MH 7550

It will be convenient to think of an ordinal number as the set of its predecessors (as in done in modern set theory). So, 0 is the empty set, $1 = \{0\}$, $2 = \{0, 1\}$, $\dots$, $n = \{0, 1, 2, \dots, n-1\}$, $\omega = \{0, 1, 2, \dots\}$ and is the set of natural numbers, ω_1 is the set of countable ordinals, etc.. Note $\beta < \alpha$ iff $\beta \in \alpha$ holds for any pair of ordinals α and β.

A binary relation $\leq$ on a set X is a *partial order* if it is reflexive ($x \leq x$ for all x), antisymmetric ($x \leq y$ and $y \leq x$ implies $x = y$), and transitive ($x \leq y$ and $y \leq z$ implies $x \leq z$). It is a *linear order* if also for any $x, y \in X$, either $x < y, y < x$, or $x = y$. A linear order is a *well-order* of X if every nonempty subset of X has a least element.

A partially ordered set $(T, \leq)$ is a *tree* if for all $t \in T$, the set $P_t = \{s \in T : s < t\}$ is well-ordered by . The elements of T are called the *nodes* of T. A maximal linearly ordered subset of T is called a *branch* of T.

Recall that any well-ordered set is order-isomorphic to (the predecessors of) an ordinal. If $(T, \leq)$ is a tree, then for any ordinal α, the set $L_\alpha = \{t \in T : P_t \text{ is isomorphic to } \alpha\}$ is called the α^{th} *level* of T. The least α such that $L_a = \emptyset$ is called the *height* of T.

Example: the Cantor tree. Let T be the set of all finite sequences of 0's and 1's (including the empty sequence). We can equivalently describe T as the set of all functions σ from some natural number n into 2, where here we think of n as the set $\{0, 1, 2, \dots, n-1\}$ and 2 as the set $\{0, 1\}$. If $\sigma, \tau \in T$, define $\sigma < \tau$ iff σ is an initial segment of τ. (E.g., $110 < 1100$, $01 < 0111010$, etc.) Then $(T, \leq)$ is called the *Cantor tree*.

If we let B^A denote all functions from set A into set B, then 2^n denotes the set of all functions from $n = \{0, 1, \dots, n-1\}$ into $2 = \{0, 1\}$. Let $2^{<\omega} = \cup\{2^n : n < \omega\}$. Then the set $2^{<\omega}$, ordered by extension, is another way to describe the Cantor tree.

Note that the branches of the Cantor tree can be identified with 2^ω, the set of all functions $f : \omega \to 2$. Given $f \in 2^\omega$, $\{f \restriction n : n < \omega\}$ is a branch, and given a branch b, then $\cup b$ is a function from ω to 2. Note $b = \{\cup b \restriction n : n < \omega\}$.

The Cantor tree $(T, \leq)$ is completely described (up to isomorphism) as follows:

(i) The least level L_0 of T has exactly one node;
(ii) Each node σ has exactly two successors;
(iii) T has height ω.

Definition of the Cantor "middle thirds" set $\mathbb{C}$. For each finite sequence σ of 0's and 1's (including the empty sequence $\emptyset$), we define a closed subinterval I_σ of $[0, 1]$ as follows. Start by setting $I_\emptyset = [0, 1]$. Then if I_σ has been defined, let $I_{\sigma^\frown 0}$ and $I_{\sigma^\frown 1}$ be the left and right thirds, respectively, of I_σ. Thus $I_0 = [0, 1/3]$, $I_1 = [2/3, 1]$, $I_{00} = [0, 1/9]$, $I_{01} = [2/9, 1/3]$, etc. For each n, let $C_n = \cup\{I_\sigma : \sigma \text{ has length } n\}$. So, $C_0 = [0, 1]$, $C_1 = [0, 1/3] \cup [2/3, 1]$, etc. Finally, $\mathbb{C} = \bigcap_{n \in \mathbb{N}} C_n$.

Theorem 1. *The Cantor set $\mathbb{C}$ as defined above is (as a subspace of the real line $\mathbb{R}$) compact, metrizable, and has no isolated points. $\mathbb{C}$ is an uncountable closed subset of $\mathbb{R}$ with empty interior in $\mathbb{R}$. If $\mathcal{B} = \{I_\sigma \cap \mathbb{C} : \sigma \in 2^{<\omega}\}$, then $\mathcal{B}$ is a* *countable base of open and closed (clopen) sets in $\mathbb{C}$ such that $(\mathcal{B}, \supseteq)$ is isomorphic to the Cantor tree.*

Remark. A space which has a base of open and closed sets is sometimes called *zero-dimensional*, or more precisely, is said to have *small inductive dimension zero*, denoted by $ind(X) = 0$. (There are several concepts of dimension.)

Theorem 2. *A space X is homeomorphic to the Cantor set $\mathbb{C}$ iff X has a base $\mathcal{B}$ consisting of clopen sets such that $(\mathcal{B}, \supseteq)$ is isomorphic to the Cantor tree and the intersection of each branch of this tree is a single point.*

Theorem 3. *A space X is homeomorphic to $\mathbb{C}$ iff X is compact Hausdorff, has no isolated points, and has a countable base of clopen sets.*

Hint. Let B_n, $n < \omega$, be the clopen base. Construct a Cantor tree of clopen sets such that every node at level n either meets B_n or is disjoint from B_n.

Corollary 4. *The following are homeomorphic to $\mathbb{C}$: $\mathbb{C}^2$, $\mathbb{C}^\omega$, 2^ω (where 2 denotes the two-point discrete space $\{0,1\}$), and any product of the form $\Pi_{n\in\omega} F_n$, where F_n is finite discrete space with at least two points.*

Let $\leq$ be a linear order on a set X. For $a, b \in X$, let $(a,b) = \{x \in X : a < x < b\}$. Also let $(-\infty, a) = \{x \in X : x < a\}$ and $(b, \infty) = \{x \in X : b < x\}$. Let $\mathcal{B} = \{(a,b) : a, b \in X\} \cup \{(-\infty, a) : a \in X\} \cup \{(b,\infty) : b \in X\}$. Then $\mathcal{B}$ is a base for a topology τ on X, and τ is called the *order topology on X induced by* $\leq$. A topological space (X, τ) is called a *linearly ordered space* if there is a linear order on X which induces the topology τ.

