Continuous Time Random Walk Limits in Bounded Domains

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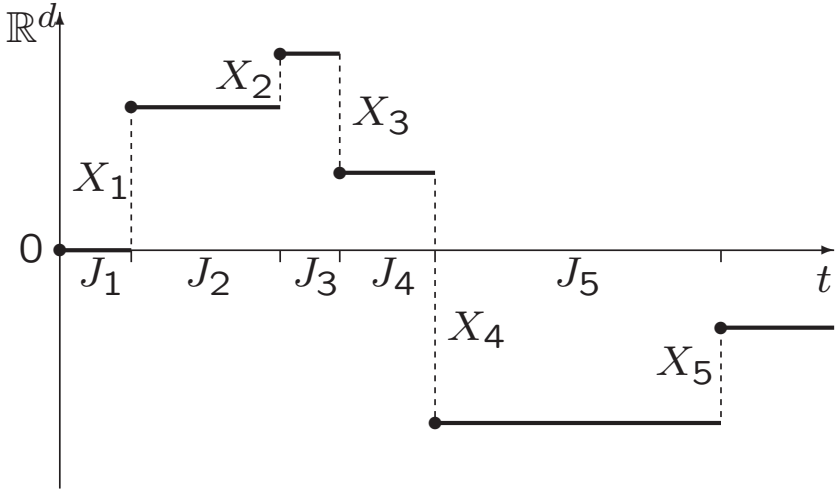
Continuous time random walks



The CTRW is a random walk with jumps X_n separated by random waiting times J_n. The random vectors (X_n, J_n) are i.i.d.

Waiting time process

J_n's are nonnegative iid.
$T_n = J_1 + J_2 + \cdots + J_n$ is the time of the nth jump.
$N(t) = \max\{n \geq 0 : \ T_n \leq t\}$ is the number of jumps by time $t > 0$.
Suppose $P(J_n > t) \approx Ct^{-\beta}$ for $0 < \beta < 1$.
Scaling limit

$$c^{-1/\beta} T_{[ct]} \implies D(t)$$

is a β-stable subordinator.
Since $\{T_n \leq t\} = \{N(t) \geq n\}$

$$c^{-\beta} N(ct) \implies E(t) = \inf\{u > 0 : \quad D(u) > t\}.$$

The self-similar limit $E(ct) \stackrel{d}{=} c^{\beta} E(t)$ is non-Markovian.

Continuous time random walks (CTRW)

$S(n) = X_1 + \cdots + X_n$ particle location after n jumps
has scaling limit $c^{-1/2}S([ct]) \implies B(t)$ a Brownian motion.
Number of jumps has scaling limit $c^{-\beta}N(ct) \implies E(t)$.
CTRW is a random walk time-changed by (a renewal process) $N(t)$

$$S(N(t)) = X_1 + X_2 + \cdots + X_{N(t)}.$$

$S(N(t))$ is particle location at time $t > 0$.
CTRW scaling limit is a time-changed process:

$$\begin{aligned} c^{-\beta/2}S(N(ct)) &= (c^\beta)^{-1/2}S(c^\beta \cdot c^{-\beta}N(ct)) \\ &\approx (c^\beta)^{-1/2}S(c^\beta E(t)) \implies B(E(t)). \end{aligned}$$

The self-similar limit $B(E(ct)) \stackrel{d}{=} c^{\beta/2}B(E(t))$ is non-Markovian.

