-disjoint family of subsets of ω. We will prove that it cannot be maximal.

First let's note that each finite union $\bigcup_{i\leq n} A_i$ has infinite complement, for otherwise A_{n+1} would have to intersect some A_i, $i \leq n$, in an infinite set. Now for each n choose $x_n \in \omega \setminus (\{x_i : i < n\} \cup \bigcup_{i\leq n} A_i)$. Let $X = \{x_n : n \in \omega\}$. Then X is infinite but $X \cap A_n$ is finite for each n. Hence $\{A_n : n \in \omega\}$ is not maximal. $\square$

Theorem 87. *Assume MA. Then every infinite maximal almost-disjoint family of subsets of ω has cardinality 2^ω.*

Proof. Let $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ be an almost-disjoint family (of infinite subsets of ω) of cardinality κ, where $\omega \leq \kappa < 2^\omega$. Let P be all pairs of the form $(\vec{x}, F)$, where $\vec{x} = \langle x_0, x_1, \ldots, x_n\rangle$ is a finite sequence of members of ω and F is a finite subset of κ. For $(\vec{x}', F')$ and $(\vec{x}, F)$ in P, define $(\vec{x}', F') \leq (\vec{x}, F)$ if and only if $\vec{x}'$ extends $\vec{x}$, $F' \supset F$, and for each $i \in \mathrm{dom}(\vec{x}') \setminus \mathrm{dom}(\vec{x})$ we have $x_i \notin \cup_{\alpha\in F} A_\alpha$.

To see that $(P, \leq)$ is a poset we need to show transitivity. Suppose $(\vec{x}'', F'') \leq (\vec{x}', F') \leq (\vec{x}, F)$. Then $\vec{x}''$ extends $\vec{x}'$ extends $\vec{x}$ and $F'' \supset F' \supset F$, so $\vec{x}''$ extends $\vec{x}$ and $F'' \supset F$. If $i \in \mathrm{dom}(\vec{x}'') \setminus \mathrm{dom}(\vec{x})$, then either $i \in \mathrm{dom}(\vec{x}'') \setminus \mathrm{dom}(\vec{x}')$ in which case $x_i \notin \cup_{\alpha\in F} A_\alpha$ since $x_i \notin \cup_{\alpha\in F'} A_\alpha$ and $F \subset F'$, or $i \in \mathrm{dom}(\vec{x}') \setminus \mathrm{dom}(\vec{x})$ in which case $x_i \notin \cup_{\alpha\in F} A_\alpha$. Thus $(\vec{x}'', F'') \leq (\vec{x}, F)$.

To see that $(P, \leq)$ has the *ccc*, suppose $(\vec{x}_\alpha, F_\alpha) \in P$ for each $\alpha < \omega_1$. There are only countably many finite sequences in ω, so there exist $\alpha \neq \beta < \omega_1$ with $\vec{x}_\alpha = \vec{x}_\beta = \vec{x}$. Certainly $(\vec{x}, F_\alpha \cup F_\beta) \leq (\vec{x}_\alpha, F_\alpha)$ and $(\vec{x}, F_\alpha \cup F_\beta) \leq (\vec{x}_\beta, F_\beta)$, so $(\vec{x}_\alpha, F_\alpha)$ and $(\vec{x}_\beta, F_\beta)$ are compatible.