Theorem 5. *Let $(T, \leq)$ be the Cantor tree. Let X be the set of all branches of T. Note that if b is a branch of T, then $\cup b \in 2^\omega$. Define a linear order $\prec$ on X as follows: if $b, b' \in X$, and n is the least integer such that $\cup b(n) \neq \cup b'(n)$, then $b \prec b'$ iff $\cup b(n) = 0$ and $\cup b'(n) = 1$. Then X with the topology induced by this order is homeomorphic to $\mathbb{C}$.*

Hint: Show that 2^ω with the topology induced by the lexicographic order is the same as the usual Tychonoff product topology.

Theorem 6. *The following are continuous images of $\mathbb{C}$:*

(i) *the unit interval* $[0,1]$*;*
(ii) *the Hilbert cube* $[0,1]^\omega$*;*
(iii) *any closed subset of* $\mathbb{C}$*;*
(iv) *any compact metric space.*

Hint for (i): Define $f : 2^\omega \to [0,1]$ by $f(\vec{x}) = \sum_{n<\omega} \frac{x_n}{2^{n+1}}$, where x_n is the n^{th}-coördinate of $\vec{x}$.

Hint for (iii): If K is a closed subset of $\mathbb{C}$, note that $K \times \mathbb{C}$ is homeomorphic to $\mathbb{C}$.

For each $n \in \omega$, let ω^n denote the set of all functions from n (i.e., from the set $\{0, 1, 2, \ldots n-1\}$) into ω, and let $\omega^{<\omega} = \bigcup_{n\in\omega} \omega^n$. (In other words, ω^n is the set of all n-length sequences of natural numbers, and $\omega^{<\omega}$ is the set of all finite sequences of natural numbers.) If $\sigma, \tau \in \omega^{<\omega}$, let $\sigma < \tau$ iff σ is an initial segment of τ. Then $(\omega^{<\omega}, <)$ is a tree of height ω with one node at the least level (the empty sequence) and such that every node has a countable infinite number of immediate successors.

Exercise. Let $\mathbb{P}$ denote the irrationals (as a subspace of $\mathbb{R}$). Show that $\mathbb{P}$ has a base $\mathcal{B}$ of clopen sets such that $(\mathcal{B}, \supseteq)$ is a tree isomorphic to $(\omega^{<\omega}, \leq)$, and such that the intersection of each branch is a singleton.

A space X is said to be *nowhere-locally-compact* if no point of X has a compact neighborhood.

Theorem 7. *The following are equivalent for a space X:*

(i) *X is homeomorphic to the space if irrationals (as a subspace of the real line with the usual Euclidean topology);*
(ii) *X has a base $\mathcal{B}$ consisting of clopen sets such that $(\mathcal{B}, \supseteq)$ is isomorphic to the tree $(\omega^{<\omega}, \leq)$, and the intersection of each branch of this tree is a single point;*
(iii) *X is a nowhere-locally-compact complete separable metric space and has a countable base of clopen sets.*

Corollary 8. *Let $\mathbb{P}$ denote the space of irrationals. Then $\mathbb{P}$ is homeomorphic to $\mathbb{P}^2$, $\mathbb{P}^\omega$, and ω^ω (where ω is given the discrete topology).*

Remark. A subset A of the real line $\mathbb{R}$ is said to be *analytic* if there is a continuous surjection $f : \mathbb{P} \to A$. It is known that every Borel set is analytic, every analytic set is Lebesgue measurable, and that there are analytic sets that are not Borel and measurable sets that are not analytic. Analytic sets play a major role in the field of "descriptive set theory".

Theorem 9. *Suppose $(X, <)$ and $(Y, \prec)$ are countable linearly ordered sets with no first or last point, and both are densely ordered (i.e., between any two points there is another point). Then there is an order-preserving bijection $f : X \to Y$.*

Theorem 10. *A linearly ordered space X is connected iff the following hold:*

(i) *X is densely ordered;*
(ii) *every bounded subset of X has a least upper bound.*

Theorem 11. *Suppose X is a connected linearly ordered space with no first or last point, and is separable. Then X is order-isormorphic to the real line $\mathbb{R}$ with the usual order.*

Corollary 12. *Suppose X is a separable compact connected linearly ordered space. Then X is homeomorphic to the unit interval $[0, 1]$.*

Lemma 13. *If X is completely regular and $|X| < |\mathbb{R}|$, then X has a base of clopen sets.*

Lemma 14. *Suppose κ is an infinite cardinal. If κ is the least cardinal of a base for a space X, then for every base $\mathcal{B}$, there is a base $\mathcal{C} \subset \mathcal{B}$ such that $|\mathcal{C}| = \kappa$.*

Hint: For every infinite cardinal κ, the set of all finite subsets of κ also has cardinality κ.

Remark. The least cardinal of a base for a space X is called the *weight* of X and is denoted by $w(X)$.

Theorem 15. *Suppose X is countable, regular, first-countable, and has no isolated points. Then X is homeomorphic to the rationals $\mathbb{Q}$.*

Hint: Construct a base $\mathcal{B}$ for X consisting of clopen sets such that $(\mathcal{B}, \supseteq)$ is isomorphic to the Cantor tree, and such that for each $x \in X$, the sequence of 0's and 1's which codes the branch $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$ has infinitely many 0's and infinitely many 1's. Use this to define a linear order on X which generates the topology and is densely ordered with no first or last point.

Corollary 16. *The following are homeomorphic to the space $\mathbb{Q}$ of rationals:*

(a) $\mathbb{Q}^n$ *for every positive integer n;*
(b) *Any countable dense subset of $\mathbb{R}^n$ for any $1 \leq n \leq \omega$, or of the Cantor set $\mathbb{C}$;*
(c) $\mathbb{Q}$ *with the right half-open interval topology.*

Lemma 17. *Let X be a compact Hausdorff space, and suppose there is a countable collection $\mathcal{U}$ of open sets such that, whenever $x \neq y \in X$, there is some $U \in \mathcal{U}$ with $x \in U$ and $y \notin \overline{U}$. Then X is metrizable.*

Hint. Let $\mathcal{B} = \{X \setminus \overline{\cup \mathcal{V}} : \mathcal{V} \text{ is a finite subset of } \mathcal{U}\}$. Show that $\mathcal{B}$ is a countable base for X.

Definition. A space X is said to have a G_δ*-diagonal* if the diagonal $\Delta = \{(x,x) \in X^2 : x \in X\}$ is a G_δ-set in X^2 (i.e., there are open subsets U_n, $n \in \omega$, of X^2 such that $\Delta = \bigcap_{n\in\omega} U_n$).

Theorem 18. *A compact Hausdorff space X has a G_δ-diagonal iff X is metrizable.*

Definition. A collection $\mathcal{N}$ of subsets of a space X is a *network* for X if whenever $x \in U$, where U is open in X, then there is some $N \in \mathcal{N}$ with $x \in N \subset U$.

