MH 2660 Exam 4

Show work on all problems.

1. Consider the vector space $C[0,1]$ with inner product defined by $< f, g >= \int_0^1 f(x)g(x)dx$. Let $f(x) = 2x$ and $g(x) = x^2$.
(a) Find $< f, g >$.
(b) Find the length $||g||$ of g.
(c) Find the vector projection of f on g.

2. Let $\vec{u}_1 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}^T$ and $\vec{u}_2 = \begin{bmatrix} 2/3 & 2/3 & 1/3 \end{bmatrix}^T$.
(a) Show that $\vec{u}_1$, $\vec{u}_2$ is an orthonormal set of vectors in $\mathbb{R}^3$.
(b) Find the projection of $\vec{v} = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}^T$ on the plane spanned by $\vec{u}_1$ and $\vec{u}_2$.
(c) If $\vec{u}_3 = \begin{bmatrix} 1/3\sqrt{2} & 1/3\sqrt{2} & -4/3\sqrt{2} \end{bmatrix}^T$, and $\vec{u}_1$ and $\vec{u}_2$ are as above, then $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$ form an orthonormal basis for $\mathbb{R}^3$. Write $\vec{v} = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}^T$ as a linear combination $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$.

3. The functions $sin\ x$, $sin\ 2x$, $sin\ 3x$, and $sin\ 4x$ form an orthonormal set in $C[-\pi, \pi]$ with inner product defined by $< f, g >= 1/\pi \int_{-\pi}^{\pi} f(x)g(x)dx$. Find the length $||f(x)||$ of $f(x)$ if $f(x) = 3sin\ x - sin\ 2x + sin\ 3x - 5sin\ 4x$.

4. Find all eigenvalues of A if $A = \begin{bmatrix} 3 & -1 \\ 5 & 1 \end{bmatrix}$.

5. Given that $\lambda = 2$ is an eigenvalue of $A = \begin{bmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$, find a basis for the corresponding eigenspace.

6. (a) What does it mean to say that λ is an eigenvalue of a matrix A?
(b) If λ is an eigenvalue of A, what does it mean to say that $\vec{x}$ is an eigenvector corresponding to λ?

7. The vectors $\vec{v}_1 = \begin{bmatrix} 1 & 2 & -2 \end{bmatrix}^T$, $\vec{v}_2 = \begin{bmatrix} 4 & 3 & 2 \end{bmatrix}^T$, and $\vec{v}_3 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$ are a basis for $\mathbb{R}^3$. In applying the Gram-Schmidt process to construct from these vectors an orthonormal basis for $\mathbb{R}^3$, suppose you have already found that $\vec{u}_1 = \begin{bmatrix} 1/3 & 2/3 & -2/3 \end{bmatrix}^T$ and $\vec{u}_2 = \begin{bmatrix} 2/3 & 1/3 & 2/3 \end{bmatrix}^T$. Show how the process is completed to find $\vec{u}_3$. **You do not need to verify that the given values of $\vec{u}_1$ and $\vec{u}_2$ are correct.**

Solutions to Exam 4

① (a) (6 pts) $\langle f,g\rangle = \int_0^1 2x\cdot x^2\,dx = \int_0^1 2x^3\,dx$
$= \frac{2x^4}{4}\Big|_0^1 = \frac{2}{4} = \frac{1}{2}$

(b) (6 pts) $\langle g,g\rangle = \int_0^1 x^2\cdot x^2\,dx = \int_0^1 x^4\,dx$
$= \frac{x^5}{5}\Big|_0^1 = \frac{1}{5}$ $\quad \|g\| = \sqrt{\langle g,g\rangle} = \frac{1}{\sqrt{5}}$

(c) (6 pts) $\vec{p} = \frac{\langle f,g\rangle}{\langle g,g\rangle} g = \frac{1/2}{1/5} x^2 = \frac{5}{2}x^2$

② (a) (8 pts) $\|\vec u_1\| = \sqrt{\frac12+\frac12+0} = \sqrt{1} = 1$

$\|\vec u_2\| = \sqrt{\frac49+\frac49+\frac19} = \sqrt{1} = 1$

$\vec u_1\cdot\vec u_2 = \frac{2}{3\sqrt2} - \frac{2}{3\sqrt2} + 0 = 0$

$\vec u_1 \perp \vec u_2$ and $\|\vec u_1\| = \|\vec u_2\| = 1$ so orthonormal

(b) (8 pts) $\vec p = \langle \vec v,\vec u_1\rangle \vec u_1 + \langle \vec v,\vec u_2\rangle \vec u_2$