For each $k \in \omega$, put $D_k = \{(\vec{x}, F) \in P : |\{x_i : i \in \mathrm{dom}(\vec{x})\}| > k\}$. For each $\alpha < \kappa$, put $E_\alpha = \{(\vec{x}, F) \in P : \alpha \in F\}$. Let $(\vec{x}, F) \in P$. Then $(\vec{x}, F \cup \{\alpha\}) \leq (\vec{x}, F)$, so E_α is dense in P for each $\alpha < \kappa$. Fix $k \in \omega$ and suppose $\vec{x} = \langle x_0, x_1, \ldots, x_n\rangle$. Take $\{x_{n+1}, \ldots, x_{n+k+1}\} \subset \omega \setminus (\cup_{\alpha\in F} A_\alpha \cup \{x_i : i \in \mathrm{dom}(\vec{x})\})$ so that $x_i \neq x_j$ whenever $n + 1 \leq i < j \leq n + k + 1$. This is possible since any finite subcollection of $\mathcal{A}$ must leave infinitely many members of ω uncovered. Put $\vec{x}' = \langle x_0, \ldots, x_n, x_{n+1}, \ldots, x_{n+k+1}\rangle$. Then $(\vec{x}', F) \in D_k$ and $(\vec{x}', F) \leq (\vec{x}, F)$, so D_k is dense in P.

Let G be a filter in P meeting E_α and D_k for each $\alpha < \kappa$ and each $k \in \omega$. If $(\vec{x}, F), (\vec{x}', F') \in G$ then there exists $(\vec{x}'', F'') \in G$ less than or equal to both of them, whence $\vec{x}''$ extends both $\vec{x}'$ and $\vec{x}$. It follows that $\vec{x}'$ extends $\vec{x}$, or vise-versa. Let $\vec{x}_G = \langle x_0^G, x_1^G, \ldots \rangle$, where x_i^G is the i^{th} term of $\vec{x}$ for some $(\vec{x}, F) \in G$ with $i \in \mathrm{dom}(\vec{x})$. Put $A = \{x_i^G : i \in \omega\}$. Since G meets every D_k, we know $|A| = \omega$.

Finally, fix $\alpha < \kappa$ and let $(\vec{x}, F) \in G \cap E_\alpha$, so $\alpha \in F$. Let $\vec{x} = \langle x_0, x_1, \ldots, x_n \rangle$. Suppose $i > n$ and let $(\vec{x}', F') \in G$ such that $i \in \mathrm{dom}(\vec{x}')$. Take $(\vec{x}'', F'') \in G$ less than or equal to both $(\vec{x}', F')$ and $(\vec{x}, F)$. Then $x_i^G = x_i'' \notin A_\alpha$, so $|A \cap A_\alpha| < \omega$. It follows that $\mathcal{A} \cup \{A\}$ is an almost-disjoint family, hence $\mathcal{A}$ is not maximal. □

Example. Let $\mathcal{A}$ be an almost-disjoint family of subsets of ω. Define a space $\psi(\mathcal{A})$ as follows. The underlying set for $\psi(\mathcal{A})$ is $\omega \cup \{x_A : A \in \mathcal{A}\}$, where $\{x_A : A \in \mathcal{A}\}$ is a set of distinct points not in ω. Define the topology by declaring the points of ω to be isolated, and the n^{th} member of a local base at x_A to be

$$b(x_A, n) = \{x_A\} \cup (A \setminus n).$$

.

Theorem 88. *$\psi(\mathcal{A})$ is a locally compact Hausdorff space, and $\{x_A : A \in \mathcal{A}\}$ is a closed discrete subset of $\psi(\mathcal{A})$. Thus, if $\mathcal{A}$ is infinite, then $\psi(\mathcal{A})$ is not countably compact.*

Proof. Consider the basic open nbhd $b(x_A, 0) = \{x_A\} \cup A$ of x_A. Since every open set U containing x_A contains $b(x_A, n)$ for some n, U therefore contains all but at most finitely many points of $b(x_A, 0)$. Thus $b(x_A, 0)$ is compact. Since all points other than the x_A's are isolated, we have that $\psi(\mathcal{A})$ is locally compact.

To see Hausdorff, suppose $A \neq B \in \mathcal{A}$. Then $A \cap B$ is finite, so there is some $n \in \omega$ such that $b(x_A, n) \cap b(x_B, n) = \emptyset$. Since all "non-$x_A$'s" are isolated, it easily follows that $\psi(\mathcal{A})$ is Hausdorff. (Remark. This is the only place in the proof where "almost-disjoint" is needed.)