Theorem 19. *If X has a countable network, then so does every continuous image of X.*

Theorem 20. *The following are equivalent for a T_1-space X:*

(a) *X has a countable network;*
(b) *X is the continuous image of a separable metric space.*

Hint for (a)⇒(b): If $\mathcal{N}$ is a countable network for X, show that the topology on X obtained by taking $\mathcal{N} \cup \{X \setminus N : N \in \mathcal{N}\}$ as a subbase is a separable metrizable topology finer (i.e., it has more open sets) than the original topology on X. (Recall that $\mathcal{B}$ is a *subbase* for a space X if the collection of all intersections of finite subsets of $\mathcal{B}$ is a base for X.)

Remark. Because of Theorem 19, spaces having a countable network are sometimes called *cosmic* spaces.

Theorem 21. *If X is regular and has a countable network, then X has a G_δ-diagonal.*

Theorem 22. *If X is a compact metrizable space, then so is every Hausdorff continuous image of X.*

Recall that a mapping $f : X \to Y$ is *closed* if $f(H)$ is closed in Y whenever H is closed in X.

Theorem 23. *Let $f : X \to Y$ be a continuous surjection. Then the following are equivalent:*

(a) *f is closed;*
(b) *Whenever $y \in Y$ and U is an open set in X containing $f^{-1}(y)$, there is an open set V in Y containing y with $f^{-1}(V) \subset U$;*
(c) *For each open set U in X, the set $f^*(U) = \{y \in Y : f^{-1}(y) \subset U\}$ is open in Y.*

Definition. A continuous surjection $f : X \to Y$ is said to be *perfect* if f is closed, and $f^{-1}(y)$ is compact for each $y \in Y$.

Theorem 24. *The perfect image of a separable metrizable space is separable metrizable.*

Definition. A space X is said to have the *countable chain condition (ccc)* if every pairwise-disjoint collection of open subsets of X is countable.

The following is an easy observation:

Theorem 25. *Every separable space has the ccc.*

Let $I = [0,1]$.

Theorem 26. I^I *is separable.*

Remark. Let $\mathfrak{c} = |I|$. By a similar argument, any product of $\mathfrak{c}$ (or fewer) separable spaces is separable.

Let $\mathfrak{c}^+$ denote the least cardinal greater than $\mathfrak{c}$.

Theorem 27. $I^{\mathfrak{c}^+}$ *is not separable.*

Theorem 28(Δ-system lemma). *Let $\mathcal{F}$ be an uncountable collection of finite sets. Then there is an uncountable subcollection $\mathcal{G}$ of $\mathcal{F}$ and a set R such that $G_1 \cap G_2 = R$ for any two distinct $G_1, G_2 \in \mathcal{G}$.*

Hint. W.l.o.g., every member of $\mathcal{F}$ has the same cardinality k. Induct on k.

Remark. A collection $\mathcal{G}$ satisfying the conclusion of Theorem 28 is called a Δ-*system* and the set R is called the *root* of the Δ-system.

Theorem 29. *Let $\{X_\alpha : \alpha \in \kappa\}$ be a collection of ccc spaces. Then $\Pi_{\alpha\in\kappa} X_\alpha$ has the ccc iff every finite subproduct has the ccc. (A finite subproduct is a product of the form $\Pi_{\alpha \in F} X_\alpha$ for some finite subset F of κ.)*

Corollary 30. $I^{\mathfrak{c}^+}$ *has the ccc but is not separable.*

Definition. A collection $\mathcal{H}$ of subsets of a space X is said to be *discrete* in X if every point of X has a nbhd meeting at most one member of $\mathcal{H}$. A T_1-space X is said to be *collectionwise-normal(CWN)* if, given any discrete collection $\mathcal{H}$ of closed sets, there is a pairwise-disjoint collection $\{U_H : H \in \mathcal{H}\}$ of open sets with $H \subset U_H$ for every $H \in \mathcal{H}$.

Example. Let S be the Sorgenfrey line. Let

$$\mathcal{H} = \{\{p\} : p \in S^2 \text{ is on the line } y = -x\}.$$

Then $\mathcal{H}$ is a discrete collection of closed subsets of S^2. Note that there is no pairwise-disjoint collection of open sets separating the member of $\mathcal{H}$. So S^2 is not collectionwise-normal. (We already know it's not even normal, but not collectionwise normal is easier to see.)

Lemma 31. *If $\mathcal{H}$ is a discrete collection of closed sets, then $\mathcal{H}$ is pairwise-disjoint, and $\cup\mathcal{H}'$ is closed for every subcollection $\mathcal{H}'$ of $\mathcal{H}$.*

Theorem 32. *Every paracompact T_2-space is collectionwise-normal.*

Corollary 33. *Every metrizable space is collectionwise-normal.*

Example (Bing's G). *Let A be an uncountable set, and let $\mathcal{P}(A)$ be the set of all subsets of A. For each $\alpha \in A$, define $e_\alpha : \mathcal{P}(A) \to \{0,1\}$ by $e_\alpha(B) = 1$ if $\alpha \in B$ and $e_\alpha(B) = 0$ if $\alpha \notin B$. Let $E = \{e_\alpha : \alpha \in A\}$. Note that E can be considered to be a subset of $2^{\mathcal{P}(A)}$. Let X be the set $2^{\mathcal{P}(A)}$ with the topology defined by declaring every point of $X \setminus E$ to be isolated, while each e_α has its usual product nbhds in $2^{\mathcal{P}(A)}$.*

Then X is normal but not collectionwise-normal.

Hint: For non-collectionwise-normal, show that $\{\{e_\alpha\} : \alpha \in A\}$ is a discrete collection of singleton sets and use the fact that the usual product topology on $2^{\mathcal{P}(A)}$ is ccc.

Definition. A space X is said to be *monotonically normal* if to each pair (H, K) of disjoint closed sets, one can assign an open set $U(H, K)$ such that

(i) $H \subset U(H,K) \subset \overline{U(H,K)} \subset X \setminus K$;
(ii) If $H \subset H'$ and $K \supset K'$, then $U(H,K) \subset U(H',K')$.

An operator $U(H, K)$ satisfying conditions (i) and (ii) is called a *monotone normality operator* for X.