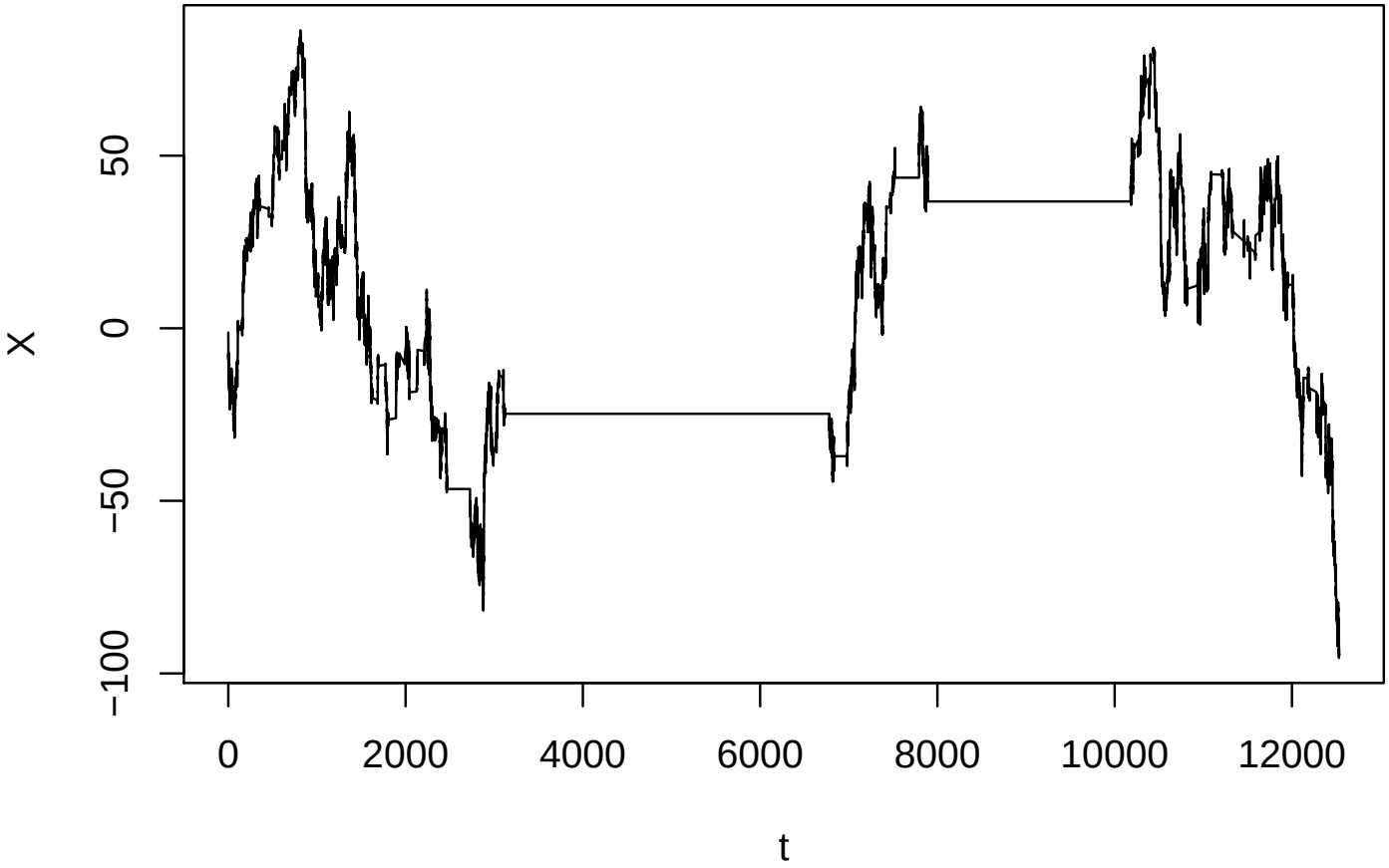


Figure: Typical sample path of the time-changed process $B(E(t))$. Here $B(t)$ is a Brownian motion and $E(t)$ is the inverse of a $\beta = 0.8$-stable subordinator. Graph has dimension $1 + \beta/2 = 1 + 0.4$. The limit process retains long resting times.

Power law waiting times

- Wait between solar flares $1 < \beta < 2$
- Wait between raindrops $\beta = 0.68$
- Wait between money transactions $\beta = 0.6$
- Wait between emails $\beta \approx 1.0$
- Wait between doctor visits $\beta \approx 1.4$
- Wait between earthquakes $\beta = 1.6$
- Wait between trades of German bond futures $\beta \approx 0.95$
- Wait between Irish stock trades $\beta = 0.4$ (truncated)

CTRW with serial dependence

Particle jumps $X_n = \sum_{j=0}^{\infty} c_j Z_{n-j}$ with Z_n IID.