Let $D = \{x_A : A \in \mathcal{A}\}$. Every point outside of D is isolated, so D is closed. Also, $b(x_A, 0) \cap D = \{x_A\}$, so D is discrete. Thus D is closed discrete. □

A space X is said to be *pseudocompact* if every continuous $f : X \to \mathbb{R}$ has bounded range.

Theorem 89. *Every countably compact space is pseudocompact.*

Proof. Let X be countably compact, and suppose $f : X \to \mathbb{R}$ is continuous. Then $f(X)$ is a countably compact subset of $\mathbb{R}$. But countable compactness and compactness are equivalent in metric spaces, so $f(X)$ is compact, hence bounded. Thus X is pseudocompact. □

Theorem 90. *If $\mathcal{A}$ is an infinite maximal almost-disjoint family of subsets of ω, then $\psi(\mathcal{A})$ is pseudocompact (but not countably compact).*

Proof. Let $\mathcal{A}$ be an infinite MAD family of infinite subsets of ω. Let $f : \psi(\mathcal{A}) \to \mathbb{R}$ be continuous and suppose on the contrary that f has unbounded range. Without loss of generality, we may assume there exists an increasing sequence $(x_n)_{n\in\omega}$ in ω such that $(f(x_n))_{n\in\omega}$ is increasing and unbounded. Let $X = \{x_i : i \in \omega\}$. Since $\mathcal{A}$ is MAD, there exists $A \in \mathcal{A}$ with $|X \cap A| = \omega$. Let $(a_n)_{n\in\omega}$ be the order-preserving indexing of A. Then $(a_n)_{n\in\omega} \to x_A$ but $(f(a_n))_{n\in\omega} \not\to f(x_A)$, a contradiction. □

13. CH contstruction of a compact S-space

An *S-space* is a regular hereditarily separable space which is not hereditarily Lindelöf; an *L-space* is a regular hereditarily Lindelöf space which is not hereditarily separable. By Theorem 75, assuming $MA(\omega_1)$ there are no compact L-spaces. $MA(\omega_1)$ also implies there are no compact S-spaces. On the other hand, a Suslin line (compactified by adding a first and last point) is a compact L-space. In this section, we show how the Continuum Hypothesis (CH) can be used to construct a compact S-space.[3]

Throughout this section, $2^\omega = \omega_1$ is assumed. Let $\mathbb{R} = \{x_\alpha : \alpha < \omega_1\}$, where $\mathbb{Q} = \{x_n : n < \omega\}$.

Lemma 91. *Let*

$$\mathcal{A} = \{C \subset \mathbb{R} : |C| = \omega \text{ and } |\overline{C}| > \omega\}.$$

It is possible to enumerate $\mathcal{A}$ as $\{A_\alpha : \omega \leq \alpha < \omega_1\}$ such that:

(∗) $\forall \alpha \geq \omega (A_\alpha \subset \{x_\beta : \beta < \alpha\})$.

Proof. By CH, $\mathcal{A}$ has cardinality ω_1, so we can let $\{A'_\alpha : \alpha < \omega_1\}$ be a listing of $\mathcal{A}$ in type ω_1. Let $A_\omega = A'_\alpha$ where α is least such that $A'_\alpha \subset \{x_n : n < \omega\}$; since $\{x_n : n < \omega\}$ is the rationals, there will always be such an A'_α; indeed, there are ω_1-many α such that $A'_\alpha \subset \{x_n : n < \omega\}$.

Now suppose A_β has been defined for each $\beta < \alpha$, where $\alpha < \omega_1$. Let $A_\alpha = A'_\delta$, where δ is least such that $A'_\delta \subset \{x_\beta : \beta < \alpha\}$ and $A'_\delta \neq A_\beta$ for any $\beta < \alpha$. This defines A_α for all $\alpha < \omega_1$.