$= \left(\frac{3}{\sqrt2} - \frac{2}{\sqrt2}\right) u_1 + \left(2 + \frac43 - \frac13\right) u_2$

$= \frac{1}{\sqrt2}\left(\frac{1}{\sqrt2}\ \ -\frac{1}{\sqrt2}\ \ 0\right)^T + 3\left(\frac23\ \ \frac23\ \ \frac13\right)^T$

$= \left(\frac12\ \ -\frac12\ \ 0\right)^T + (2\ \ 2\ \ 1)^T = \left(\frac52\ \ \frac32\ \ 1\right)^T$

(c) (8 pts) $\vec v = \langle \vec v,\vec u_1\rangle \vec u_1 + \langle \vec v,\vec u_2\rangle \vec u_2 + \langle v,u_3\rangle u_3 = \frac{1}{\sqrt2}u_1 + 3\vec u_2 + \left(\frac{3}{3\sqrt2} + \frac{2}{3\sqrt2} + \frac{4}{3\sqrt2}\right)u_3$

$= \frac{1}{\sqrt2}\vec u_1 + 3\vec u_2 + \frac{3}{\sqrt2}\vec u_3$

③ (9 pts) $\|f(x)\| = \sqrt{9+1+1+25}$
$= \sqrt{36} = 6$

④ (10 pts) $|A-\lambda I| = \begin{vmatrix} 3-\lambda & -1 \\ 5 & 1-\lambda \end{vmatrix}$

$= (3-\lambda)(1-\lambda) + 5 = (3-4\lambda+\lambda^2)+5$

$= \lambda^2 - 4\lambda + 8 = 0$

$\lambda = \frac{4\pm\sqrt{16-32}}{2} = \frac{4\pm\sqrt{-16}}{2} = \frac{4\pm 4i}{2} = 2\pm 2i$

⑤ (10 pts) $A-2I = \begin{bmatrix} 2 & -5 & 1 \\ 1 & -2 & -1 \\ 0 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 \\ 2 & -5 & 1 \\ 0 & 1 & -3 \end{bmatrix}$

$\xrightarrow{-2r_1+r_2} \begin{bmatrix} 1 & -2 & -1 \\ 0 & -1 & +3 \\ 0 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

⑤ (continued) $\xrightarrow{2r_2+r_1} \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

$x_3 = \alpha \quad x_2 = 3\alpha \quad x_1 = 7\alpha$

$(x_1, x_2, x_3)^T = (7\alpha, 3\alpha, \alpha)^T = \alpha(7, 3, 1)^T$

basis

⑥ (a) (7 pts) It means that there is a nonzero vector $\vec x$ such that $A\vec x = \lambda\vec x$.

(b) (7 pts) It means that $\vec x$ is a nonzero vector such that $A\vec x = \lambda\vec x$.

⑦ (15 pts) $\vec p = \langle v_3,u_1\rangle u_1 + \langle v_3,u_2\rangle u_2$

$= \left(\frac13 + \frac43 - \frac23\right)u_1 + \left(\frac23+\frac23+\frac23\right)u_2$

$= u_1 + 2u_2 = \left(\frac13\ \ \frac23\ \ -\frac23\right)^T + 2\left(\frac23\ \ \frac13\ \ \frac23\right)^T$

$= \left(\frac13\ \ \frac23\ \ -\frac23\right)^T + \left(\frac43\ \ \frac23\ \ \frac43\right)^T$

$= \left(\frac53\ \ \frac43\ \ \frac23\right)^T$

$\vec v_3 - \vec p = (1\ \ 2\ \ 1)^T - \left(\frac53\ \ \frac43\ \ \frac23\right)^T$

$= \left(-\frac23\ \ \frac23\ \ \frac13\right)^T$

$\vec u_3 = \frac{\vec v_3 - \vec p}{\|\vec v_3 - \vec p\|}$

$\|\vec v_3 - \vec p\| = \sqrt{\frac49 + \frac49 + \frac19} = 1$

$\vec u_3 = \left(-\frac23\ \ \frac23\ \ \frac13\right)^T$