Short range dependence: $\sum_{n=1}^{\infty} |\mathbb{E}(X_n X_0)| < \infty \implies$ the usual limit and PDE.

Long range dependence:
If Z_n has light tails: **time-changed fractional Brownian motion** limit $B_H(E(t))$. Hahn, Kobayashi and Umarov (2010) established a governing equation.

For heavy tails: **time-changed linear fractional stable motion** $L_{\alpha,H}(E(t))$.

Open problems: Governing equations, dependent waiting times.

Meerschaert, Nane and Xiao (2009).

Fractional time derivative: Two approaches

- Riemann-Liouville fractional derivative of order $0 < \beta < 1$;

$$\mathbb{D}_t^\beta g(t) = \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial t} \left[\int_0^t g(s) \frac{ds}{(t-s)^\beta} \right]$$

  with Laplace transform $s^\beta \tilde{g}(s)$, $\tilde{g}(s) = \int_0^\infty e^{-st} g(t) dt$ denotes the usual Laplace transform of g.

- Caputo fractional derivative of order $0 < \beta < 1$;

$$D_t^\beta g(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{dg(s)}{ds} \frac{ds}{(t-s)^\beta} \tag{1}$$

  was invented to properly handle initial values (Caputo 1967). Laplace transform of $D_t^\beta g(t)$ is $s^\beta \tilde{g}(s) - s^{\beta-1} g(0)$ incorporates the initial value in the same way as the first derivative.

examples

- $$D_t^\beta(t^p) = \frac{\Gamma(1+p)}{\Gamma(p+1-\beta)} t^{p-\beta}$$
- $$D_t^\beta(e^{\lambda t}) = \lambda^\beta e^{\lambda t} - \frac{t^{-\beta}}{\Gamma(1-\beta)}?$$
- $$D_t^\beta(\sin t) = \sin(t + \frac{\pi\beta}{2})$$

Diffusion

Let L_x be the generator of some continuous Markov process $X(t)$. Then $p(t,x) = \mathbb{E}_x[f(X(t))]$ is the unique solution of the heat-type Cauchy problem

$$\partial_t p(t,x) = L_x p(t,x), \;\; t > 0, x \in \mathbb{R}^d; \quad p(0,x) = f(x), \;\; x \in \mathbb{R}^d$$

Examples:
X: Brownian motion then $L_x = \Delta_x$, BM is a stochastic solution of the heat equation
X: Symmetric α-stable process then $L_x = -(-\Delta)^{\alpha/2}$

Time-fractional model for Anamolous sub-diffusion

Let $0 < \beta < 1$. Nigmatullin (1986) gave a physical derivation of fractional diffusion

$$\partial_t^\beta u(t,x) = L_x u(t,x); \quad u(0,x) = f(x) \tag{2}$$

Zaslavsky (1994) used this to model Hamiltonian chaos.
(2) has the unique solution

$$u(t,x) = \mathbb{E}_x[f(X(E(t)))] = \int_0^\infty p(s,x) g_{E(t)}(s) ds$$

where $p(t,x) = \mathbb{E}_x[f(X(t))]$ and $E(t) = \inf\{\tau > 0 : \ D_\tau > t\}$, $D(t)$ is a stable subordinator with index β and $\mathbb{E}(e^{-sD(t)}) = e^{-ts^\beta}$ (Baeumer and Meerschaert, 2002).
$\mathbb{E}_x(B(E(t)))^2 = \mathbb{E}(E(t)) \approx t^\beta$.

Equivalence to higher order PDE's

- For any $m = 2, 3, 4, \ldots$ both the Cauchy problem

$$\partial_t u(t,x) = \sum_{j=1}^{m-1} \frac{t^{j/m-1}}{\Gamma(j/m)} L_x^j f(x) + L_x^m u(t,x); \quad u(0,x) = f(x) \tag{3}$$

and the fractional Cauchy problem:

$$\partial_t^{1/m} u(t,x) = L_x u(t,x); \quad u(0,x) = f(x), \tag{4}$$

have the same unique solution given by

$$u(t,x) = \mathbb{E}_x[f(X(E^{1/m}(t)))] = \int_0^\infty p(s,x) g_{E^{1/m}(t)}(s)\, ds$$

- Allouba and Zheng (2001), Baeumer, Meerschaert and Nane (2007), Keyantuo and Lizama (2009), Li et al. (2010).