Theorem 34. *Metrizable spaces are monotonically normal.*

Lemma 35. *If X is monotonically normal, then there is a monotone normality operator $U(H, K)$ for X satisfying $U(H,K) \cap U(K,H) = \emptyset$ for any pair H, K of disjoint closed sets.*

Theorem 36. *Monotonically normal spaces are collectionwise-normal.*

Theorem 37. *TFAE for a T_1-space X:*

(a) *X is monotonically normal*
(b) *To each $x \in X$ and open nbhd U of x, one can assign an open nbhd U_x of x satisfying:*

$$U_x \cap V_y \neq \emptyset \Rightarrow x \in V \text{ or } y \in U;$$

(c) *Same as (b), but with the nbhds U restricted to members of a given base $\mathcal{B}$.*

Theorem 38. *Every subspace of a monotonically normal space is monotonically normal.*

Theorem 39. *Every linearly ordered space is monotonically normal.*

Hint: Let $\prec$ be any well-ordering of the linearly ordered space X. For $a < x < b$, define a_x to be a if $(a,x) = \emptyset$, else let a_x be the $\prec$-least element of (a,x). Define b_x analogously, and let $(a,b)_x = (a_x, b_x)$.

We now discuss paracompactness of ordered spaces. The following result is a corollary of a couple of results from first year topology:

Theorem 40. *The space ω_1 of countable ordinals is not paracompact.*

Theorem 41.

(i) *Let C and D be closed (in the order topology) and unbounded subsets of ω_1. Then $C \cap D \neq \emptyset$;*
(ii) *Let C_n, $n \in \omega$, be closed unbounded subsets of ω_1. Then $\bigcap_{n\in\omega} C_n \neq \emptyset$.*

Theorem 42. *ω_1 is not perfectly normal.*

Hint. Show that the set H of all limit ordinals is a closed set, and any open superset of H contains all but countably many points of the space.

Theorem 43. *Let C be closed unbounded in ω_1. Then C is homeomorphic to ω_1.*

Theorem 44. *Suppose $f : \omega_1 \to \omega_1$ (not necessarily continuous). Let $C = \{\alpha : \forall\beta < \alpha(f(\beta) < \alpha)\}$. Then C is closed unbounded.*

A subset S of ω_1 is said to be *stationary* in ω_1 if $S \cap C \neq \emptyset$ for every closed unbounded set C.

Theorem 45. *(Pressing Down Lemma) Let S be a stationary set in ω_1. If for each ordinal α in S, $\alpha > 0$, we choose an ordinal $\beta_\alpha < \alpha$, then there is some $\beta < \omega_1$ such that $\beta = \beta_\alpha$ for uncountably many $\alpha \in S$.*

Theorem 46. *Let S be a stationary subset of ω_1. Then S is not paracompact.*

Theorem 47. *Let C be a closed subset of a linearly ordered space X. Then $X \setminus C$ is the union of a disjoint collection of convex open sets.*

Theorem 48. *A subspace S of ω_1 is metrizable iff S is paracompact iff S is non-stationary.*

Example 49. *The space $\omega_1 \times (\omega_1 + 1)$ is not normal.*

Hint. Let $H = \{(\alpha, \alpha) : \alpha < \omega_1\}$ and $K = [0, \omega_1) \times \{\omega_1\}$. Show H and K are disjoint closed sets which can't be separated.

It will be convenient to define an ordinal κ to be a *cardinal* if there is no function from an ordinal $\alpha < \kappa$ onto κ. With this notation, ω and ω_1 denote the least infinite cardinal and least uncountable cardinal, resp. (as well as the least ordinals with infinitely many and uncountably many predecessors, resp.). A cardinal κ is called a *successor cardinal* if there is a cardinal $\lambda < \kappa$ such that κ is the least cardinal greater than λ. In this case, κ is often denoted by λ^+. A cardinal κ which is not a successor cardinal is called a *limit cardinal*.

E.g., $\omega_1, \omega_2, \omega_3, \ldots$ are successor cardinals, while ω and $\omega_\omega = sup\{\omega_n : n < \omega\}$ are limit cardinals.

Let λ be a limit ordinal. We define the *cofinality* of λ, denoted $cf(\lambda)$, to be the least cardinal κ such that there is a function $f : \kappa \to \lambda$ such that $\{f(\alpha) : \alpha < \kappa\}$ is unbounded in λ.

Example. If λ is any countable limit ordinal, then $cf(\lambda) = \omega$. Also, $cf(\omega_\omega) = \omega$.

An infinite cardinal κ is said to be *regular* if $cf(\kappa) = \kappa$. Otherwise, κ is called *singular*.

Clearly, ω_ω is singular. It is not difficult to see from the fact that a countable union of countable sets is countable that $cf(\omega_1) = \omega_1$, hence ω_1 is a regular cardinal.

Lemma 50.

(i) *If κ is any infinite cardinal, then the union of $\leq \kappa$-many sets, each of cardinality $\leq \kappa$, has cardinality $\leq \kappa$;*
(ii) *If κ is a successor cardinal, then κ is regular;*
(iii) *For any limit ordinal λ, $cf(\lambda)$ is a regular cardinal.*

Remark. 50(i) is essentially equivalent to $\kappa = |\kappa \times \kappa|$.

Theorem 51. *The following are equivalent for an infinite cardinal κ:*

(i) *κ is regular;*
(ii) *For any $A \subset \kappa$, if $|A| < \kappa$, then $sup(A) < \kappa$;*
(iii) *The union of $< \kappa$-many sets, each of cardinality $< \kappa$, has cardinality $< \kappa$.*

Closed unbounded and stationary sets are defined for any uncountable regular cardinal κ in the same way they were defined for ω_1, and the analogues of Theorems 41, 44, and the Pressing Down Lemma hold by similar arguments.

Theorem 52. *Let κ be an uncountable regular cardinal.*

(i) *If $\mathcal{C}$ is a collection of $< \kappa$-many closed unbounded subsets of κ, then $\cap\mathcal{C}$ is closed unbounded;*
(ii) *If $f : \kappa \to \kappa$ is a function, then the set $C = \{\alpha < \kappa : \forall\beta < \alpha(f(\beta) < \alpha)\}$ is closed unbounded;*
(iii) *If $S \subset \kappa$ is stationary, and to each $\alpha \in S$ with $\alpha > 0$ we choose $\beta_\alpha < \alpha$, then there is some $\beta < \kappa$ such that $\beta_\alpha = \beta$ for κ-many $\alpha \in S$.*

Theorem 53. *Let S be a stationary subset of a regular cardinal κ. Then S is not paracompact.*

Theorem 54. *A linearly ordered space X is compact iff every subset of X has a least upper bound and a greatest lower bound.*

Theorem 55. *Every linearly ordered space X is a dense subset of a compact linearly ordered space $\hat{X}$.*

Hint. Given X, call a subset A of X *left-closed* if A is closed, and $a \in A$ and $b < a$ implies $b \in A$. For example, for each $x \in X$, the set $\{a \in X : a \leq x\}$ is left-closed. (Other left-closed sets are often called "gaps".) Let $\hat{X}$ be all left-closed sets, ordered by $\subseteq$. Include the empty set in $\hat{X}$ iff X has no least element.

