TWO-TERM TRACE ESTIMATES FOR RELATIVISTIC STABLE PROCESSES

RODRIGO BAÑUELOS, JEBESSA MIJENA, AND ERKAN NANE

Abstract. We prove trace estimates for the relativistic $\alpha$-stable process extending the result of Bañuelos and Kulczycki (2008) in the stable case.

1. Introduction and statement of main results

In Ryznar [13] Green function estimates of the Schödinger operator with the free Hamiltonian of the form

$$(−\Delta + m^{2/\alpha})^{\alpha/2} − m,$$

were investigated, where $m > 0$ and $\Delta$ is the Laplace operator acting on $L^2(\mathbb{R}^d)$. Some of these estimates (see Lemma 2.6 below) and (essentially the same) proof in Bañuelos and Kulczycki (2008) can be used to provide an extension of the asymptotics in [3] to the relativistic $\alpha$ stable processes for any $0 < \alpha < 2$.

An $\mathbb{R}^d$-valued process with independent, stationary increments having the following characteristic function

$$\mathbb{E}e^{i\xi \cdot X^{\alpha,m}_t} = e^{-t\{(m^{2/\alpha}|\xi|^2)\alpha/2 − m\}}, \, \xi \in \mathbb{R}^d$$

is called relativistic $\alpha$-stable process. We assume that sample paths of $X^{\alpha,m}_t$ are right continuous and have left-hand limits a.s. If we put $m = 0$ we obtain the symmetric rotation invariant $\alpha$-stable process with the characteristic function $e^{-t|\xi|^\alpha}, \xi \in \mathbb{R}^d$. We refer to this process as standard $\alpha$-stable process. For the rest of the paper we keep $\alpha, m$ and $d \geq 2$ fixed and drop $\alpha, m$ in the notation, when it does not lead to confusion. Hence from now on the relativistic $\alpha$-stable process is denoted by $X_t$ and its standard $\alpha$-stable counterpart by $\tilde{X}_t$. We keep this notational convention consistently throughout the paper, e.g., if $p_t(x − y)$ is the transition density of $X_t$ then $\tilde{p}_t(x − y)$ is the transition density of $\tilde{X}_t$.

Brownian motion has characteristic function

$$\mathbb{E}^0 e^{i\xi \cdot B_t} = e^{-t|\xi|^2}, \, \xi \in \mathbb{R}^d$$

Let $\alpha = 2\beta$. Ryznar showed that $X_t$ is subordinated to Brownian motion. Let $T_\beta(t), \, t > 0$, denote the strictly $\beta$-stable subordinator with the following Laplace transform

$$\mathbb{E}^0 e^{-\lambda T_\beta(t)} = e^{-t\lambda^\beta}, \, \lambda > 0.$$
Let \( \theta_\beta(t, u), \ u > 0 \), denote the density function of \( T_\beta(t) \). Then the process \( B_{T_\beta(t)} \) is the standard symmetric \( \alpha \)-stable process.

Ryznar [13, Lemma 1] showed that we can obtain \( X_t = B_{T_\beta(t, m)} \), where \( T_\beta(t, m) \) is a positive infinitely divisible process with stationary increments with probability density function

\[
\theta_\beta(t, u, m) = e^{-m^{1/\beta}u + mt}\theta_\beta(t, u), \quad u > 0.
\]

Transition density of \( T_\beta(t, m) \) is given by \( \theta_\beta(t, u - v, m) \). Hence the transition density of \( X_t \) is given by

\[
p(t, x) = e^{mt} \int_0^\infty \frac{1}{(4\pi u)^{d/2}} e^{-|x|^2/4u} e^{-m^1/\beta}u \theta_\beta(t, u) du.
\]

Then \( p(t, x, y) = p(t, x - y) \) Since the transition density is obtained from the characteristic function by inverse Fourier transform, it follows that \( p(t, x) \) is a radially symmetric decreasing function and that

\[
p(t, x) \leq p(t, 0) \leq e^{mt} \int_0^\infty \frac{1}{(4\pi u)^{d/2}} \theta_\beta(t, u) du = e^{mt - d/\alpha} \omega_d \Gamma(\alpha/d) \frac{(2\pi)^d}{\Gamma(d/\alpha)}
\]

where \( \omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \) is the surface area of the unit sphere in \( \mathbb{R}^d \). For \( A \subset \mathbb{R}^d \) we define the first exit time from \( A \) by \( \tau_A = \inf\{t \geq 0 : \ X_t \notin A\} \).

Let \( D \subset \mathbb{R}^d \) be a domain.