Connections to iterated Brownian motions

- Orsingher and Beghin (2004, 2008) show that the solution to

$$\partial_t^{1/2^n} u(t,x) = \Delta_x u(t,x); \quad u(0,x) = f(x), \tag{5}$$

is given by running

$$I_{n+1}(t) = B_1(|B_2(|B_3(|\cdots(B_{n+1}(t))\cdots|)|)|)$$

Where B_j's are independent Brownian motions, i.e., $u(t,x) = \mathbb{E}_x(f(I_{n+1}(t)))$ solves (5), and solves (3) for $m = 2^n$.

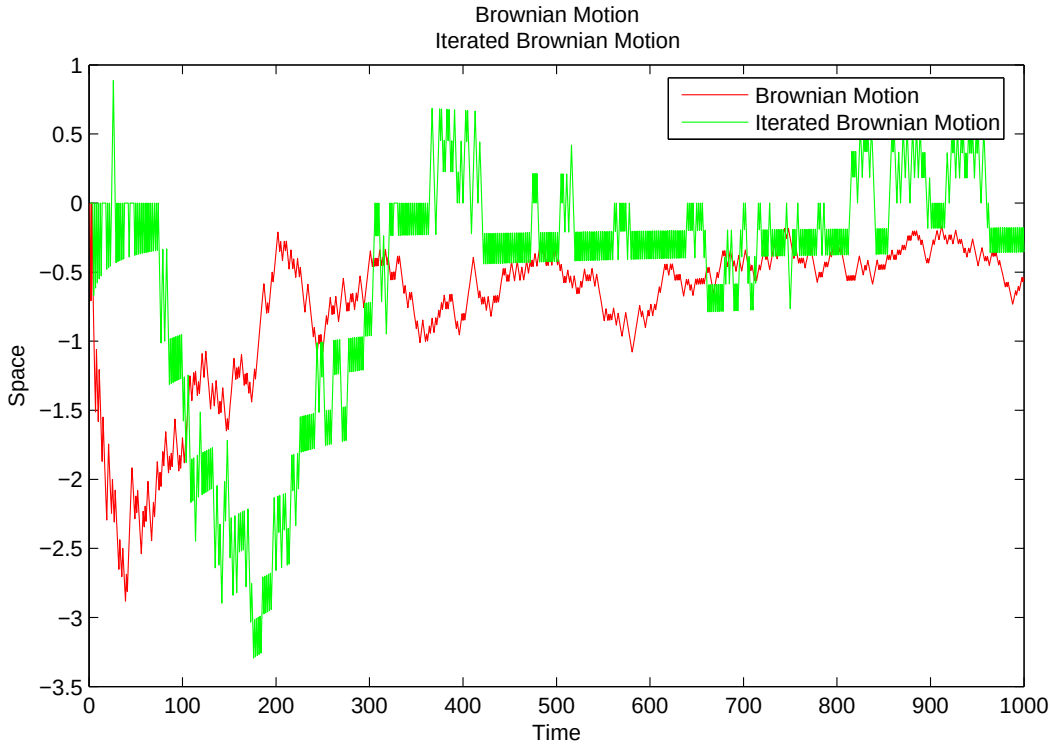


Figure: Simulations of iterated Brownian motions

Heat equation in bounded domains

Denote the eigenvalues and the eigenfunctions of Δ on a bounded domain D with Dirichlet boundary conditions by $\{\mu_n, \phi_n\}_{n=1}^{\infty}$;

$$\Delta\phi_n(x) = -\mu_n\phi_n(x), \quad x \in D; \phi_n(x) = 0, \quad x \in \partial D.$$