Clearly (∗) holds by construction; it remains to prove that $\{A_\alpha : \omega \leq \alpha < \omega_1\} = \mathcal{A}$, which will be the case if for every $\gamma < \omega_1$, there is some $\alpha < \omega_1$ such that $A_\alpha = A'_\gamma$. To this end, fix $\gamma < \omega_1$. There will be some $\rho < \omega_1$ such that $A'_\gamma \subset \{x_\beta : \beta < \rho\}$. Then at any step $\alpha \geq \rho$, $A_\alpha = A'_\gamma$ if γ is least such that A'_γ has not already been chosen. Since the induction goes on for ω_1 many steps and there are only countably many ordinals $\mu < \gamma$, A'_γ must get chosen at some stage. Hence $\{A_\alpha : \omega \leq \alpha < \omega_1\} = \mathcal{A}$. □

Let $X = \mathbb{R}$, and for each $\alpha \leq \omega_1$ let $X_\alpha = \{x_\beta : \beta < \alpha\}$. Let $\mathcal{A}$ be indexed as in Lemma 91. We inductively define a topology τ_α on X_α. Let τ_ω be the discrete topology on X_ω. Note that $A_\omega \subset X_\omega$. Choose a sequence $n_0 < n_1 < ...$ such that $x_{n_i} \to x_\omega$ in the Euclidean topology and such that $x_{n_i} \in A_\omega$ if $x_\omega \in \overline{A}_\omega$. Let

$$\mathcal{B}_\omega = \{\{x_{n_i}\}_{i \geq k} \cup \{x_\omega\} : k \in \omega\},$$

and let $\tau_{\omega+1}$ be the topology on $X_{\omega+1}$ generated by $\tau_\omega \cup \mathcal{B}_\omega$. Note that $\tau_{\omega+1}$ is locally compact Hausdorff, has a countable base of clopen sets, and every subset of $X_{\omega+1}$ which is open in the Euclidean topology of $X_{\omega+1}$ is open in $\tau_{\omega+1}$ (in short, $\tau_{\omega+1}$ is finer than the Euclidean topology). Also, $x_\omega \in \overline{A}_\omega$ in $\tau_{\omega+1}$ if this is true in $\mathbb{R}$.

Lemma 92. *Suppose $\alpha < \omega_1$ and for all $\beta < \alpha$ we have defined a toplogy τ_β on X_β satisfying:*

[3] The results in these notes related to S-and L-spaces were proven by the late 1970's. In the early 1980's, S. Todorcevic proved that under the Proper Forcing Axiom PFA (an axiom stronger than $MA(\omega_1)$), there are no S-spaces. Whether or not there are L-spaces in ZFC remained unsettled until 2006 when J. Moore surprised everyone by constructing an L-space without assuming any special axioms of set theory.

(i) (X_β, τ_β) *is a locally compact Hausdorff space with a countable base of clopen sets, and* τ_β *is finer than the Euclidean topology on* X_β*;*
(ii) $\gamma < \beta \Rightarrow \tau_\gamma \subseteq \tau_\beta$ *and* $\tau_\gamma = \tau_\beta \upharpoonright X_\gamma (= \{U \cap X_\gamma : U \in \tau_\beta\})$*;*
(iii) *If* $\gamma < \delta < \beta$ *and* x_δ *is in the Euclidean closure of* A_γ*, then* $x_\delta \in cl_{\tau_{\delta+1}}(A_\gamma)$*.*

Then there is a topology τ_α *on* X_α *satisfying conditions (i)–(iii), and hence this defines* τ_α *for all* $\omega \leq \alpha < \omega_1$*.*

Proof. For limit α, let τ_α be the topology generated by $\bigcup_{\beta<\alpha} \tau_\beta$. It is straightforward to check that conditions (i)-(iii) hold with $\beta = \alpha$.