We set

\[
r_D(t, x, y) = \mathbb{E}^x[p(t - \tau_D, X_{\tau_D}, y); \tau_D < t]
\]

and

\[
p_D(t, x, y) = p(t, x, y) - r_D(t, x, y)
\]

for any \( x, y \in \mathbb{R}^d, \ t > 0 \). For a nonnegative Borel function \( f \) and \( t > 0 \), let

\[
P_t^D f(x) = \mathbb{E}^x[f(X_t) ; t < \tau_D] = \int_D p_D(t, x, y) f(y) dy
\]

be the semigroup of the killed process acting on \( L^2(D) \), see, Ryznar [13, Theorem 1].

Let \( D \) be a bounded domain (or of finite volume). Then the operator \( P_t^D \) maps \( L^2(D) \) into \( L^\infty(D) \) for every \( t > 0 \). This follows from (1.3), (1.4), and the general theory of heat semigroups as described in [10]. It follows that there exists an orthonormal basis of eigenfunctions \( \{ \varphi_n : n = 1, 2, 3, \cdots \} \) for \( L^2(D) \) and corresponding eigenvalues \( \{ \lambda_n : n = 1, 2, 3, \cdots \} \) of the generator of the semigroup \( P_t^D \) satisfying

\[
\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots
\]

with \( \lambda_n \to \infty \) as \( n \to \infty \). Hence the pair \( \{ \varphi_n, \lambda_n \} \) satisfies

\[
P_t^D \varphi_n(x) = e^{-\lambda_n t} \varphi_n(x), \quad x \in D, \ t > 0.
\]
Under such assumptions we have

\[ p_D(t, x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y) \]  

(1.6)

In this paper we are interested in the behavior of the trace of this semigroup

\[ Z_D(t) = \int_D p_D(t, x, x) dx. \]  

(1.7)

Because of (1.6) we can write (1.7) as

\[ Z_D(t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \int_D \varphi_n^2(x) dx = \sum_{n=1}^{\infty} e^{-\lambda_n t}. \]  

(1.8)

We denote \( d \)-dimensional volume of \( D \) by \( |D| \).

Our first result is the Weyl’s asymptotic for the eigenvalues of the relativistic Laplacian

**Proposition 1.1.**

\[ \lim_{t \to 0} t^{d/\alpha} e^{-m t} Z_D(t) = C_1 |D| \]  

(1.9)

where \( C_1 = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^{d/2}} \).

Let \( N(\lambda) \) be the number of eigenvalues \( \{\lambda_j\} \) which do not exceed \( \lambda \), it follows from (1.9) and the classical Tauberian theorem (see for example [11], p.445 Theorem 2) where \( L(t) = C_1 |D| e^{m/t} \) is our slowly varying function at infinity that

\[ \lim_{\lambda \to \infty} \lambda^{-d/\alpha} e^{-m/\lambda} N(\lambda) = \frac{C_1 |D|}{\Gamma(1+d/\alpha)} \]  

(1.10)

This is the analogue for relativistic stable process of the celebrated Weyl’s asymptotic formula for the eigenvalues of the Laplacian.

**Remark 1.2.** The first author presented (1.10) at a conference in Vienna at the Schrödinger Institute in 2009 ((see [1]) and at the 34th conference in stochastic processes and their applications in Osaka in 2010 (see [2]). Thus this result has been known for to the authors, and perhaps others, for number of years.

Our goal in this paper is to obtain the second term in the asymptotics of \( Z_D(t) \) under some additional assumptions on the smoothness of \( D \). Our result is inspired by result for Trace estimates for stable processes by Bañuelos and Kulczycki [3].

To state our main result we need the following definition of the domain \( D \).

**Definition 1.3.** The boundary, \( \partial D \), of an open set \( D \) in \( \mathbb{R}^d \) is said to be \( \mathcal{R} \)--smooth if for each point \( x_0 \in \partial D \) there are two open balls \( B_1 \) and \( B_2 \) with radii \( \mathcal{R} \) such that \( B_1 \subset D, B_2 \subset \mathbb{R}^d \setminus (D \cup \partial D) \) and \( \partial B_1 \cap \partial B_2 = x_0 \).
Theorem 1.4. Let \( D \subset \mathbb{R}^d, d \geq 2 \), be an open bounded set with \( R - \) smooth boundary. Let \(|D|\) denote the volume \((d-\)dimensional Lebesgue measure) of \( D \) and \( \partial D \) denote its surface area \(((d-1)-\)dimensional Lebesgue measure) of its boundary. Suppose \( \alpha \in (0, 2) \). Then

\[
|Z_D(t) - \frac{C_1(t)e^{mt}|D|}{td^\alpha} + C_2(t)|\partial D| |D|^t^{2/\alpha} \left( \frac{R^2}{t^d/\alpha} \right), \quad t > 0,
\]

where

\[
C_1(t) = \frac{1}{(4\pi)^{d/2}} \int_0^\infty z^{-d/2} e^{-(mt)^{1/\alpha}} \theta(1, z) dz \rightarrow \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^{d/\alpha}} \quad \text{as } t \to 0,
\]

\[
C_2(t) = \int_0^\infty r_H(t, (x_1, 0, \ldots, 0), (x_1, 0, \ldots, 0)) dx_1 \leq \frac{C_4 e^{2mt} t^{1/\alpha}}{td^\alpha}, \quad t > 0
\]

\[
C_3 = C_3(d, \alpha), H = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 > 0 \text{ and } r_H \text{ is given by (1.4)}
\]

