MH 3100 Quiz #7 $\frac{12}{12}$ Name: The Prof

This is a take-home quiz, due at noon on Monday, Oct. 16.

① For each $n \in \mathbb{N}$, let $A_n = \{x \in \mathbb{R} : \frac{1}{n} \le x \le 1 + \frac{1}{n}\}$.

Find: (a) $\bigcap_{n=1}^{\infty} A_n = \{1\}$  $A_1 = [1, 2]$, $A_2 = [\frac{1}{2}, \frac{3}{2}]$, $A_3 = [\frac{1}{3}, \frac{4}{3}]$, ...

(3 pts)

(b) $\bigcup_{n=1}^{\infty} A_n = \{x \in \mathbb{R} : 0 < x \le 2\} = (0, 2]$

(3 pts)

② Let $\mathcal{A}$ be a family of sets. Suppose that the intersection of any two members of $\mathcal{A}$ is also in $\mathcal{A}$. Prove the following: If $n \in \mathbb{N}$ and $A_1, A_2, \dots, A_n$ are in $\mathcal{A}$, then $\bigcap_{i=1}^{n} A_i$ is in $\mathcal{A}$.

(6 pts) <u>Proof.</u> Let $n \in \mathbb{N}$, and let $A_1, A_2, \dots, A_n$ be in $\mathcal{A}$.

Need to prove: $\bigcap_{i=1}^{n} A_i$ is in $\mathcal{A}$.

Proof by induction: If $n = 1$, then $\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{1} A_i = A_1$, which is in $\mathcal{A}$.

So true for $n = 1$.

Now assume $\bigcap_{i=1}^{n} A_i$ is in $\mathcal{A}$.

Need to prove: $\bigcap_{i=1}^{n+1} A_i$ is in $\mathcal{A}$.

$$\bigcap_{i=1}^{n+1} A_i = \left(\bigcap_{i=1}^{n} A_i\right) \cap A_{n+1}$$

Since $\bigcap_{i=1}^{n} A_i$ is in $\mathcal{A}$ (by assumption) and $A_{n+1}$ is in $\mathcal{A}$, and the intersection of any two members of $\mathcal{A}$ is in $\mathcal{A}$, we have

$$\bigcap_{i=1}^{n+1} A_i = \left(\bigcap_{i=1}^{n} A_i\right) \cap A_{n+1} \text{ is in } \mathcal{A}.$$

$\therefore$ by PMI, $\bigcap_{i=1}^{n} A_i$ is in $\mathcal{A}$ for all $n \in \mathbb{N}$.