If α is a successor, say $\alpha = \beta+1$, then $X_\alpha = X_\beta \cup \{x_\beta\}$, and the task is to define a nbhd base $\mathcal{B}_\beta$ at x_β so that the topology generated by $\tau_\beta \cup \mathcal{B}_\beta$ satisfies (i)–(iii).

If α is a successor, say $\alpha = \beta + 1$, then $X_\alpha = X_\beta \cup \{x_\beta\}$. Enumerate $\{\gamma < \alpha : x_\beta \in \overline{A}_g\}$ as $\{\gamma_i : i \in \omega\}$. Let $x_{0,0} \in B(x_\beta, 1) \cap A_{\gamma_0}$ and let $C_0 = \{x_{0,0}\}$. (Here $B(x, \epsilon)$ is the ball about x of radius ϵ with respect to the Euclidean distance d.) If C_i and $x_{j,i}$ has been defined for $i \leq n$ and $j \leq i$, for each $j \leq n+1$ let $x_{j,n+1} \in B(x_\beta, 1/2^{n+1}) \cap A_{\gamma_j}$. Then let $C_{n+1} = C_n \cup \{x_{j,n+1} : j \leq n+1\}$.

Let $C = \{x_{j,n} : n \in \omega \text{ and } j \leq n\}$. Every ball about x_β contains $x_{j,n}$ for all sufficiently large n, and hence contains all but finitely many members of C. So $C = \{c_i : i \in \omega\}$ is a sequence which converges to x_β is the Euclidean topology. Let K_i be a compact open set in X_β containing c_i and of diameter $\leq \frac{1}{i+1}$; this is possible since the topology of X_β is a locally compact 0-dimensional topology finer than the Euclidean topology. Then let sets $U_{\beta,m} = \{x_\beta\} \cup \bigcup_{i \geq m} K_i$ for $m \in \omega$ be a local base $\mathcal{B}_\beta$ at x_β in X_α.

Let τ_α be the topology on X_α generated by $\tau_\beta \cup \mathcal{B}_\beta$. □

Theorem 93. *Let* X_α *and* τ_α*,* $\alpha < \omega_1$*, be as in Lemma 92. Let* τ_{ω_1} *be the topology on* $X = \mathbb{R}$ *generated by* $\bigcup_{\omega \leq \alpha < \omega_1} \tau_\alpha$*. Then* $(\mathbb{R}, \tau_{\omega_1})$ *is locally compact Hausdorff, hereditarily separable, and non-Lindelöf, hence its one-point compactification is a compact S-space.*

Proof. That $(\mathbb{R}, \tau_{\omega_1})$ is locally compact Hausdorff follows from 92 (i) and (ii). Certainly $\{X_\alpha : \alpha < \omega_1\}$ is an open cover with no countable subcover, so $(\mathbb{R}, \tau_{\omega_1})$ is not Lindelöf.

Let $Y \subset \mathbb{R}$ be uncountable and let D be a countable dense (in the Euclidean topology) subset thereof. Since $|\overline{D}| > \omega$—with $\overline{D}$ the Euclidean closure—there exists $\gamma < \omega_1$ such that $D = A_\gamma$. Let $\delta \in (\gamma, \omega_1)$. By 92 (iii),

$$x_\delta \in \overline{A}_\gamma = Y \implies x_\delta \in \mathrm{cl}_{\tau_{\delta_1}}(A_\delta) \implies x_\delta \in \mathrm{cl}_{\tau_{\omega_1}}(A_\gamma).$$

It follows that $|Y \setminus \mathrm{cl}_{\tau_{\omega_1}}(A_\gamma)| \leq \omega$, so $D \cup (Y \setminus \mathrm{cl}_{\tau_{\omega_1}}(A_\gamma))$ is a countable dense (in τ_{ω_1}) subset of Y. Thus, $(\mathbb{R}, \tau_{\omega_1})$ is hereditarily separable. □