Let $\mathcal{U}$ be a collection of sets, and let $U, V \in \mathcal{U}$. A *finite linked chain in* $\mathcal{U}$ *from* U *to* V is a sequence $U_1, U_2, \ldots, U_n$ of members of $\mathcal{U}$ such that $U_1 = U$, $U_n = V$, and $U_i \cap U_{i+1} \neq \emptyset$ for any $i = 1, 2, \ldots, n-1$. $\mathcal{U}$ is said to be *connected* if there is a finite linked chain in $\mathcal{U}$ between any two members of $\mathcal{U}$.

The next result has nothing to do with ordered spaces, but is good to know.

Theorem 56. *A space X is connected iff every cover of X by nonempty open sets is connected.*

Theorem 57. *A linearly ordered space X is paracompact iff X does not contain a closed subspace homeomorphic to a stationary subset of a regular uncountable cardinal.*

Hint: For the "if" direction, first show that it suffices to construct a locally finite open refinement of $\mathcal{U}$ on $\cup\mathcal{U}$, where $\mathcal{U}$ is a connected collection of open intervals.

Lemma 58. *The following are equivalent for a space X:*

(a) *X is hereditarily Lindelöf;*
(b) *Every open subspace of X is Lindelöf;*
(c) *For any collection $\mathcal{U}$ of open subsets of X, there is a countable $\mathcal{V} \subset \mathcal{U}$ such that $\cup\mathcal{V} = \cup\mathcal{U}$;*
(d) *There is no subset $\{x_\alpha : \alpha < \omega_1\}$ of X with the property that, for each $\alpha < \omega_1$, $x_\alpha \notin \overline{\{x_\beta : \beta > \alpha\}}$.*

Remark. A set $\{x_\alpha : \alpha < \omega_1\}$ satisfying the conditions of Lemma 58(d) is said to be *right-separated in type* ω_1. *Left-separated in type* ω_1 is defined analogously, and the analogous result is that a space is hereditarily separable iff it does not contain a subspace which is left-separated in type ω_1.

Theorem 59. *Suppose X is a ccc linearly ordered space. Then X is hereditarily Lindelöf.*

Hint: Since open subspaces of *ccc* linearly ordered spaces are also *ccc* linearly ordered, it suffices to show X is Lindelöf. Let $\mathcal{U}$ be an open cover of X. Define $x \sim y$ iff $[x, y]$ is covered by some countable subcollection of $\mathcal{U}$. Show that each equivalence class E is open and is covered by some countable subcollection of $\mathcal{U}$.

A linearly ordered space X is a *Suslin line* if X is *ccc*, connected, has no first or last point, and is not homeomorphic to the real line. By Theorem 11, a Suslin line cannot be separable.

Theorem 60. *If there is a ccc nonseparable linearly ordered space, then:*

(i) *there is one which is densely ordered and such that no nonempty open interval is separable;*
(ii) *there is a Suslin line.*

Hint for (i). Define $x \sim y$ iff the interval from x to y is separable. Note that equivalence classes are convex. Show that the set of equivalence classes with the natural order satisfies the desired conditions.

Remark. Sometimes a Suslin line is defined to be a nonseparable *ccc* linearly ordered space. By Theorem 60, this is equivalent to our definition (in the sense that one exists iff the other exists).

Theorem 61. *If T is a tree of height ω, and every level of T is finite, then T has an infinite branch.*

An *Aronszajn tree* is a tree of height ω_1 such that every branch and every level is countable. A *Suslin tree* is a tree of height ω_1 such that every branch and every antichain is countable. Note that every Suslin tree is Aronszajn.

Theorem 62. *There is an Aronszajn tree.*

Theorem 63. *If there is a Suslin line, then there is a Suslin tree.*

Hint. Let X be a Suslin line. Define closed subsets C_α, $\alpha < \omega_1$, of X as follows. Let $C_0 = \emptyset$. If α is a limit ordinal and C_β has been defined for all $\beta < \alpha$, let $C_\alpha = \overline{\bigcup_{\beta<\alpha} C_\beta}$. If $\alpha = \gamma + 1$ and C_γ has been defined, let L_γ be the collection of convex components of $X \setminus C_\gamma$. For each $I \in L_\gamma$, choose a countable sequence of points converging to each endpoint of I. Let C'_γ be the collection of these chosen points for all $I \in L_\gamma$. Then let $C_\alpha = \overline{C_\gamma \cup C'_\gamma}$.

Now let L_α be the set of all convex components of $X \setminus C_\alpha$, and let $T = \bigcup_{\alpha<\omega_1} L_\alpha$. Show that T ordered by $\supseteq$ is a Suslin tree.

Theorem 64. *If there is a Suslin tree, then there is a Suslin line.*

Hint. Let T be a Suslin tree, and let $\prec$ be an arbitrary linear order on T. Let X be the set of all branches of T. For any branch b, let $b(\alpha)$ be the member of b in level α of the tree. Given $b_1, b_2 \in X$, let α be minimal such that $b_1(\alpha) \neq b_2(\alpha)$, and then define $b_1 < b_2$ iff $b_1(\alpha) \prec b_2(\alpha)$. Show that X with this order is *ccc* and nonseparable.

Theorem 65. *If S is a Suslin line, then S^2 does not have the ccc.*

Hint. Let $T = \bigcup_{\alpha<\omega_1} L_\alpha$ be as in the hint for Theorem 63. For each α, choose $I_\alpha \in L_\alpha$. There are disjoint $I_\alpha^0, I_\alpha^1 \in L_{\alpha+1}$ contained in I_α. Show that $\{I_\alpha^0 \times I_\alpha^1 : \alpha < \omega_1\}$ is pairwise-disjoint.

Theorem 66. *If there is a Suslin line, there is one such that no nondegenerate interval is separable. Such a Suslin line is the union of ω_1-many nowhere-dense sets.*

Hint for the second part: study the hint for Theorem 63.

MARTIN'S AXIOM

Let $(P, \leq)$ be a partially ordered set. We say $D \subset P$ is *dense* in P if for any $p \in P$, there is $q \leq p$ with $q \in D$. A subset G of P is called a *filter* in P if

(i) For each $p, q \in G$, there is $r \in G$ with $r \leq p$ and $r \leq q$;
(ii) For each $p \in G$, if $p \leq q$ then $q \in G$.

