**MH 2660 Exam 3**

1. (a) Find the coördinates of $\vec{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ with respect to the basis $\vec{u_1} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\vec{u_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for $\mathbb{R}^2$.

(b) If $\vec{v_1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{u_1}, \vec{u_2}$ are as in part (a), find the transition matrix from the basis $\vec{v_1}, \vec{v_2}$ to the basis $\vec{u_1}, \vec{u_2}$.

(c) **Use your answer to (b)** to find the coördinates of $\vec{x} = \vec{v_1} - 3\vec{v_2}$ with respect to $\vec{u_1}, \vec{u_2}$.

2. Let $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 2 & 4 & 1 & 1 \end{bmatrix}$.

(a) Find a basis for the nullspace $N(A)$ of $A$.
(b) Find a basis for the row space of $A$.
(c) Find a basis for the column space of $A$.

3. Define a transformation $L$ from $\mathbb{R}^2$ to $\mathbb{R}^3$ by $L(\vec{x}) = \begin{bmatrix} 3x_2 \\ 0 \\ x_1 \end{bmatrix}$.

(a) Show that $L$ is a linear transformation.
(b) Find the standard matrix representation of $L$.

4. Let $L$ be the transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that first rotates a vector $90^o$ counterclockwise and then reflects it about the $x_1$-axis. Find the standard matrix representation of $L$.

5. Let $\vec{u} = \begin{bmatrix} 1 & -2 & 0 & -2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$. Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

6. Find the point on the line $y = 4x$ which is closest to the point $(2, 1)$.

7. Let S be a subspace of the vector space V. What is meant by the "orthogonal complement" of S?

8. Let $S$ be the subspace of $\mathbb{R}^3$ spanned by $\vec{v_1} = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix}^T$, $\vec{v_2} = \begin{bmatrix} 2 & 3 & 2 \end{bmatrix}^T$. Find a basis for the orthogonal complement $S^\perp$ of $S$.

9. Find the least squares solution to the system

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_2 &= 1 \\ 2x_1 - x_2 &= 0 \end{aligned}$$

<u>Formulae that you may use if you wish:</u>

(1) If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $A^{-1} = 1/\det A \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$.

(2) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates a vector in the plane by angle $\theta$ counterclockwise.

# Solutions to Exam 3

① (a) (8 pts) $U = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \Rightarrow U^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

$U^{-1}\vec{b} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$

(b) (6 pts) $U^{-1}V = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -8 & -1 \end{bmatrix}$

(c) (6 pts) $\begin{bmatrix} 5 & 1 \\ -8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

② (a) (10 pts) $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 2 & 4 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -1 & -3 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad x_2 = \alpha,\ x_4 = \beta$

$x_1 + 2x_2 - x_4 = 0 \Rightarrow x_1 = -2\alpha + \beta$

$x_3 + 3x_4 = 0 \rightarrow x_3 = -3\beta$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2\alpha + \beta \\ \alpha \\ -3\beta \\ \beta \end{bmatrix} = \begin{bmatrix} -2\alpha \\ \alpha \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \beta \\ 0 \\ -3\beta \\ \beta \end{bmatrix} = \alpha\underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \end{bmatrix}}_{\text{basis for } N(A)}$$

(b) (4 pts) $[1\ 2\ 0\ -1]$, $[0\ 0\ 1\ 3]$

(c) (4 pts) Leading 1's in columns 1 & 3.

Basis: $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

③ (a) (10 pts) $L(\alpha\vec{x} + \beta\vec{y}) = L\left(\begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{bmatrix}\right)$

$= \begin{bmatrix} 3(\alpha x_2 + \beta y_2) \\ 0 \\ \alpha x_1 + \beta y_1 \end{bmatrix} = \begin{bmatrix} 3\alpha x_2 + 3\beta y_2 \\ 0 \\ \alpha x_1 + \beta y_1 \end{bmatrix}$

$= \alpha\begin{bmatrix} 3x_2 \\ 0 \\ x_1 \end{bmatrix} + \beta\begin{bmatrix} 3y_2 \\ 0 \\ y_1 \end{bmatrix} = \alpha L(\vec{x}) + \beta L(\vec{y})$

(b) (6 pts) $L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$

④ (6 pts) $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{\text{rotate}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{\text{reflect}} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{\text{rotate}} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \xrightarrow{\text{reflect}} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

⑤ (6 pts) $u \cdot v = \|u\|\|v\| \cos\theta$

$u \cdot v = 1 - 2 + 0 - 2 = -3$. $\|u\| = \sqrt{1+4+4} = \sqrt{9} = 3$

$\|v\| = \sqrt{1+1+1+1} = \sqrt{4} = 2$

$-3 = 3 \cdot 2\cos\theta \Rightarrow \cos\theta = -\frac{1}{2}$

⑥ (6 pts) Let $\vec{x} = (2,1)$. Pick $\vec{y} = (1,4)$ on the line. Then $\vec{p} = \frac{\vec{x}\cdot\vec{y}}{\vec{y}\cdot\vec{y}}\vec{y}$

$= \frac{6}{17}(1,4) = \left(\frac{6}{17}, \frac{24}{17}\right)$

⑦ (6 pts) The orthogonal complement of $S$ is the set of all vectors $\vec{x}$ in $V$ that are orthogonal to every vector in $S$.

⑧ (10 pts) Let $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & 2 \end{bmatrix}$. Then

$S^{\perp} = N(A)$. $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix}$

$x_3 = \alpha \quad x_1 = 2\alpha \quad x_2 = -2\alpha$

$(2\alpha, -2\alpha, \alpha) = \alpha(2,-2,1)$

Basis is $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

⑨ (10 pts) System is $A\vec{x} = \vec{b}$, where

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

$A^TA = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -1 & 3 \end{bmatrix}$

$A^T\vec{b} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

We need to solve $\begin{bmatrix} 5 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\left[\begin{array}{cc|c} 5 & -1 & 2 \\ -1 & 3 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -1 & 3 & 3 \\ 5 & -1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -3 & -3 \\ 5 & -1 & 2 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & -3 & -3 \\ 0 & 14 & 17 \end{array}\right]$ $\quad x_1 - 3x_2 = -3$, $\quad 14x_2 = 17$

$x_2 = \frac{17}{14} \qquad x_1 = -3 + 3x_2 = -3 + \frac{51}{14}$

$x_1 = -\frac{42}{14} + \frac{51}{14} = \frac{9}{14}$

$x_1 = \frac{9}{14}, \ x_2 = \frac{17}{14}$