When \( m = 0, 0 < \alpha \leq 2 \) our result becomes for bounded domains with \( R - \) smooth boundary

\[
|Z_D(t) - \frac{C_5|D|}{td^\alpha} + C_4|\partial D|^{1/\alpha} |D|^{t^{2/\alpha}} \left( \frac{R^2}{t^d/\alpha} \right), \quad t > 0
\]

where \( c_5 = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^{d/\alpha}} \), \( C_4 \) as in Theorem 1.4. This was established by Bañuelos and Kulczycki [3] recently.

The asymptotic for the trace of the heat kernel when \( \alpha = 2 \) (the case of the Laplacian with Dirichlet boundary condition in a domain of \( \mathbb{R}^d \)), have been extensively studied by many authors. The van den Berg [5] result states that under the \( R - \) smoothness condition when \( \alpha = 2 \),

\[
|Z_D(t) - (4\pi t)^{-d/2} \left( |D| - \frac{\sqrt{\pi t}}{2} |\partial D| \right) \left( \frac{R^2}{t^{d/\alpha}} \right), \quad t > 0.
\]

For domains with \( C^1 \) boundaries the result

\[
Z_D(t) = (4\pi t)^{-d/2} \left( |D| - \frac{\sqrt{\pi t}}{2} |\partial D| + o(t^{1/2}) \right)
\]

was proved by Brossard and Carmona [8].
2. Preliminaries

Next we introduce some notations. For \( x \in \mathbb{R}^d \), let \( \delta_D(x) \) denote the Euclidean distance between \( x \) and \( \partial D \) and the ball in \( \mathbb{R}^d \) center at \( x \) and radius \( r \), \( \{ y : |y-x| < r \} \) will be denoted by \( B(x,r) \). Define

\[
\psi(\theta) = \int_0^\infty e^{-v^{p-1/2}(\theta + v/2)^{p-1/2}} dv, \theta \geq 0,
\]

where \( p = (d + \alpha)/2 \). We put \( R(\alpha,d) = A(-\alpha,d)/\psi(0) \), where \( A(v,d) = (\Gamma((d-v)/2))/((\pi^{d/2}|\Gamma(v/2)|)) \). Let \( \nu(x) \), \( \tilde{\nu}(x) \) be the densities of the Lévy measures of the relativistic \( \alpha \)-stable process and the standard \( \alpha \)-stable process, respectively. These densities, are given by

\begin{align}
\nu(x) &= \frac{R(\alpha,d)}{|x|^{d+\alpha}} e^{-m^{1/\alpha}|x|/\psi(m^{1/\alpha}|x|)} \\
\tilde{\nu}(x) &= \frac{A(-\alpha,d)}{|x|^{d+\alpha}}
\end{align}

We need the following estimate of the transition probabilities of the process \( X_t \) which is given in ([14], Lemma 2.2): For any \( x, y \in \mathbb{R}^d \) and \( t > 0 \) there exist constants \( c_1 > 0 \) and \( c_2 > 0 \),

\[
p(t,x,y) \leq c_1 e^{mt} \min \left\{ \frac{t}{|x-y|^{d+\alpha}} e^{-c_2|x-y|}, t^{-d/\alpha} \right\}
\]

We will use the fact([7], Lemma 6) that if \( D \subset \mathbb{R}^d \) is an open bounded set satisfying a uniform outer cone condition, then \( P^x(X(\tau_D) \in \partial D) = 0 \) for all \( x \in D \). For the open bounded set \( D \) we will be denoted by \( G_D(x,y) \) the Green function for the set \( D \) equal to

\[
G_D(x,y) = \int_0^\infty p_D(t,x,y) dt, x, y \in \mathbb{R}^d
\]

and for any such \( D \) the expectation of the exit time of the processes \( X_t \) from \( D \) is given by the integral of the Green function over the domain. That is,

\[
E^x(\tau_D) = \int_D G_D(x,y) dy.
\]

**Lemma 2.1.** Let \( D \subset \mathbb{R}^d \) be an open set. For any \( x, y \in D \) we have

\[
r_D(t,x,y) \leq c_1 e^{mt} \left( \frac{t}{\delta_D^{d+\alpha}(x)} e^{-c_2 \delta_D(x) \wedge t^{-d/\alpha}} \right)
\]
Proof. Using (1.4) and (2.3) we have
\[ r_D(t, x, y) = E_y(p(t - \tau_D, X(\tau_D), x); \tau_D < t) \]
\[ \leq c_1 e^{mt} E_y \left( \frac{t}{|x - X(\tau_D)|^{d+\alpha}} e^{-c_2|x - X(\tau_D)|} \wedge t^{-d/\alpha} \right) \]
\[ \leq c_1 e^{mt} \left( \frac{t}{\delta_D(x)} e^{-c_2\delta_D(x)} \wedge t^{-d/\alpha} \right) \]
\[ \square \]

We need the following result for the proof of Proposition 1.1.

