Solutions to Exam 2

(1) (a) (6pts) $\emptyset$, $\{a\}$, a

(b) (6pts) $\{1\}$

(2) (6 pts each) (a) F (b) F (c) T (d) F (e) F

(3) (10 pts) Prove: $(A-B)\cap(A-C) \subseteq A-(B\cup C)$.

Proof: Let $x \in (A-B)\cap(A-C)$.

$\therefore$ $x\in A-B$ and $x\in A-C$

$\therefore$ $x\in A$, and $x\notin B$, and $x\notin C$.

$\therefore$ $x\in A$, and $x\notin B\cup C$.

$\therefore$ $x\in A-(B\cup C)$.

Q.E.D.

(4) (a) (7pts) $\bigcap_{n=1}^{\infty} A_n = \{2\}$.

(b) (7 pts) $\bigcup_{n=1}^{\infty} A_n = \{n\in\mathbb{N} : n \text{ is even}\}$.

(5) (a) (7pts). Suppose for each $n\in\mathbb{N}$, we have a proposition $P(n)$. If

(i) $P(1)$ holds, and

(ii) $P(n)$ holds $\Rightarrow$ $P(n+1)$ holds,

Then $P(n)$ holds $\forall n\in\mathbb{N}$.

(b) (7pts) Every nonempty subset of $\mathbb{N}$ has a least element.

(6) (10pts). Define sequence by: $a_1=2, a_2=4$, and $a_{n+2}=5a_{n+1}-6a_n$.

Prove: $a_n=2^n$ $\forall n\in\mathbb{N}$.

Proof. Check $n=1$: $a_1=2=2^1$

" $n=2$ $a_2=4=2^2$

Assume true for all $i<n$, i.e., assume $a_i=2^i$ for all $i<n$.

Need to prove: $a_n=2^n$.

$$a_n = 5a_{n-1}-6a_{n-2}$$
$$= 5\cdot 2^{n-1}-6\cdot 2^{n-2} \quad (\text{by} \quad)$$
$$= 2^{n-2}(5\cdot 2-6)$$
$$= 2^{n-2}\cdot 4$$
$$= 2^{n-2}\cdot 2^2 = 2^n.$$

$\therefore$ by PCI, $a_n=2^n$ $\forall n\in\mathbb{N}$.

(7) (10 pts).

Prove that $2+2^2+2^3+\dots+2^n=2^{n+1}-2$ for all $n\in\mathbb{N}$.

Proof. Check $n=1$: $2 \overset{?}{=} 2^2-2$ yes.

Assume true for some n. ← the inductive assumption

Need to prove: True for $n+1$, i.e.,

need to prove that

$$2+2^2+2^3+\dots+2^n+2^{n+1}=2^{n+2}-2.$$

We work on the left side & show that it's equal to the right side.

$$2+2^2+2^3+\dots+2^n+2^{n+1}$$
$$=(2+2^2+2^3+\dots+2^n)+2^{n+1}$$
$$=(2^{n+1}-2)+2^{n+1} \quad (\text{by the inductive assumption})$$
$$=2\cdot 2^{n+1}-2$$
$$=2^{n+2}-2$$

$\therefore$ true for n $\Rightarrow$ true for $n+1$.

$\therefore$ by PMI, it's true $\forall n\in\mathbb{N}$.

Q.E.D.

MH 3100 Exam 2

1. Name all of the elements of the following sets:
   (a) $\{\emptyset, \{a\}, a\}$
   (b) $\{\{1\}\}$

2. True or False:
   (a) $\{3\} \in \mathbb{N}$;
   (b) $X - \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty}(X - A_n)$;
   (c) $\emptyset \subseteq \{1/n : n = 1, 2, 3, ...\}$;
   (d) $A \times (B \cap C) = (A \cap B) \times (A \cap C)$;
   (e) If $A \subseteq B \cup C$, then $A \subseteq B$ or $A \subseteq C$.

3. Let A, B, and C be sets. Prove that $(A - B) \cap (A - C) \subseteq A - (B \cup C)$.

4. For each $n \in \mathbb{N}$, let $A_n = \{2, 4, 6, ..., 2n\}$. Find:
   (a) $\bigcap_{n=1}^{\infty} A_n$;
   (b) $\bigcup_{n=1}^{\infty} A_n$.

5. State the following:
   (a) The Principle of Mathematical Induction;
   (b) The Well-ordering Principle.

6. Define a sequence recursively by: $a_1 = 2$, $a_2 = 4$, and $a_{n+2} = 5a_{n+1} - 6a_n$. Use induction to prove that $a_n = 2^n$ for every $n \in \mathbb{N}$.

7. Prove that the following holds for every natural number n:

$$2 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 2.$$