For example, let X be any set, and let $\mathcal{P}(X) \setminus \{\emptyset\}$ be the collection of all nonempty subsets of X. For $p, q \in \mathcal{P}(X) \setminus \{\emptyset\}$, define $p \leq q$ iff $p \subset q$. Then G is a filter on $\mathcal{P}(X) \setminus \{\emptyset\}$ iff G is a filter of subsets of X in the usual sense ((i) and (ii) tell you G is closed under finite intersections and under supersets).

We say that two elements p, q of P are *comparable* if $p \leq q$ or $q \leq p$, *compatible*(abbreviated $p \not\perp q$) if there is $r \in P$ with $r \leq p$ and $r \leq q$, and *incompatible* (abbreviated $p \perp q$) if they are not compatible.

In the above example of $\mathcal{P}(X) \setminus \{\emptyset\}$, p and q are comparable iff one is a subset of the other, compatible iff they have nonempty intersection, and thus incompatible iff they are disjoint.

An *antichain* in P is a subset A of P such that every two elements of A are incompatible. We say that $(P, \leq)$ has the *ccc* if every antichain is countable, or equivalently, every uncountable subset of P has a pair of compatible elements.

Note that $\mathcal{P}(X)$ will not have the *ccc* if X is uncountable. But now let X be a topological space, and let $\mathcal{O}(X)$ be the collection of all nonempty open sets, and define $p \leq q$ iff $p \subset q$. Then the partial order $\mathcal{O}(X)$ has the *ccc* iff the space X has the *ccc*. Note that a subset $\mathcal{D}$ of $\mathcal{O}(X)$ is dense in this partial order iff $\mathcal{D}$ is a collection of nonempty open sets such that every nonempty open set contains a member of $\mathcal{D}$. (Such a collection $\mathcal{D}$ is sometimes called a π-*base* for the space X.)

Finally, we now define Martin's Axiom(MA): Let κ be a cardinal. $MA(\kappa)$ is the following statement: Whenever $(P, \leq)$ is a *ccc* partial order, and $\mathcal{D}$ is a family of $\leq \kappa$-many dense sets, then there is a filter G in P such that $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.

MA is the statement that $MA(\kappa)$ holds for every $\kappa < 2^\omega$.

Theorem 67. *$MA(\omega)$ is true, and hence the Continuum Hypothesis (CH) implies MA.*

Theorem 68. *Assume $MA(\kappa)$. Let X be a compact Hausdorff ccc space. If $\mathcal{U}$ is a collection of $\leq \kappa$-many dense open sets, then $\cap\mathcal{U} \neq \emptyset$.*

Hint: Use the partial order $\mathcal{O}(X)$ defined above.

Corollary 69. *$MA(2^\omega)$ is false.*

Corollary 70. *Assume $MA(\omega_1)$. Then there are no Suslin lines.*

The topological statement in Theorem 67 is actually equivalent to $MA(\kappa)$. So Martin's Axiom is equivalent to the statement that no compact Hausdorff *ccc* space is the union of fewer than 2^ω-many nowhere-dense sets. In particular, MA implies that the real line is not the union of fewer than 2^ω-many nowhere-dense sets. One can also show that MA implies the real line is not the union of fewer than 2^ω-many Lebesgue measure zero sets.

Lemma 71. *Assume $MA(\omega_1)$. Suppose X is ccc and $\{U_\alpha : \alpha < \omega_1\}$ is a collection of nonempty open subsets of X. Then there is an uncountable subset A of ω_1 such that $\{U_\alpha : \alpha \in A\}$ has the f.i.p..*

Hint. First show that there is $\alpha_0 < \omega_1$ such that $\forall \alpha > \alpha_0$

$$\overline{\bigcup_{\beta > \alpha} U_\beta} = \overline{\bigcup_{\beta > \alpha_0} U_\beta}.$$

Then apply MA with $P = \{O \in \mathcal{O}(X) : O \subset \bigcup_{\beta > \alpha} U_{\alpha_0}\}$.

A space X is said to have *property K* if every uncountable collection $\mathcal{U}$ of nonempty open sets contains an uncountable subcollection $\mathcal{V}$ such that $V_1 \cap V_2 \neq \emptyset$ for every $V_1, V_2 \in \mathcal{V}$.

Corollary 72. *Assume $MA(\omega_1)$. Then every ccc space has property K.*

Theorem 73. *Assume $MA(\omega_1)$. If X and Y are ccc, so is $X \times Y$. Hence any product of ccc spaces is ccc.*

Lemma 74. *(a)A regular Lindelöf space X is hereditarily Lindelöf iff X is perfectly normal.*

(b) If X is compact Hausdorff, then X is first-countable iff every point of X is a G_δ-set. Consequently, compact Hausdorff perfectly normal spaces are first-countable.

Theorem 75. *Assume $MA(\omega_1)$. If X is a compact Hausdorff hereditarily Lindelöf space, then X is hereditarily separable.*

Hint. Suppose not. Then there are points x_α, $\alpha < \omega_1$, in X such that $x_\alpha \notin \overline{\{x_\beta : \beta < \alpha\}}$. Let $Y = \overline{\{x_\alpha : \alpha < \omega_1\}}$. It follows from first-countability that every point of Y is in $\overline{\{x_\beta : \beta < \alpha\}}$ for some $\alpha < \omega_1$. Use compactness of Y and Lemma 71 applied to Y to get a contradiction.

Lemma 76. *Assume $MA(\kappa)$. Let $\{U_\alpha : \alpha < \kappa\}$ be a collection of dense open subsets of the real line $\mathbb{R}$. Then there is a dense G_δ-set G such that $G \subset \bigcap_{\alpha<\kappa} U_\alpha$.*

Proof. Let $\mathcal{B}$ be a countable base for $\mathbb{R}$. Let P be all pairs of the form $(\vec{C}, F)$, where $\vec{C}$ is a finite sequence $\langle C_0, C_1, \ldots, C_n\rangle$ of members of $\mathcal{B}$, and F is a finite subset of κ. For $(\vec{C}', F'), (\vec{C}, F) \in P$, define $(\vec{C}', F') \leq (\vec{C}, F)$ iff $\vec{C}'$ extends $\vec{C}$, $F' \supseteq F$, and for each $i \in dom(\vec{C}') \setminus dom(\vec{C})$, we have

$$C_i \subset \bigcap_{\alpha \in F} U_\alpha$$

.