Lemma 2.2.
\[ (2.4) \lim_{t \to 0} p(t, 0) e^{-mt} t^{d/\alpha} = C_1 \]
where
\[ C_1 = \left( \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^{d/\alpha}} \right) \int_0^{\infty} u^{-d/2} \theta_\beta(1, u) du = (4\pi)^{d/2} \int_0^\infty u^{-d/2} \theta_\beta(1, u) du \]

Proof. By (1.2) we have
\[ p(t, x, x) = p(t, 0) = e^{mt} \int_0^{\infty} \frac{1}{(4\pi u)^{d/2}} e^{-z^{1/\beta} u} \theta_\beta(1, u) du \]

Now using the scaling of stable subordinator \( \theta_\beta(t, u) = t^{-1/\beta} \theta_\beta(1, ut^{-1/\beta}) \) and a change of variables we get
\[ p(t, 0) = \frac{e^{mt}}{(4\pi)^{d/2} t^{d/\alpha}} \int_0^{\infty} z^{-d/2} e^{-z^{1/\beta} t^{1/\beta} \theta_\beta(1, z)} dz \]
then by dominated convergence theorem we obtain
\[ \lim_{t \to 0} p(t, 0) e^{-mt} t^{d/\alpha} = \frac{1}{(4\pi)^{d/2}} \int_0^{\infty} z^{-d/2} \theta_\beta(1, z) dz \]
and this last integral is equal to the density of \( \alpha \)-stable process at time 1 and \( x = 0 \) which was calculated in [3] to be
\[ \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^{d/\alpha}}. \]
\[ \square \]

We next give the proof of Proposition 1.1.

Proof of Proposition 1.1. By (1.4) we see that
\[ p_D(t, x, x) = \frac{p(t, 0)}{C_1 e^{mt} t^{-d/\alpha}} - \frac{r_D(t, x, x)}{C_1 e^{mt} t^{-d/\alpha}}. \]

(2.5)
Since the limit of the first term tend to 1 as $t \to 0$ by (2.4), in order to prove (1.9), we must show that
\begin{equation}
\frac{t^{d/\alpha}}{C_1e^{mt}} \int_D r_D(t, x, x) dx \to 0, \text{ as } t \to 0.
\end{equation}

For $0 < t < 1$, consider the subdomains $D_t = \{x \in D : \delta_D(x) \geq t^{1/2\alpha}\}$ and its complement $D_t^C = \{x \in D : \delta_D(x) < t^{1/2\alpha}\}$. By Lebesgue dominated convergence Theorem and recalling that $|D| < \infty$ we get $|D_t^C| \to 0$ as $t \to 0$. Since $p_D(t, x, x) \leq p(t, x, x)$, by (1.3) we see that
\begin{equation}
\int_{D_t^C} r_D(t, x, x) dx \to 0, \text{ as } t \to 0.
\end{equation}

On the other hand, by Lemma 2.2 in [14] we obtain
\begin{align*}
\frac{t^{d/\alpha}}{C_1e^{mt}} \int_{D_t^C} r_D(t, x, x) dx & \leq c e^{t^{1+d/\alpha} \min \left\{ t^{1+d/\alpha}/\delta_D(x), 1 \right\}} \\
& \leq c \min \left\{ t^{1+d/\alpha}/\delta_D(x), 1 \right\}.
\end{align*}

For $x \in D_t$ and $0 < t < 1$, the right hand side of (2.8) is bounded above by $ct^{d(2\alpha+1)/2}$ and hence
\begin{equation}
\frac{t^{d/\alpha}}{C_1e^{mt}} \int_{D_t} r_D(t, x, x) dx \leq ct^{d(2\alpha+1)/2} |D|
\end{equation}
and this last quantity goes to 0 as $t \to 0$. \qed

For an open set $D \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, the distribution $P^x(\tau_D < \infty, X(\tau_D) \in \cdot)$ will be called the relativistic $\alpha$–harmonic measure for $D$. The following Ikeda-Watanabe formula recovers the relativistic $\alpha$–harmonic measure for the set $D$ from the Green function.

