MH 2660 Exam 2

Give reasoning/show work on all problems except definitions and True-False.

1. Determine whether or not the following sets of vectors are subspaces of $\mathbb{R}^3$:
(a) The set S of all $\vec{x} = (x_1, x_2, x_3)^T$ in $\mathbb{R}^3$ such that $x_3 = 2x_2$.
(b) The set S of all $\vec{x}$ in $\mathbb{R}^3$ such that $x_1 = 0$ or $x_3 = 0$.

2. Let S be the set of all matrices A such that $a_{11} = 0$ and $a_{22} = 0$. Is S a subspace of $\mathbb{R}^{2\times 2}$?

3. Let $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, and $\vec{v_4} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(a) Do $\vec{v_1}, \vec{v_2}$ span $\mathbb{R}^3$? Are they linearly independent?
(b) Do $\vec{v_1}, \vec{v_2}, \vec{v_3}$ span $\mathbb{R}^3$? Are they linearly independent?
(c) What is the dimension of Span(v_1, v_2, v_3)?
(d) Do $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$ span $\mathbb{R}^3$? Are they linearly independent?
(e) Are v_1, v_2, v_4 a basis for $\mathbb{R}^3$?

4. (a) What does it mean to say the vectors $\vec{v_1}, \vec{v_2}, ..., \vec{v_n}$ span a vector space V?
(b) What does it mean to say the vectors $\vec{v_1}, \vec{v_2}, ..., \vec{v_n}$ are linearly independent?

5. True or False:
(a) The set of all 4×6 matrices with the usual scalar multiplication and matrix addition is a vector space.
(b) Four vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$ in $\mathbb{R}^3$ can never span $\mathbb{R}^3$.
(c) If $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly independent in $\mathbb{R}^3$, then they must also span $\mathbb{R}^3$.
(d) Every basis for $\mathbb{R}^3$ must have exactly 3 vectors.
(e) The set of all functions f in $C[0,1]$ such that $f(0) = 1$ and $f(1) = 1$ is a subspace of $C[0,1]$.

6. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix}$. Determine whether or not A, B, and C are linearly independent in the vector space $\mathbb{R}^{2\times 2}$ of all 2×2 matrices.

Solutions to Exam 2, MH2660

(1) (a) (6 pts) Let $\vec{x}$ be in S, and α a scalar,

$$\alpha\vec{x} = \begin{bmatrix}\alpha x_1\\ \alpha x_2\\ \alpha x_3\end{bmatrix} = \begin{bmatrix}\alpha x_1\\ \alpha x_2\\ \alpha(2x_2)\end{bmatrix} = \begin{bmatrix}\alpha x_1\\ \alpha x_2\\ 2(\alpha x_2)\end{bmatrix} \in S$$

Closed under scalar mult. Check addition: Let $\vec{x}$ & $\vec{y}$ be in S.

$$\vec{x}+\vec{y} = \begin{bmatrix}x_1+y_1\\ x_2+y_2\\ x_3+y_3\end{bmatrix} = \begin{bmatrix}x_1+y_1\\ x_2+y_2\\ 2x_2+2y_2\end{bmatrix} = \begin{bmatrix}x_1+y_1\\ x_2+y_2\\ 2(x_2+y_2)\end{bmatrix} \in S$$

Closed under addition. Yes, it is a subspace

(b) (6 pts) Not a subspace because: $\vec{x}=\begin{bmatrix}0\\1\\1\end{bmatrix}$ and $\vec{y}=\begin{bmatrix}1\\1\\0\end{bmatrix}$ are in S, but $\vec{x}+\vec{y}=\begin{bmatrix}1\\2\\1\end{bmatrix}$ is not. Not closed under addition.

(2) (5 pts) S is all 2x2 matrices of the form $\begin{bmatrix}0 & a_{12}\\ a_{21} & 0\end{bmatrix}$. $\alpha\begin{bmatrix}0 & a_{12}\\ a_{21} & 0\end{bmatrix} = \begin{bmatrix}0 & \alpha a_{12}\\ \alpha a_{21} & 0\end{bmatrix}$ is in S. Closed under scalar mult.

$$\begin{bmatrix}0 & a_{12}\\ a_{21} & 0\end{bmatrix} + \begin{bmatrix}0 & b_{12}\\ b_{21} & 0\end{bmatrix} = \begin{bmatrix}0 & a_{12}+b_{12}\\ a_{21}+b_{21} & 0\end{bmatrix} \in S$$

Closed under addition. Is a subspace

(3) (a) (10 pts) Don't span: 2 vectors can't span $\mathbb{R}^3$. Are indep.: neither is a scalar mult. of the other.

(b) (10 pts) $\begin{vmatrix}1 & 2 & 0\\ 1 & 0 & 2\\ 0 & 1 & -1\end{vmatrix} = \begin{vmatrix}0 & 2\\ 1 & -1\end{vmatrix} - 2\begin{vmatrix}1 & 2\\ 0 & -1\end{vmatrix}$

$= (-2) - 2(-1) = -2+2 = 0.$

Det $=0$, so they do not span and are not lin. indep.

(c) (5 pts) The dimension is 2 because: They don't span $\mathbb{R}^3$ so the dimension must be less than 3, and the first two are lin. indep. so they at least span a plane in $\mathbb{R}^3$ (which they do)

(d) (10 pts). They are not lin. indep.: 4 vectors in $\mathbb{R}^3$ are never indep.

(continued top of next column)

(3) (d) (continued).

$$|\vec{v}_1\ \vec{v}_2\ \vec{v}_4| = \begin{vmatrix}1 & 2 & 1\\ 1 & 0 & 0\\ 0 & 1 & 1\end{vmatrix} = -\begin{vmatrix}2 & 1\\ 1 & 1\end{vmatrix} = -(2-1) = -1 \neq 0$$

$\vec{v}_1, \vec{v}_2, \vec{v}_4$ span, so $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ span too

(e) (5 pts). By 3(d) above, $\det[v_1\ v_2\ v_4] \neq 0$ so they span and are lin. indep. Yes, they are a basis for $\mathbb{R}^3$

(4) (a) (6 pts) It means that every vector in V is some linear combination of $\vec{v}_1, \dots, \vec{v}_n$.

(b) (6 pts) It means that no one of $\vec{v}_1, \dots, \vec{v}_n$ is a lin. combination of the others. Equivalently, the only sol'n of $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \dots + \alpha_n\vec{v}_n = 0$ is $\alpha_1=0, \alpha_2=0, \dots, \alpha_n=0$

(5) (5 pts each) (a) T (b) F (c) T (d) T (e) F

(6) They are indep. if the only sol'n of $\alpha_1 A + \alpha_2 B + \alpha_3 C = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ is $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$.

$$\alpha_1\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \alpha_2\begin{bmatrix}2 & 0\\ -1 & 1\end{bmatrix} + \alpha_3\begin{bmatrix}3 & 0\\ 2 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$2\alpha_2 + 3\alpha_3 = 0$ ← upper left entries

$\alpha_1 = 0$ ← upper right

$\alpha_1 - \alpha_2 + 2\alpha_3 = 0$ ← lower left

$\alpha_2 = 0$ ← lower right.

We have $\alpha_1 = 0$ & $\alpha_2 = 0$,

From top equation, $2\alpha_2 + 3\alpha_3 = 0$

$3\alpha_3 = -2\alpha_2 = -2\cdot 0 = 0$

$\alpha_3 = 0$

So $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$ is the only sol'n.

They are linearly independent

(6 pts for this one)