Claim 1. $(P, \leq)$ *is a partially ordered set.*

We need to prove transitivity. Suppose $(\vec{C}'', F'') \leq (\vec{C}', F') \leq (\vec{C}, F)$. Then $\vec{C}''$ extends $\vec{C}'$ extends $\vec{C}$ and $F'' \supseteq F' \supseteq F$, so $\vec{C}''$ extends $\vec{C}$ and $F'' \supset F$. If $i \in dom(\vec{C})'' \setminus dom(\vec{C})$, then either $i \in dom(\vec{C}'') \setminus dom(\vec{C}')$ in which case $C_i \subset \bigcap_{\alpha \in F'} U_\alpha \subset \bigcap_{\alpha \in F} U_\alpha$, or $i \in dom(\vec{C}') \setminus dom(\vec{C})$ in which case $C_i \subset \bigcap_{\alpha \in F} U_\alpha$. Thus $(\vec{C}'', F'') \leq (\vec{C}, F)$.

Claim 2. $(P, \leq)$ *has the ccc.*

Suppose $(\vec{C}_\alpha, F_\alpha)$ is in P for each $\alpha < \omega_1$. We need to show that there is $\alpha \neq \beta$ such that $(\vec{C}_\alpha, F_\alpha)$ and $(\vec{C}_\beta, F_\beta)$ are compatible. Since $\mathcal{B}$ is countable, so is the collection of finite sequences from $\mathcal{B}$, so there are $\alpha \neq \beta$ such that $\vec{C}_\alpha = \vec{C}_\beta = \vec{C}$. Then it is easy to see that $(\vec{C}, F_\alpha \cup F_\beta)$ is less than or equal to both $(\vec{C}_\alpha, F_\alpha)$ and $(\vec{C}_\beta, F_\beta)$, and hence they are compatible.

Now we define some dense sets. For each $B \in \mathcal{B}$ and $k \in \omega$, let

$$D_{B,k} = \{(\vec{C}, F) \in P : \exists i > k(C_i \subset B)\}.$$

Remark: The $D_{B,k}$'s serve two purposes. One, they will make sure that the generic filter G contains elements whose first coördinate is a sequence of arbitrarily long length, and thus G will determine an infinite sequence $< C_0^G, C_1^G, \cdots >$ of members of $\mathcal{B}$. Two, they make sure that $\bigcup_{i>n} C_i^G$ is dense in $\mathbb{R}$ for each n. (We'll argue this later.)

Also, for each $\alpha \in \kappa$, let

$$E_\alpha = \{(\vec{C}, F) \in P : \alpha \in F\}.$$

The E_α's will make sure that, for each α, there is some n so that $\bigcup_{i>n} C_i \subset U_\alpha$.

Let us show that the $D_{B,k}$'s and the E_α's are indeed dense in P. Let $(\vec{C}, F) \in P$. Then $(\vec{C}, F \cup \{\alpha\}) \leq (\vec{C}, F)$, so E_α is dense. Now fix $B \in \mathcal{B}$ and $k \in \omega$. Suppose $\vec{C} =< C_0, C_1, ..., C_n >$. Let $m > max\{n, k\}$ and choose C_i for $i = n+1, ..., m$ such that $C_i \subset \bigcap_{\alpha \in F} U_\alpha$, and also $C_m \subset B \cap \bigcap_{\alpha \in F} U_\alpha$. This is possible since $\bigcap_{\alpha \in F} U_\alpha$ is dense open. Let $\vec{C}' =< C_0, C_1, \ldots, C_m >$. Then $(\vec{C}', F) \in D_{B,k}$ and $(\vec{C}', F) \leq (\vec{C}, F)$. So $D_{B,k}$ is dense.

Let G be a filter in P meeting all E_α's, $\alpha < \kappa$, and all $D_{B,k}$'s, $B \in \mathcal{B}$ and $k \in \omega$.

Claim 3. If $(\vec{C}, F)$ and $(\vec{C}', F')$ are in G, then the sequence $\vec{C}$ extends $\vec{C}'$ or vice-versa; i.e., if $i \in dom(\vec{C}) \cap dom(\vec{C}')$, then $C_i = C'_i$. Well, if $(\vec{C}, F)$ and $(\vec{C}', F')$ are in G, then there is some $(\vec{C}'', F'')$ in G such that $(\vec{C}'', F'')$ is less than or equal to both $(\vec{C}, F)$ and $(\vec{C}', F')$, and hence the sequence $\vec{C}''$ extends both $\vec{C}$ and $\vec{C}'$. The claim follows.

Now, since G meets all $D_{B,k}$'s, there are arbitrarily long sequences appearing as first coördinates of members of G, so we can define an infinite sequence $\vec{C}_G =< C_0^G, C_1^G, \cdots >$ of members of $\mathcal{B}$ by defining C_i^G to be the i^{th} term of $\vec{C}$ for some $(\vec{C}, F) \in G$ such that $i \in dom(\vec{C})$. It follows from Claim 3 that it doesn't matter which member of G we use to define C_i^G. Note that

$$\vec{C}_G = \bigcup \{\vec{C} : \exists F((\vec{C}, F) \in G)\},$$

where $\vec{C}$ is viewed as a set of ordered pairs.

Claim 4. For each $n \in \omega$, $\bigcup_{i>n} C_i^G$ is dense in $\mathbb{R}$. If $(\vec{C}, F) \in G \cap D_{B,n}$, then we have $C_i \subset B$ for some $i > n$. It follows that every basic open set $B \in \mathcal{B}$ contains C_i^G for some $i > n$, from which it easily follows that $\bigcup_{i>n} C_i^G$ is dense in $\mathbb{R}$.

Claim 5. For each $\alpha < \kappa$, there is some $n \in \omega$ such that $\bigcup_{i>n} C_i^G \subset U_\alpha$. Fix $\alpha < \kappa$, and let $(\vec{C}, F) \in G \cap E_\alpha$. Then $\alpha \in F$. Let $\vec{C} =< C_0, C_1, \ldots, C_n >$. Suppose $i > n$, and let $(\vec{C}', F') \in G$ such that $i \in dom(\vec{C}')$. There is some $(\vec{C}'', F'') \in G$ with $(\vec{C}'', F'')$ less than or equal to both $(\vec{C}', F')$ and $(\vec{C}, F)$. Then $C_i^G = C_i''$ and $C_i'' \subset U_\alpha$ because $(\vec{C}'', F'') \leq (\vec{C}, F)$ and $\alpha \in F$.