**Proposition 2.3 ([14]).** Assume that $D$ is an open, nonempty, bounded subset of $\mathbb{R}^d$, and $A$ is a Borel set such that $\text{dist}(D, A) > 0$. Then
\begin{equation}
P^x(X(\tau_D) \in A, \tau_D < \infty) = \int_D G_D(x, y) \int_A v(y - z) dz dy, x \in D
\end{equation}
Here we need the following generalization already stated and used in [3].
Proposition 2.4. [14, Proposition 2.5] Assume that \( D \) is an open, nonempty, bounded subset of \( \mathbb{R}^d \), and \( A \) is a Borel set such that \( A \subset D^c \setminus \partial D \) and \( 0 \leq t_1 < t_2 < \infty, x \in D \). Then we have

\[
P^x(X(\tau_D) \in A, t_1 < \tau_D < t_2) = \int_D \int_{t_1}^{t_2} p_D(s, x, y) ds \int_A v(y - z) dz dy.
\]

The following proposition holds for a large class of Lévy processes

Proposition 2.5. [3, Proposition 2.3] Let \( D \) and \( F \) be open sets in \( \mathbb{R}^d \) such that \( D \subset F \). Then for any \( x, y \in \mathbb{R}^d \) we have

\[
p_F(t, x, y) - p_D(t, x, y) = E^x(\tau_D < t, X(\tau_D) \in F \setminus D; p_F(t - \tau_D, X(\tau_D), y))
\]

Lemma 2.6. [13, Lemma 5] Let \( D \subset \mathbb{R}^d \) be an open set. For any \( x, y \in D \) and \( t > 0 \) the following estimates hold;

\[
(2.11)
\]

\[
p_D(t, x, y) \leq e^{mt} \tilde{p}_D(t, x, y)
\]

\[
r_D(t, x, y) \leq e^{2mt} \tilde{r}_D(t, x, y)
\]

We need the following lemma given by Van den Berg in [5].

Lemma 2.7. [5, Lemma 5] Let \( D \) be an open bounded set in \( \mathbb{R}^d \) with \( R \)-smooth boundary \( \partial D \) and define for \( 0 \leq q < R \)

\[
D_q = \{ x \in D : \delta_D(x) > q \}
\]

and denote the area of its boundary \( \partial D_q \) by \( |\partial D_q| \). Then

\[
(2.12)
\]

\[
\left( \frac{R - q}{R} \right)^{d-1} |\partial D| \leq |\partial D_q| \leq \left( \frac{R}{R - q} \right)^{d-1} |\partial D|, 0 \leq q < R.
\]

Corollary 2.8. ([3], Corollary 2.14) Let \( D \) be an open bounded set in \( \mathbb{R}^d \) with \( R \)-smooth boundary. For any \( 0 < q \leq R \) we have

(i)

\[
2^{-d+1} |\partial D| \leq |\partial D_q| \leq 2^{d-1} |\partial D|,
\]

(ii)

\[
|\partial D| \leq \frac{2^d |D|}{R},
\]

(iii)

\[
|\partial D_q| - |\partial D| \leq \frac{2^d dq |\partial D|}{R \leq \frac{2^{2d} dq |D|}{R^2}}.
\]
3. Proof of main result

**Proof of Theorem 1.4.** For the case \( t^{1/\alpha} > R/2 \) the theorem holds trivially. This is true because for such \( t \)'s we have by Equation (1.3)

\[
Z_D(t) \leq \int_D p(t, x, x) dx \leq \frac{c_1e^{mt}|D|}{t^{d/\alpha}} \leq \frac{c_1e^{mt}|D|t^{2/\alpha}}{R^{2t^{d/\alpha}}}
\]

By Corollary 2.8 and Lemma 2.6 we also have

\[
C_2(t)|\partial D| \leq \frac{C_4e^{2mt}|\partial D|t^{1/\alpha}}{t^{d/\alpha}} \leq \frac{2^dC_4e^{2mt}|D|t^{1/\alpha}}{R^{t^{d/\alpha}}} \leq \frac{2^{d+1}C_4e^{2mt}|D|t^{2/\alpha}}{R^{2t^{d/\alpha}}}
\]

\[
C_1(t)e^{mt}|D| \leq \frac{C_1(t)e^{mt}|D|t^{2/\alpha}}{t^{d/\alpha}} \leq \frac{C_1(t)e^{mt}|D|t^{2/\alpha}}{R^{2t^{d/\alpha}}}
\]

Therefore for \( t^{1/\alpha} > R/2 \) (1.11) holds. Here and in sequel we consider the case \( t^{1/\alpha} \leq R/2 \). From (1.5) and the fact that \( p(t, x, x) = \frac{C_1(t)e^{mt}}{t^{d/\alpha}} \), we have that

\[
Z_D(t) - \frac{C_1(t)e^{mt}|D|}{t^{d/\alpha}} = \int_D p_D(t, x, x) dx - \int_D p(t, x, x) dx
\]

\[
= -\int_D r_D(t, x, x) dx,
\]

where \( C_1(t) \) is as stated in the theorem. Therefore we must estimate (3.1). We break our domain into two pieces, \( D_{R/2} \) and its complement. We will first consider the contribution of \( D_{R/2} \).

