Continuous Time Random Walk Limits in Bounded Domains

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The CTRW is a random walk with jumps $X_n$ separated by random waiting times $J_n$. The random vectors $(X_n, J_n)$ are i.i.d.
Scaling limits
Initial-Boundary value problems

Waiting time process

\( J_n \)'s are nonnegative iid.
\( T_n = J_1 + J_2 + \cdots + J_n \) is the time of the \( n \)th jump.
\( N(t) = \max\{n \geq 0 : T_n \leq t\} \) is the number of jumps by time \( t > 0 \).

Suppose \( P(J_n > t) \approx Ct^{-\beta} \) for \( 0 < \beta < 1 \).
Scaling limit

\[ c^{-1/\beta} T_{[ct]} \xrightarrow{d} D(t) \]

is a \( \beta \)-stable subordinator.
Since \( \{ T_n \leq t \} = \{ N(t) \geq n \} \)

\[ c^{-\beta} N(ct) \xrightarrow{d} E(t) = \inf\{u > 0 : D(u) > t\} \]

The self-similar limit \( E(ct) \xrightarrow{d} c^\beta E(t) \) is non-Markovian.
Continuous time random walks (CTRW)

\[ S(n) = X_1 + \cdots + X_n \] particle location after \( n \) jumps

has scaling limit \( c^{-1/2} S([ct]) \rightarrow B(t) \) a Brownian motion.

Number of jumps has scaling limit \( c^{-\beta} N(ct) \rightarrow E(t) \).

CTRW is a random walk time-changed by (a renewal process) \( N(t) \)

\[ S(N(t)) = X_1 + X_2 + \cdots + X_{N(t)}. \]

\( S(N(t)) \) is particle location at time \( t > 0 \).

CTRW scaling limit is a time-changed process:

\[
c^{-\beta/2}S(N(ct)) = (c^\beta)^{-1/2}S(c^\beta \cdot c^{-\beta}N(ct)) \\
\approx (c^\beta)^{-1/2}S(c^\beta E(t)) \rightarrow B(E(t)).
\]

The self-similar limit \( B(E(ct)) \overset{d}{=} c^{\beta/2}B(E(t)) \) is non-Markovian.
Figure: Typical sample path of the time-changed process $B(E(t))$. Here $B(t)$ is a Brownian motion and $E(t)$ is the inverse of a $\beta = 0.8$-stable subordinator. Graph has dimension $1 + \beta/2 = 1 + 0.4$. The limit process retains long resting times.
Power law waiting times

- Wait between solar flares $1 < \beta < 2$
- Wait between raindrops $\beta = 0.68$
- Wait between money transactions $\beta = 0.6$
- Wait between emails $\beta \approx 1.0$
- Wait between doctor visits $\beta \approx 1.4$
- Wait between earthquakes $\beta = 1.6$
- Wait between trades of German bond futures $\beta \approx 0.95$
- Wait between Irish stock trades $\beta = 0.4$ (truncated)
Particle jumps $X_n = \sum_{j=0}^{\infty} c_j Z_{n-j}$ with $Z_n$ IID.

**Short range dependence:** $\sum_{n=1}^{\infty} |\mathbb{E}(X_n X_0)| < \infty \implies$ the usual limit and PDE.

**Long range dependence:**
If $Z_n$ has light tails: **time-changed fractional Brownian motion** limit $B_H(E(t))$. Hahn, Kobayashi and Umarov (2010) established a governing equation.

For heavy tails: **time-changed linear fractional stable motion** $L_{\alpha,H}(E(t))$.

Open problems: Governing equations, dependent waiting times.

Meerschaert, Nane and Xiao (2009).
Fractional time derivative: Two approaches

- **Riemann-Liouville** fractional derivative of order $0 < \beta < 1$;

  $$D^\beta_t g(t) = \frac{1}{\Gamma(1 - \beta)} \frac{\partial}{\partial t} \left[ \int_0^t g(s) \frac{ds}{(t-s)^\beta} \right]$$

  with Laplace transform $s^\beta \tilde{g}(s)$, $\tilde{g}(s) = \int_0^\infty e^{-st} g(t) dt$
  denotes the usual Laplace transform of $g$.