Finally, from Claims 4 and 5, it follows that $\bigcap_{n\in\omega}(\bigcup_{i>n} C_i^G)$ is a dense G_δ subset of $\mathbb{R}$ which is contained in $\bigcap_{\alpha<\kappa} U_\alpha$. $\square$

A subset X of $\mathbb{R}$ is said to be *first category* in $\mathbb{R}$ if X is contained in the union of countably many nowhere-dense subsets of $\mathbb{R}$. Since the closure of a nowhere-dense set is nowhere-dense, it is equivalent to say that X is first category iff the complement of X contains a dense G_δ-set. (Some texts call sets of first category *meager*, and their complements *comeager*.)

Theorem 77. *Assume MA. Then the union of $< 2^\omega$-many first category subsets of $\mathbb{R}$ is first category. In particular, any subset of $\mathbb{R}$ of cardinality $< 2^\omega$ is first category.*

Remark. It is also true that assuming MA, the union of $< 2^\omega$-many Lebesgue measure zero subsets of $\mathbb{R}$ has measure zero, and, in particular, any subset of $\mathbb{R}$ of cardinality $< 2^\omega$ has measure zero.

Example. Let A be a subset of the real line $\mathbb{R}$. Let $X(A)$ be the space whose set is

$$(A \times \{0\}) \cup \{(q, r) : q, r \in \mathbb{Q}, r > 0\}.$$

That is, $X(A)$ consists of the points on the x-axis corresponding to $a \in A$, together with all points in the upper half-plane with rational coördinates.

Let the points in the upper half-plane be isolated, and for each $a \in A$ and $n > 0$, let a basic neighborhood of $(a, 0)$ be

$$D(a, n) = \{(a, 0)\} \cup \{(q, r) : q, r \in \mathbb{Q} \text{ and } \sqrt{(q-a)^2 + (r - 1/n)^2} < 1/n\}.$$

That is, $D(a, n)$ consists of the point $(a, 0)$ together with all points with rational coördinates in the interior of a disk of radius $1/n$ tangent to the x-axis at $(a, 0)$.

Note that $A \times \{0\}$ is closed in $X(A)$, and discrete as a subspace. It follows that the collection of singletons $\{\{(a, 0)\} : a \in A\}$ is a discrete collection of closed sets, and hence the union of any subcollection, i.e., $B \times \{0\}$ for any $B \subset A$, is closed in $X(A)$.

Theorem 78. *Let $A \subset \mathbb{R}$, and let $X(A)$ be as defined above. Then:*

(a) *If $A = \mathbb{R}$, then $X(A)$ is not normal;*
(b) *If A is uncountable, then $X(A)$ is not collectionwise-normal;*
(c) *If every subset of A is a G_δ-set in A, then $X(A)$ is normal.*

Hint for (c): Recall when working with Bing's G, we noted that to prove normality for a space consisting of a closed discrete set plus isolated points, we need only show that any subset of the closed discrete set, and its complement in the closed discrete set, can be put into disjoint open sets.

An uncountable subset A of $\mathbb{R}$ whose every subset is G_δ in A is called a *Q-set.*

Theorem 79. *If $2^\omega < 2^{\omega_1}$, then there are no Q-sets.*

Hint: How many G_δ-sets can there be in a space with a countable base?

Theorem 80. *Let κ be an uncountable cardinal, and assume $MA(\kappa)$. Then every subset of $\mathbb{R}$ of cardinality κ is a Q-set.*

Hint. Let $A \subset \mathbb{R}$ have cardinality κ, and let $B \subset A$. Use $MA(\kappa)$ to show that there is a sequence $\langle C_0, C_1, \dots \rangle$ of basic open sets of A such that every $x \in B$ is in infinitely many C_i's, while every $y \in A \setminus B$ is in only finitely many C_i's. Then note that $B = \bigcap_{n\in\omega}(\bigcup_{i>n} C_i)$.

Corollary 81. *Assume $MA(\omega_1)$. Then there is a Q-set, and hence there is a normal first-countable separable non-collectionwise-normal space.*

Corollary 82. *Assume MA. Then $2^\kappa = 2^\omega$ for every infinite cardinal $\kappa < 2^\omega$.*

Remark. Bob Heath showed that if there is a normal first-countable separable non-collectionwise-normal space, then there is a Q-set. So, by 78(c), the existence of such a space is equivalent to the existence of a Q-set.

Lemma 83. *Let X be a separable space, and let D be a closed discrete subset of X. If $2^{|D|} > 2^\omega$, then X is not normal.*

Remark. Since $2^{2^\omega} > 2^\omega$, Lemma 83 gives another way to show Theorem 78(a), the square of the Sorgenfrey line is not normal, and the like.

Corollary 84. *If $2^\omega < 2^{\omega_1}$, then every normal separable space is collectionwise-normal.*

Hint. In a normal space, any *countable* discrete collection of closed sets can be separated by disjoint open sets.

A collection $\mathcal{A}$ of infinite subsets of ω is said to be *almost-disjoint* if the intersection of any two distinct members of $\mathcal{A}$ is finite. By a standard Zorn's Lemma argument, every almost-disjoint family $\mathcal{A}$ is contained in a maximal almost-disjoint family $\mathcal{A}'$.

Theorem 85. *There is an almost-disjoint family of subsets of ω of cardinality 2^ω.*

It is easy to check that any finite partition of ω into infinite sets is a maximal almost-disjoint family, but ...

Lemma 86. *No countably infinite almost-disjoint family of subsets of ω is maximal.*

Theorem 87. *Assume MA. Then every infinite maximal almost-disjoint family of subsets of ω has cardinality 2^ω.*

Example. Let $\mathcal{A}$ be an almost-disjoint family of subsets of ω. Define a space $\psi(\mathcal{A})$ as follows. The underlying set for $\psi(\mathcal{A})$ is $\omega \cup \{x_A : A \in \mathcal{A}\}$, where $\{x_A : A \in \mathcal{A}\}$ is a set of distinct points not in ω. Define the topology by declaring the points of ω to be isolated, and the n^{th} member of a local base at x_A to be

$$b(x_A, n) = \{x_A\} \cup (A \setminus n).$$

.

Theorem 88. *$\psi(\mathcal{A})$ is a locally compact Hausdorff space, and $\{x_A : A \in \mathcal{A}\}$ is a closed discrete subset of $\psi(\mathcal{A})$. Thus, if $\mathcal{A}$ is infinite, then $\psi(\mathcal{A})$ is not countably compact.*

A space X is said to be *pseudocompact* if every continuous $f : X \to \mathbb{R}$ has bounded range.

Theorem 89. *Every countably compact space is pseudocompact.*

Theorem 90. *If $\mathcal{A}$ is an infinite maximal almost-disjoint family of subsets of ω, then $\psi(\mathcal{A})$ is pseudocompact (but not countably compact).*