**Claim 1:**

\[
\int_{D_{R/2}} r_D(t, x, x) dx \leq \frac{ce^{2mt}|D|t^{2/\alpha}}{R^{2t^{d/\alpha}}}
\]

for \( t^{1/\alpha} \leq R/2 \).

**Proof of Claim 1:** By Lemma 2.6 we have

\[
\int_{D_{R/2}} r_D(t, x, x) dx \leq e^{2mt} \int_{D_{R/2}} \tilde{r}_D(t, x, x) dx,
\]

and by scaling of the stable density the right hand side of (3.3) equals

\[
\frac{e^{2mt}}{t^{d/\alpha}} \int_{D_{R/2}} \tilde{r}_{D/t^{1/\alpha}}(1, \frac{x}{t^{1/\alpha}}, \frac{x}{t^{1/\alpha}}) dx.
\]

For \( x \in D_{R/2} \) we have \( \delta_{D/t^{1/\alpha}}(x/t^{1/\alpha}) \geq R/(2t^{1/\alpha}) \geq 1 \). By [3, Lemma 2.1], we get

\[
\tilde{r}_{D/t^{1/\alpha}}(1, \frac{x}{t^{1/\alpha}}, \frac{x}{t^{1/\alpha}}) \leq \frac{c}{\delta_{D/t^{1/\alpha}}(x/t^{1/\alpha})} \leq \frac{c}{\delta_{D/t^{1/\alpha}}^2(x/t^{1/\alpha})} \leq \frac{ct^{2/\alpha}}{R^2}.
\]
Using the above inequality, we get
\[ \int_{D_{R/2}} r_D(t, x, x) dx \leq \frac{e^{2mt}}{t^{d/\alpha}} \int_{D_{R/2}} \frac{ct^{2/\alpha}}{R^2} dx \leq \frac{ce^{2mt}|D|t^{2/\alpha}}{R^2 t^{d/\alpha}}, \]
which proves (3.2).

Now we will introduce the following notation. Since \( D \) has \( R \)-smooth boundary, for any point \( y \in \partial D \) there are two open balls \( B_1 \) and \( B_2 \) both of radius \( R \) such that \( B_1 \subset D, B_2 \subset \mathbb{R}^d \setminus (D \cup \partial D), \partial B_1 \cap \partial B_2 = y \). For any \( x \in D_{R/2} \) there exist a unique point \( x_* \in \partial D \) such that \( \delta_D(x) = |x - x_*| \). Let \( B_1 = B(z_1, R), B_2 = B(z_2, R) \) be the balls for the point \( x_* \). Let \( H(x) \) be the half-space containing \( B_1 \) such that \( \partial H(x) \) contains \( x_* \) and is perpendicular to the segment \( z_1z_2 \).

We will need the following very important proposition in the proof of Theorem 1.2. Such a proposition has been proved for the stable process in [3, Proposition 3.1].

**Proposition 3.1.** Let \( D \subset \mathbb{R}^d, d \geq 2, \) be an open bounded set with \( R \)-smooth boundary \( \partial D \). Then for any \( x \in D \setminus D_{R/2} \) and \( t > 0 \) such that \( t^{1/\alpha} \leq R/2 \) we have
\[(3.5) \quad |r_D(t, x, x) - r_{H(x)}(t, x, x)| \leq \frac{ce^{2mt}t^{1/\alpha}}{Rt^{d/\alpha}} \left( \left( \frac{t^{1/\alpha}}{\delta_D(x)} \right)^{d+\alpha/2-1} \land 1 \right) \]

First let us assume the proposition 3.1 and use it to estimate the contribution from \( D \setminus D_{R/2} \) to the integral of \( r_D(t, x, x) \) in (3.1).

**Claim 2:**
\[(3.6) \quad \left| \int_{D \setminus D_{R/2}} r_D(t, x, x) dx - \int_{D \setminus D_{R/2}} r_{H(x)}(t, x, x) dx \right| \leq \frac{ce^{2mt}|D|t^{2/\alpha}}{R^2 t^{d/\alpha}} \]
for \( t^{1/\alpha} \leq R/2 \).

