MH 3100 Exam 3

1. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. Let $R = \{(1, a), (2, a), (2, c), (3, c)\}$ be a relation from A to B.
   (a) Find Dom(R) and Rng(R).
   (b) Is R a function? Explain your answer.
   (c) Find R^{-1}.
   (d) Let $C = \{x, y\}$ and define $S = \{(a, y), (b, y), (c, y)\}$. Find $S \circ R$.

2. Let $f(x) = 2ln\ x$ and $g(x) = x^2 - x - 2$.
   (a) Find Dom(f) and Rng(f).
   (b) Find f^{-1}, including a formula for $f^{-1}(x)$.
   (c) Show that f is one-to-one.
   (d) Explain why g is not one-to-one.
   (e) Find $g \circ f$.

3. For real numbers x and y, define the relation $\sim$ as follows: $x \sim y$ iff $x - y$ is an integer.
   (a) Show that $\sim$ is an equivalence relation.
   (b) Name two members of the equivalence class of π, other than π itself.

4. Let x be a sequence of real numbers with n^{th} term x_n, and let $L \in \mathbb{R}$. Define $lim_{n\to\infty} x_n = L$ (i.e., define what "the sequence x converges to L" means).

5. Let $x_n = 1/\sqrt{n}$. Prove $x_n \to 0$.

6. True or False:
   (a) $lim_{n\to\infty}(-1)^n/ln\ n$ does not exist.
   (b) The relation S defined in 1(d) is a function.
   (c) Let g be the function defined in 2. Then g^{-1} is not a function.

BONUS. Suppose $x_n \to L$ and $y_n \to M$. Prove that $x_n + y_n \to L + M$.

Solutions to Exam 3

① (a) (6 pts) Dom $R = \{1,2,3\}$
Rng $R = \{a, c\}$

(b) (6 pts) No, because there is an element of the domain, namely 2 which is paired with more than one element of the range, namely a and c.

(c) $R^{-1} = \{(a,1),(a,2),(c,2),(c,3)\}$
(5pts)

(d)
(5 pts)

$S \circ R = \{(1,y),(2,y),(3,y)\}$

② (a) Dom $f = \{x \in \mathbb{R} : x > 0\} = (0, \infty)$
(6 pts) Rng $f = \mathbb{R}$

(b) $f^{-1} = \{(y,x) : y = 2\ln x\}$
(5 pts) $= \{(x,y) : x = 2\ln y\}$
$= \{(x,y) : \frac{x}{2} = \ln y\}$
$= \{(x,y) : y = e^{x/2}\}$
$f^{-1}(x) = e^{x/2}$

(c) (6 pts) Suppose $f(x) = f(y)$.
Then $2\ln x = 2\ln y$
$\therefore \ln x = \ln y$
$\therefore e^{\ln x} = e^{\ln y}$
$\therefore x = y$
Since $f(x) = f(y) \Rightarrow x = y$, f is 1-1.

(d) (5pts) Possible answers:
- $g(2) = g(-1) = 0$.
- the graph is a parabola opening up, so it fails the horizontal line test.

(e) (5pts) $g \circ f(x) = g(f(x))$
$= g(2\ln x) = (2\ln x)^2 - 2\ln x - 2$
$= 4\ln^2 x - 2\ln x - 2$

③ (16 pts) Thrown out — in the sense that everyone got 16 pts for this one.

④ $\lim_{n\to\infty} x_n = L$ means that for every positive number ε, there is a positive integer N such that, if $n > N$ then $|x_n - L| < \varepsilon$.
(10 pts)

⑤ (10 pts). Prove $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Proof. Let $\varepsilon > 0$.
$\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon$ iff $\frac{1}{\sqrt{n}} < \varepsilon$ iff $\frac{1}{n} < \varepsilon^2$
iff $n > \frac{1}{\varepsilon^2}$.

Let $N \in \mathbb{N}$ be such that $N \geq \frac{1}{\varepsilon^2}$.
Suppose $n > N$.
Then $n > N \geq \frac{1}{\varepsilon^2}$, so $n > \frac{1}{\varepsilon^2}$.
$\therefore \left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon$ (by)
So $n > N \Rightarrow \left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon$.
$\therefore \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

⑥ (5 pts each).
(a) False ($\lim_{n\to\infty} \frac{(-1)^n}{\ln n} = 0$)
(b) True
(c) True