SOME SET THEORY WE SHOULD KNOW

CARDINALITY AND CARDINAL NUMBERS

Definition. Two sets A and B are said to have the *same cardinality*, and we write $|A| = |B|$, if there exists a one-to-one onto function $f : A \to B$. We also say $|A| \leq |B|$ if there exists a one-to-one (but not necessarily onto) function $f : A \to B$. Then the Schröder-Bernstein Theorem says: $|A| \leq |B|$ and $|B| \leq |A|$ implies $|A| = |B|$:

Schröder-Bernstein Theorem. *If there are one-to-one maps $f : A \to B$ and $g : B \to A$, then $|A| = |B|$.*

A set is called *countable* if it is either finite or has the same cardinality as the set $\mathbb{N}$ of positive integers.

Theorem ST1.

(a) *A countable union of countable sets is countable;*
(b) *If $A_1, A_2, ..., A_n$ are countable, so is $\Pi_{i\leq n} A_i$;*
(c) *If A is countable, so is the set of all finite subsets of A, as well as the set of all finite sequences of elements of A;*
(d) *The set $\mathbb{Q}$ of all rational numbers is countable.*

Theorem ST2. *The following sets have the same cardinality as the set $\mathbb{R}$ of real numbers:*

(a) *The set $\mathcal{P}(\mathbb{N})$ of all subsets of the natural numbers $\mathbb{N}$;*
(b) *The set of all functions $f : \mathbb{N} \to \{0, 1\}$;*
(c) *The set of all infinite sequences of 0's and 1's;*
(d) *The set of all infinite sequences of real numbers.*

The cardinality of $\mathbb{N}$ (and any countable infinite set) is denoted by $\aleph_0$. $\aleph_1$ denotes the next infinite cardinal, $\aleph_2$ the next, etc.

Cantor's Continuum Hypothesis (CH) says that the cardinality of $\mathbb{R}$ is $\aleph_1$. It turns out that CH is undecidable from the usual axioms of set theory.

The next theorem shows that for any cardinal, there is a bigger one.

Theorem ST3. *For any set X, we have $|X| < |\mathcal{P}(X)|$, where $\mathcal{P}(X)$ is the set of all subsets of X.*

If the set X has cardinality κ, then the cardinal 2^κ is defined to be the cardinality of the set $\mathcal{P}(X)$ (which has the same cardinality as the set of all functions from X into $\{0, 1\}$, by the same argument that gets ST2(a) $\iff$ (b)). So by Theorem ST2, $2^{\aleph_0}$ is the cardinality of $\mathbb{R}$, and Cantor's Continuum Hypothesis can be stated as $2^{\aleph_0} = \aleph_1$.

ORDINAL NUMBERS AND TRANSFINITE INDUCTION

Definition. A *relation* R on a set X is a subset of $X \times X$. We write xRy to mean $(x, y) \in R$. A *partial order* on X is a relation $<$ satisfying, for every $x, y, z \in X$:

(i) $x < y$ and $y < x$ are not both true (antireflexive);
(ii) $x < y$ and $y < z$ implies $x < z$ (transitive).

We also write $x \leq y$ to mean "$x < y$ or $x = y$". We say that $<$ is a *linear order* if the following also holds for any $x, y \in X$:

(iii) Either $x < y$, $y < x$, or $x = y$ (trichotomy).

And finally, a linear order $<$ is a *well-order* if also:

(iv) Every non-empty subset of X has a least element.

Two linearly ordered sets $(A, <_A)$ and $(B, <_B)$ are *order- isomorphic* if there is a one-to-one onto order-preserving map $h : A \to B$.

We will assume the following:

Theorem ST4. *There is a class Ord and a linear ordering $<$ on Ord with the following properties:*

(i) *Every nonempty subset of Ord has a least element;*
(ii) *Given any well-ordered set $(X, \prec)$, there is $\alpha \in Ord$ such that $(X, \prec)$ is order-isomorphic to $(P_\alpha, <)$, where $P_\alpha = \{\beta \in Ord : \beta < \alpha\}$.*

Elements of Ord are called *ordinals*. Intuitively, an ordinal number denotes position in a well-ordered sequence. The least ordinal is denoted by 0, the next by 1, the next by 2, etc. The least ordinal with infinitely many predecessors, i.e, the least ordinal greater than all the finite ordinals $0, 1, 2, \ldots,$ is denoted by ω. Then comes $\omega + 1, \omega + 2, \ldots.$ The least ordinal bigger than $\omega + n$ for every $n < \omega$ is denoted by $\omega \cdot 2$. One can similarly define $\omega \cdot 3, \omega \cdot 4, \ldots.$ The least ordinal bigger than $\omega \cdot n$ for all $n < \omega$ is denoted by $\omega \cdot \omega$ or ω^2. Etc.