**Proof of Claim 2:** By Proposition 3.1 the left hand side of (3.6) is bounded above by
\[ \frac{ce^{2mt}t^{1/\alpha}}{Rt^{d/\alpha}} \int_0^{R/2} |\partial D_q| \left( \left( \frac{t^{1/\alpha}}{q} \right)^{d+\alpha/2-1} \land 1 \right) dq. \]

By Corollary 2.8, (i), the last quantity is smaller than or equal to
\[ \frac{ce^{2mt} \left( e^{mt} t^{1/\alpha} |\partial D| \right)}{Rt^{d/\alpha}} \int_0^{R/2} \left( \left( \frac{t^{1/\alpha}}{q} \right)^{d+\alpha/2-1} \land 1 \right) dq. \]
The integral in last quantity is bounded above by \(c t^{1/\alpha}\). To see this observe that since \(t^{1/\alpha} \leq R/2\) the above integral is equal to
\[
\frac{ce^{2mt}t^{1/\alpha}|\partial D|}{R^{t^{d/\alpha}}} \left[ \int_0^{t^{1/\alpha}} \left( \frac{t^{1/\alpha}}{q} \right)^{d+\alpha/2-1} \wedge 1 \right] dq + \int_{t^{1/\alpha}}^{R/2} \left( \frac{t^{1/\alpha}}{q} \right)^{d+\alpha/2-1} \wedge 1 \right] dq
\]
\[
\leq \frac{ce^{2mt}t^{2/\alpha}|\partial D|}{R^{t^{d/\alpha}}}.\]

Using this and Corollary (2.8), (ii), we get (3.6).

Recall that 
\(H = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 > 0\}\). For abbreviation let us denote
\(f_H(t,q) = r_H(t,(q,0,\ldots,0),(q,0,\ldots,0)), t,q > 0\).

Of course we have \(r_H(x)(t,x,x) = f_H(t,\delta_H(x))\). In the next step we will show that
\[
\left| \int_{D \setminus D_{R/2}} r_H(x)(t,x,x) dx - |\partial D| \int_0^{R/2} f_H(t,q) dq \right| \leq \frac{ce^{2mt}|D|t^{2/\alpha}}{R^2 t^{d/\alpha}}
\]

We have
\[
\int_{D \setminus D_{R/2}} r_H(x)(t,x,x) dx = \int_0^{R/2} |\partial D_q| f_H(t,q) dq
\]

Hence the left hand side of (3.7) is bounded above by
\[
\int_0^{R/2} ||\partial D_q| - |\partial D|| f_H(t,q) dq
\]

By Corollary 2.8, (iii), this is smaller than
\[
\frac{c|D|}{R^2} \int_0^{R/2} q f_H(t,q) dq
\]
\[
\leq \frac{c|D|e^{2mt}}{R^2} \int_0^{R/2} q f_H(t,q) dq
\]
\[
= \frac{c|D|e^{2mt}}{R^2} \int_0^{R/2} qt^{-d/\alpha} f_H(1,qt^{-1/\alpha}) dq
\]
\[
= \frac{c|D|e^{2mt}}{R^{2t^{1/\alpha}}} \int_0^{R/2} qt^{2/\alpha} f_H(1,q) dq
\]
\[
\leq \frac{c|D|e^{2mt}t^{2/\alpha}}{R^{t^{d/\alpha}}} \int_0^{\infty} q (q^{-d-\alpha} \wedge 1) dq \leq \frac{c|D|e^{2mt}t^{2/\alpha}}{R^{t^{d/\alpha}}}
\]
This shows (3.7). Finally, we have
\[
\left| \partial D \right| \int_0^{R/2} f_H(t, q) dq - \left| \partial D \right| \int_0^\infty f_H(t, q) dq \leq \left| \partial D \right| \int_{R/2}^\infty f_H(t, q) dq \\
\leq \frac{c|D|}{R} \int_{R/2}^\infty f_H(t, q) dq \quad \text{by Corollary 2.8, (ii)} \\
\leq \frac{c|D|e^{2mt}t^{1/\alpha}}{R t^{d/\alpha}} \int_{R/2t^{1/\alpha}}^\infty \tilde{f}_H(1, qt^{-1/\alpha}) dq \\
= \frac{c|D|e^{2mt}t^{1/\alpha}}{R t^{d/\alpha}} \int_{R/2t^{1/\alpha}}^\infty \tilde{f}_H(1, q) dq 
\]
Since \( R/2t^{1/\alpha} \geq 1 \), so for \( q \geq R/2t^{1/\alpha} \geq 1 \) we have \( \tilde{f}_H(1, q) \leq cq^{-d-\alpha} \leq cq^{-2} \). Therefore,
\[
\int_{R/2t^{1/\alpha}}^\infty \tilde{f}_H(1, q) dq \leq c \int_{R/2t^{1/\alpha}}^\infty dq \left( \frac{d/\alpha}{q^2} \right) \leq \frac{ct^{1/\alpha}}{R}. 
\]
Hence,
\[
(3.8) \quad \left| \partial D \right| \int_0^{R/2} f_H(t, q) dq - \left| \partial D \right| \int_0^\infty f_H(t, q) dq \leq \frac{c|D|e^{2mt}t^{2/\alpha}}{R^2 t^{d/\alpha}} 
\]
Note that the constant \( C_2(t) \) which appears in the formulation of Theorem 1.4 satisfies \( C_2(t) = \int_0^\infty f_H(t, q) dq \). Now equations (3.1), (3.2), (3.6), (3.7), (3.8) give (1.11).