$\tau_D(X) = \inf\{t \geq 0 : \ X(t) \notin D\}$ is the first exit time of a process X, and let $\bar{f}(n) = \int_D f(x)\phi_n(x)dx$. The semigroup given by

$$T_D(t)f(x) \quad = \quad \mathbb{E}_x[f(B(t))I(\tau_D(B) > t)] = \sum_{n=1}^{\infty} e^{-\mu_n t}\phi_n(x)\bar{f}(n)$$

solves the heat equation in D with Dirichlet boundary conditions:

$$\begin{aligned} \partial_t u(t,x) &= \Delta_x u(t,x), \quad x \in D, \ t > 0, \\ u(t,x) &= 0, \quad x \in \partial D; \quad u(0,x) = f(x), \quad x \in D. \end{aligned}$$

Fractional diffusion in bounded domains

$$\begin{aligned} \partial_t^\beta u(t,x) &= \Delta_x u(t,x); \quad x \in D, \ t > 0 \\ u(t,x) &= 0, \ x \in \partial D, \ t > 0; \quad u(0,x) = f(x), \ x \in D. \end{aligned} \tag{6}$$

has the unique (classical) solution

$$\begin{aligned} u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x) M_\beta(-\mu_n t^\beta) = \int_0^\infty T_D(l) f(x) g_{E(t)}(l) dl \\ &= \mathbb{E}_x[f(B(E(t)))I(\tau_D(B) > E(t))] \\ &= \mathbb{E}_x[f(B(E(t)))I(\tau_D(B(E)) > t)] \end{aligned}$$

Joint work with Meerschaert and Vellaisamy (2009).

Analytic solution in intervals $(0, M) \subset \mathbb{R}$ was obtained by Agrawal (2002).
In this case, eigenfunctions and eigenvalues are

$$\phi_n(x) = \left(\frac{2}{M}\right)^{1/2} \sin(n\pi x/M), \quad \mu_n = \frac{\pi^2 n^2}{M^2}$$

The time fractional diffusion on $(0, M)$ has the solution

$$u(t,x) \quad = \quad \sum_{n=1}^{\infty} \bar{f}(n) \left(\frac{2}{M}\right)^{1/2} \sin(n\pi x/M) M_\beta(-\mu_n t^\beta)$$

here

$$M_\beta(z) = \sum_{k=0}^{\infty} \frac{z^n}{\Gamma(1+\beta n)}$$

For $\beta = 1$, $M_1(-z) = e^{-z}$, and u coincides with the solution of the heat equation on $(0, M)$.

Sketch of Proof

- Use Green's second identity and Dirichlet b.c. to write

$$\int_D \phi_n(x)\Delta_x u(t,x)dx = \int_D u(t,x)\Delta\phi_n(x)$$
$$= -\mu_n \int_D u(t,x)\phi_n(x)dx = -\mu_n \bar{u}(t,n)$$

  Apply to both sides of the fractional Cauchy problem to get

$$\partial_t^\beta \bar{u}(t,n) = -\mu_n \bar{u}(t,n). \tag{7}$$

- taking Laplace transforms on both sides of (7), we get

$$s^\beta \hat{u}(s,n) - s^{\beta-1}\bar{u}(0,n) = -\mu_n \hat{u}(s,n) \tag{8}$$

- Collecting the like terms leads to $\hat{u}(s,n) = \frac{\bar{f}(n)s^{\beta-1}}{s^\beta+\mu_n}$.

Sketch of Proof (page2)

By inverting the above Laplace transform, we obtain

$$\bar{u}(t,n) = \bar{f}(n) M_\beta(-\mu_n t^\beta)$$

in terms of the Mittag-Leffler function defined by

$$M_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\beta k)}.$$

Compute the Laplace transform of the hitting time density

$$\mathbb{E}(e^{-\mu E(t)}) = \int_0^\infty e^{-\mu l} g_{E(t)}(l) dl = M_\beta(-\mu t^\beta).$$

Inverting now the ϕ_n-transform, we get an L^2-convergent solution of Equation (6) as (for each $t \geq 0$)

$$u(t,x) = \sum_{n=1}^{\infty} \bar{f}(n) \phi_n(x) M_\beta(-\mu_n t^\beta) \tag{9}$$