Note that all ordinals mentioned in the above paragraph have only countably many predecessors. Such ordinals are called *countable* ordinals. The least ordinal with uncountably many predecessors, i.e., the least ordinal greater than all of the countable ordinals, is denoted by ω_1. A better definition of $\aleph_1$ than the one given above is that $\aleph_1$ denotes the cardinality of the set of predecessors of ω_1. $\aleph_2, \aleph_3, \ldots$ may be defined similarly. See below where $\aleph_\alpha$ is defined, by transfinite induction, for any ordinal α.

Principle of Transfinite Induction. *Let κ be an ordinal. Suppose we have for each $\alpha < \kappa$ a statement S_α such that:*

(i) *S_0 is true;*
(ii) *If $\beta < \kappa$ and S_α is true for every $\alpha < \beta$, then S_β is true.*

Then S_γ is true for every $\gamma < \kappa$.

Theorem ST5. *Let α be any ordinal. Then the set $[0, \alpha] = \{\beta \in On : 0 \leq \beta \leq \alpha\}$ with the order topology (using the usual ordering on the ordinals) is a compact Hausdorff space.*

Lemma ST6. *Let Ω denote the set of countable ordinals. If C is any countable subset of Ω, then there is a countable ordinal α such that $\alpha \geq \beta$ for any $\beta \in C$.*

Theorem ST7. *Give the set Ω of all countable ordinals the order topology (using the usual order on the ordinals). Then Ω with this topology is sequentially compact (and so also countably compact) but not compact.*

Principle of Definition by Transfinite Induction. *Let κ be an ordinal, B a set, and $\mathcal{F}$ the set of all functions (including the empty function) into B whose domain is the set $\{\beta \in Ord : \beta < \alpha\}$ for some $\alpha < \kappa$. Let $R : \mathcal{F} \to B$. Then there is a unique function G with domain $\{\alpha \in Ord : \alpha < \kappa\}$ such that, for any $\alpha < \kappa$,*

$$G(\alpha) = R(G \restriction \{\beta \in Ord : \beta < \alpha\})$$

.

Remark. The Transfinite Induction Principle and the Principle of Definition by Transfinite Induction also hold, by the same proof, if in their statements we replace κ by the class Ord of all ordinals.

Now the Principle of Definition by Transfinite Induction justifies defining ω_α and $\aleph_\alpha$ for any ordinal α as follows:

(i) $\aleph_0$ is as defined previously, and $\omega_0 = \omega$.

Suppose α is an ordinal and ω_β and $\aleph_\beta$ have been defined for every $\beta < \alpha$.

(ii) If α has an immediate predecessor γ (which would make $\alpha = \gamma + 1$), then let ω_α be the least ordinal with more than $\aleph_\gamma$-many predecessors, and let $\aleph_\alpha$ denote the cardinality of the set of predecessors of ω_α.

(iii) If α has no immediate predecessor, then let ω_α be the least ordinal greater than ω_β for every $\beta < \alpha$. Again, let $\aleph_\alpha$ denote the cardinality of the set of predecessors of ω_α.

Exercise ST8.

(a) *For each ordinal $\alpha < \omega_1$, there is a one-to-one order-preserving function from $\{\beta \in Ord : \beta < \alpha\}$ to the real line $\mathbb{R}$;*

(b) *There is no one-to-one order-preserving function from $\{\beta \in Ord : \beta < \omega_1\}$ to the real line $\mathbb{R}$.*

AXIOM OF CHOICE AND EQUIVALENTS

Axiom of Choice. *Given any collection $\mathcal{A}$ of nonempty sets, there is a function $f : \mathcal{A} \to \cup\mathcal{A}$ such that $f(A) \in A$ for every $A \in \mathcal{A}$.*

Zorn's Lemma. *Suppose $(P, <)$ is a partially ordered set with the following property:*

($*$) *If L is any subset of P linearly ordered by $<$, then there is $p \in P$ such that $l \leq p$ for any $l \in L$.*

Then there is an element q in P such that no member of P is strictly greater than q.

An element q of the partial order P that satisfies the conclusion of Zorn's Lemma is called a *maximal* element of P.

Theorem ST9. *The following are equivalent:*

(a) *The Axiom of Choice;*
(b) *Zorn's Lemma;*
(c) *Every set can be well-ordered.*