**Proof of Proposition 3.1.** Exactly as in [3], let \( x_s \in \partial D \) be a unique point such that \(|x - x_s| = \text{dist}(x, \partial D)\) and \( B_1 \) and \( B_2 \) be balls with radius \( R \) such that \( B_1 \subset D, B_2 \subset \mathbb{R}^d \setminus (D \cup \partial D), \partial B_1 \cap \partial B_2 = x_s \). Let us also assume that \( x_s = 0 \) and choose an orthonormal coordinate system \((x_1, x_2, ..., x_d)\) so that the positive axis \( 0x_1 \) is in the direction of \( 0p \) where \( p \) is the center of the ball \( B_1 \). Note that \( x \) lies on the interval \( 0p \) so \( x = (|x|, 0, 0, ..., 0) \). Note also that \( B_1 \subset D \subset (B_2)^c \) and \( B_1 \subset H(x) \subset (B_2)^c \). For any open sets \( A_1, A_2 \) such that \( A_1 \subset A_2 \) we have \( r_{A_1}(t, x, y) \geq r_{A_2}(t, x, y) \) so
\[
|r_D(t, x, x) - r_{H(x)}(t, x, x)| \leq r_{B_1}(t, x, x) - r_{(B_2)^c}(t, x, x). 
\]
So in order to prove the proposition it suffices to show that
\[
r_{B_1}(t, x, x) - r_{(B_2)^c}(t, x, x) \leq \frac{ce^{2mt}t^{1/\alpha}}{R t^{d/\alpha}} \left( \left( \frac{t^{1/\alpha}}{\delta_D(x)} \right)^{d+\alpha/2-1} \wedge 1 \right) 
\]
for any \( x = (|x|, 0, ..., 0), |x| \in (0, R/2] \). Such an estimate was proved for the case \( m = 0 \) in [3]. In order to complete the proof it is enough to prove that
\[ r_{B_1}(t, x, x) - r_{\overline{B_2}}(t, x, x) \leq ce^{2mt} \left\{ \tilde{r}_{B_1}(t, x, x) - \tilde{r}_{\overline{B_2}}(t, x, x) \right\}, \]

But this follows from Propositions 2.4, 2.5 and 2.6.

Given the ball \( B_2 \), we set \( U = (\overline{B_2})^c \). Now using the generalized Ikeda-Watanabe formula, Proposition (2.5) and Lemma 2.4 in [14] we have

\[
\begin{align*}
& r_{B_1}(t, x, x) - r_U(t, x, x) \\
& = E^x \left[ t > \tau_{B_1}, X(\tau_{B_1}) \in U \setminus B_1; p_U(t - \tau_{B_1}, X(\tau_{B_1}), x) \right] \\
& = \int_{B_1} \int_0^t p_{B_1}(s, x, y) ds \int_{U \setminus B_1} v(y - z) p_U(t - s, z, x) dz dy \\
& \leq e^{2mt} \int_{B_1} \int_0^t \tilde{p}_{B_1}(s, x, y) ds \int_{U \setminus B_1} \tilde{v}(y - z) \tilde{p}_U(t - s, z, x) dz dy \\
& \leq ce^{2mt} E^x \left[ t > \tilde{\tau}_{B_1}, \tilde{X}(\tilde{\tau}_{B_1}) \in U \setminus B_1; \tilde{p}_U(t - \tilde{\tau}_{B_1}, \tilde{X}(\tilde{\tau}_{B_1}), x) \right] \\
& = ce^{2mt} (\tilde{r}_{B_1}(t, x, x) - \tilde{r}_U(t, x, x)) \\
& \leq \frac{ce^{2mt}t^{1/\alpha}}{Rt^{d/\alpha}} \left( \frac{t^{1/\alpha}}{\delta_D(x)} \right)^{d+\alpha/2-1} \land 1
\end{align*}
\]

The last inequality follows by Proposition 3.1 in [3].

REFERENCES


Rodrigo Bañuelos, Department of Mathematics Purdue University 150 North University Street West Lafayette, Indiana 47907-2067

E-mail address: banuelos@math.purdue.edu
URL: http://www.math.purdue.edu/~banuelos

Jebessa Mijena, 221 Parker Hall, Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849

E-mail address: jbm0018@tigermail.auburn.edu

Erkan Nane, 221 Parker Hall, Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849

E-mail address: nane@auburn.edu
URL: http://www.auburn.edu/~ezn0001