Sketch of Proof (page3)

To get the probabilistic form of the solution we proceed as

$$
\begin{aligned}
u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x) M_\beta(-\mu_n t^\beta) \\
&= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x) \int_0^\infty e^{-\mu_n l} g_{E(t)}(l)dl \\
&= \int_0^\infty \left(\sum_{n=1}^{\infty} \bar{f}(n) e^{-\mu_n l} \phi_n(x) \right) g_{E(t)}(l)dl \qquad (10) \\
&= \int_0^\infty T_D(l) f(x) g_{E(t)}(l)dl \\
&= \int_0^\infty \mathbb{E}_x[f(B(l))I(\tau_D > l)] g_{E(t)}(l)dl \\
&= \mathbb{E}_x[f(B(E(t)))I(\tau_D(B) > E(t))]
\end{aligned}
$$

IBM in bounded domains

The (classical) solution of

$$\begin{aligned}
\partial_t u(t,x) &= \sum_{j=1}^{2^n-1} \frac{t^{j/2^n-1}}{\Gamma(j/m)} \Delta_x^j f(x) + \Delta_x^{2^n} u(t,x), \quad x \in D, \ t > 0; \\
u(t,x) &= \Delta_x^l u(t,x) = 0, \ t \geq 0, \ x \in \partial D, \ l = 1, \cdots 2^n - 1; \\
u(0,x) &= f(x), \ x \in D
\end{aligned}$$

is given by (running
$I_{n+1}(t) = B_1(|B_2(\cdots|B_{n+1}(t)|)|) = B_1(|I_n(t)|))$

$$\begin{aligned}
u(t,x) &= E_x[f(I_{n+1}(t))I(\tau_D(B_1) > |I_n(t)|)] \\
&= 2\int_0^\infty T_D(l) f(x) h(t,l) dl, \qquad (11)
\end{aligned}$$

where $h(t,l)$ is the transition density of $\{I_n(t)\}$.
Proof: equivalence with fractional Cauchy problem for $\beta = 1/2^n$.

Corollary

$$||u(t,\cdot)||_{L^2} \sim CM_\beta(-\mu_1 t^\beta) \sim \frac{C}{\mu_1 t^\beta}$$

In the Heat-equation case, since $\beta = 1$ we have $M_\beta(-\mu_1 t^\beta) = e^{-\mu_1 t}$ so

$$||u(t,\cdot)||_{L^2} \sim CM_1(-\mu_1 t) = Ce^{-\mu_1 t}$$

New time operators

Laplace symbol: $\psi(s)$	inverse subordinator	time operator
$\int_0^\infty (1-e^{-sy})\nu(dy)$	$E_\psi(t)$	$\psi(\partial_t) - \delta(0)\nu(t,\infty)$
s^β	$E(t)$	∂_t^β, Caputo
$\int_0^1 s^\beta \Gamma(1-\beta)\mu(d\beta)$	$E_\mu(t)$	$\int_0^1 \partial_t^\beta \Gamma(1-\beta)\mu(d\beta)$
$(s+\lambda)^\beta - \lambda^\beta$	$E_\lambda(t)$	$\partial_t^{\beta,\lambda}$ in (12)

$$\begin{aligned}\partial_t^{\beta,\lambda} g(t) &= \psi_\lambda(\partial_t) g(t) - g(0)\phi_\lambda(t,\infty) \\ &= e^{-\lambda t}\frac{1}{\Gamma(1-\beta)} d_t\left[\int_0^t \frac{e^{\lambda s} g(s)\,ds}{(t-s)^\beta}\right] - \lambda^\beta g(t) \\ &- \frac{g(0)}{\Gamma(1-\beta)}\int_t^\infty e^{-\lambda r}\beta r^{-\beta-1}\,dr.\end{aligned} \tag{12}$$