- **Caputo** fractional derivative of order $0 < \beta < 1$;

  $$D^\beta_t g(t) = \frac{1}{\Gamma(1 - \beta)} \int_0^t \frac{dg(s)}{ds} \frac{ds}{(t-s)^\beta}$$  \(1\)

  was invented to properly handle initial values (Caputo 1967). Laplace transform of $D^\beta_t g(t)$ is $s^\beta \tilde{g}(s) - s^{\beta-1} g(0)$
  incorporates the initial value in the same way as the first derivative.
examples

\[ D_t^\beta (t^p) = \frac{\Gamma(1 + p)}{\Gamma(p + 1 - \beta)} t^{p-\beta} \]

\[ D_t^\beta (e^{\lambda t}) = \lambda^\beta e^{\lambda t} - \frac{t^{-\beta}}{\Gamma(1 - \beta)} \]

\[ D_t^\beta (\sin t) = \sin(t + \frac{\pi \beta}{2}) \]
Diffusion

Let $L_x$ be the generator of some continuous Markov process $X(t)$. Then $p(t, x) = \mathbb{E}_x[f(X(t))]$ is the unique solution of the heat-type Cauchy problem

$$\partial_t p(t, x) = L_x p(t, x), \quad t > 0, \; x \in \mathbb{R}^d; \quad p(0, x) = f(x), \; x \in \mathbb{R}^d$$

Examples:
$X$: Brownian motion then $L_x = \Delta_x$, BM is a stochastic solution of the heat equation
$X$: Symmetric $\alpha$-stable process then $L_x = -(-\Delta)^{\alpha/2}$
Let $0 < \beta < 1$. Nigmatullin (1986) gave a physical derivation of fractional diffusion

$$\partial_t^\beta u(t, x) = L_x u(t, x); \quad u(0, x) = f(x) \quad (2)$$

Zaslavsky (1994) used this to model Hamiltonian chaos. (2) has the unique solution

$$u(t, x) = \mathbb{E}_x[f(X(E(t)))] = \int_0^\infty p(s, x) g_{E(t)}(s) ds$$

where $p(t, x) = \mathbb{E}_x[f(X(t))]$ and $E(t) = \inf\{\tau > 0 : D_{\tau} > t\}$, $D(t)$ is a stable subordinator with index $\beta$ and $\mathbb{E}(e^{-sD(t)}) = e^{-ts^\beta}$ (Baeumer and Meerschaert, 2002).

$$\mathbb{E}_x(B(E(t)))^2 = \mathbb{E}(E(t)) \approx t^\beta.$$
For any $m = 2, 3, 4, \ldots$ both the Cauchy problem

$$\partial_t u(t, x) = \sum_{j=1}^{m-1} \frac{t^{j/m-1}}{\Gamma(j/m)} L^j_x f(x) + L^m_x u(t, x); \quad u(0, x) = f(x)$$

and the fractional Cauchy problem:

$$\partial_t^{1/m} u(t, x) = L_x u(t, x); \quad u(0, x) = f(x),$$

have the same unique solution given by

$$u(t, x) = \mathbb{E}_x[f(X(E^{1/m}(t)))] = \int_0^\infty p(s, x) g_{E^{1/m}(t)}(s) \, ds$$

Allouba and Zheng (2001), Baeumer, Meerschaert and Nane (2007), Keyantuo and Lizama (2009), Li et al. (2010).
Connections to iterated Brownian motions

- Orsingher and Beghin (2004, 2008) show that the solution to

\[
\frac{\partial^{1/2^n}}{\partial t} u(t, x) = \Delta_x u(t, x); \quad u(0, x) = f(x), \quad (5)
\]

is given by running

\[
I_{n+1}(t) = B_1(|B_2(|B_3(| \cdots (B_{n+1}(t)) \cdots |)|)|)
\]

Where \(B_j\)'s are independent Brownian motions, i.e.,

\[
u(t, x) = \mathbb{E}_x(f(I_{n+1}(t)))\]

solves (5), and solves (3) for \(m = 2^n\).
Figure: Simulations of iterated Brownian motions
Heat equation in bounded domains

