MH 3100 Quiz #4 $\frac{12}{12}$ Name: The Prof

(1) Let a be an integer. Prove:

$$\underbrace{\text{a is odd}}_{P} \text{ if and only if } \underbrace{a^2 \text{ is odd}}_{Q}.$$

$(P \Rightarrow Q)$ Assume a is odd.

Then $a = 2k+1$ for some $k \in \mathbb{Z}$.

$\therefore a^2 = (2k+1)^2 = 4k^2 + 4k + 1$

$\therefore a^2 = 2(2k^2+2k)+1$

$2k^2+2k \in \mathbb{Z}$, so a^2 is odd.

$(Q \Rightarrow P)$ A direct proof doesn't work well, because from "a^2 is odd" you only get "$a^2 = 2k+1$ for some k", and it's not clear how to get from "$a^2 = 2k+1$" to "a is odd".

Proof by contrapositive $(\sim P \Rightarrow \sim Q)$ or proof by contradiction (Assume Q and $\sim P$ and find a contradiction) work better.

Proof by contrapositive $(\sim P \Rightarrow \sim Q)$

Assume a is not odd, i.e., a is even.

Then $a = 2k$ for some $k \in \mathbb{Z}$.

$\therefore a^2 = (2k)^2 = 4k^2 = 2(2k^2)$

$\therefore a^2$ is even

$\therefore a^2$ is not odd, so we've shown $\sim Q$.

Q.E.D.

Remark Proof by contradiction is similar:

Suppose a^2 is odd, but a is not odd.

$\therefore$ a is even, i.e., $a = 2k$ for some $k \in \mathbb{Z}$.

$\therefore a^2 = 4k^2 = 2(2k^2)$

$\therefore a^2$ is even, which is a contradiction (because we assumed a^2 is odd).

Q.E.D.