Subdiffusion: $0 < \beta < 1$, $\mathbb{E}_x(B(E(t)))^2 = \mathbb{E}(E(t)) \approx t^\beta$.
Ultraslow diffusion: For special $\mu \in RV_0(\theta - 1)$ for some $\theta > 0$: $\mathbb{E}_x(B(E_\mu(t)))^2 = \mathbb{E}(E_\mu(t)) \approx (\log t)^\theta$.
Intermediate between subdiffusion and diffusion: Tempered fractional diffusion

$$\mathbb{E}_x(B(E_\lambda(t)))^2 \approx \begin{cases} t^\beta/\Gamma(1+\beta), & t << 1 \\ t/\beta, & t >> 1. \end{cases}$$

$B(E_\lambda(t))$ occupies an intermediate place between subdiffusion and diffusion (Stanislavsky et al., 2008)

New space operators

Laplace exp.: $\psi(s)$	subord.	process	Generator
$\int_0^\infty (1-e^{-sy})\nu(dy)$	$D_\psi(t)$	$B(D_\psi(t))$	$-\psi(-\Delta)$
s^β	$D(t)$	$B(D(t))$	$-(-\Delta)^\beta$
$(s+m^{1/\beta})^\beta - m$	$T_\beta(t,m)$	$B(T_\beta(t,m))$	$-[(-\Delta+m^{1/\beta})^\beta - m]$
$\log(1+s^\beta)$	$D_{\log}(t)$	$B(D_{\log}(t))$	$-\log(1+(-\Delta)^\beta)$

$B(t)$, Brownian motion
$B(D(t))$, symmetric stable process
$B(T_\beta(t,m))$, relativistic stable process
$B(D_{\log}(t))$, geometric stable process

Space-time fractional diffusion in bounded domains

$$
\begin{aligned}
[\psi_1(\partial_t) - \delta(0)\nu(t,\infty)]u(t,x) &= -\psi_2(-\Delta_x)u(t,x); \quad x \in D, \ t > 0 \\
u(t,x) &= 0, \ x \in \partial D \ (\text{or } x \in D^C), \ t > 0; \\
u(0,x) &= f(x), \ x \in D.
\end{aligned}
$$

has the unique (classical) solution

$$
\begin{aligned}
u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)h_{\psi_1}(t,\lambda_n) \\
&= \mathbb{E}_x[f(B(D_{\psi_2}(E_{\psi_1}(t))))I(\tau_D(B(D_{\psi_2}(E_{\psi_1}))) > t)]
\end{aligned}
$$

$h_{\psi_1}(t,\lambda) = \mathbb{E}(e^{-\lambda E_{\psi_1}(t)})$ is the solution of

$$[\psi_1(\partial_t) - \delta(0)\nu(t,\infty)]h_{\psi_1}(t,\lambda) = -\lambda h_{\psi_1}(t,\lambda); \quad h_{\psi_1}(0,\lambda) = 1.$$

and

$$-\psi_2(-\Delta_x)\phi_n(x) = -\lambda_n\phi_n(x);\ \phi_n(x) = 0,\ x \in \partial D \text{ (or } x \in D^C).$$

$\psi_2(s) = s$ with

- $\psi_1(s) = s^\beta$: subdiffusion
- $\psi_1(s) = \int_0^1 s^\beta \Gamma(1-\beta)\mu(d\beta)$: Ultraslow diffusion
- $\psi_1(s) = (s+\lambda)^\beta - \lambda^\beta$: tempered fractional diffusion.

$\psi_2(s) = s^{\beta_2}$ with the three ψ_1s: space-time fractional diffusion, $(-\Delta)^{\beta_2}$

Joint work with Meerschaert, Vellaisamy and Chen (2009,2010, 2012).

Further research

- Work completed recently for the symmetric stable process as the outer process. The corresponding space operator is $(-\Delta)^{\alpha/2}$ for $0 < \alpha \leq 2$.
- Work in progress for other Subordinated Brownian motions, e.g. relativistic stable process as the outer process.... corresponding to the operator $(-\Delta + m^{1/\beta})^{\beta} - m$ for $0 < \alpha \leq 2$, $m \geq 0$.
- Extension to Neumann boundary conditions...
- Fractal properties of $B(E(t))$ and other time-changed processes
- Applications-interdisciplinary research

Thank You!