Denote the eigenvalues and the eigenfunctions of $\Delta$ on a bounded domain $D$ with Dirichlet boundary conditions by $\{\mu_n, \phi_n\}_{n=1}^{\infty}$;

$$\Delta \phi_n(x) = -\mu_n \phi_n(x), \quad x \in D; \quad \phi_n(x) = 0, \quad x \in \partial D.$$ 

$\tau_D(X) = \inf\{t \geq 0 : X(t) \notin D\}$ is the first exit time of a process $X$, and let $\bar{f}(n) = \int_D f(x) \phi_n(x) dx$. The semigroup given by

$$T_D(t)f(x) = \mathbb{E}_x[f(B(t))I(\tau_D(B) > t)] = \sum_{n=1}^{\infty} e^{-\mu_n t} \phi_n(x) \bar{f}(n)$$

solves the heat equation in $D$ with Dirichlet boundary conditions:

$$\partial_t u(t, x) = \Delta_x u(t, x), \quad x \in D, \quad t > 0,$$

$$u(t, x) = 0, \quad x \in \partial D; \quad u(0, x) = f(x), \quad x \in D.$$
Fractional diffusion in bounded domains

\[ \partial_t^\beta u(t, x) = \Delta_x u(t, x); \quad x \in D, \ t > 0 \]

\[ u(t, x) = 0, \ x \in \partial D, \ t > 0; \quad u(0, x) = f(x), \ x \in D. \]  

has the unique (classical) solution

\[ u(t, x) = \sum_{n=1}^{\infty} \tilde{f}(n) \phi_n(x) M_\beta(-\mu_n t^\beta) = \int_0^\infty T_D(l)f(x)g_{E(t)}(l)dl \]

\[ = \mathbb{E}_x[f(B(E(t)))I(\tau_D(B) > E(t))] \]

\[ = \mathbb{E}_x[f(B(E(t)))I(\tau_D(B(E)) > t)] \]

Joint work with Meerschaert and Vellaisamy (2009).
Analytic solution in intervals \((0, M) \subset \mathbb{R}\) was obtained by Agrawal (2002). In this case, eigenfunctions and eigenvalues are

\[
\phi_n(x) = \left(\frac{2}{M}\right)^{1/2} \sin\left(n\pi x / M\right), \quad \mu_n = \frac{\pi^2 n^2}{M^2}
\]

The time fractional diffusion on \((0, M)\) has the solution

\[
 u(t, x) = \sum_{n=1}^{\infty} \bar{f}(n) \left(\frac{2}{M}\right)^{1/2} \sin\left(n\pi x / M\right) M_\beta(-\mu_n t^\beta)
\]

here

\[
 M_\beta(z) = \sum_{k=0}^{\infty} \frac{z^n}{\Gamma(1 + \beta n)}
\]

For \(\beta = 1\), \(M_1(-z) = e^{-z}\), and \(u\) coincides with the solution of the heat equation on \((0, M)\).
Sketch of Proof

- Use Green’s second identity and Dirichlet b.c. to write

\[ \int_D \phi_n(x) \Delta_x u(t, x) \, dx = \int_D u(t, x) \Delta \phi_n(x) \]

\[ = -\mu_n \int_D u(t, x) \phi_n(x) \, dx = -\mu_n \bar{u}(t, n) \]

Apply to both sides of the fractional Cauchy problem to get

\[ \partial_t^\beta \bar{u}(t, n) = -\mu_n \bar{u}(t, n). \quad (7) \]

- Taking Laplace transforms on both sides of (7), we get

\[ s^\beta \hat{u}(s, n) - s^{\beta-1} \bar{u}(0, n) = -\mu_n \hat{u}(s, n) \quad (8) \]

- Collecting the like terms leads to \( \hat{u}(s, n) = \frac{\bar{f}(n)s^{\beta-1}}{s^\beta + \mu_n} \).
Sketch of Proof (page2)

By inverting the above Laplace transform, we obtain

\[ \bar{u}(t, n) = \bar{f}(n)M_\beta(-\mu_n t^\beta) \]

in terms of the Mittag-Leffler function defined by

\[ M_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + \beta k)}. \]

Compute the Laplace transform of the hitting time density

\[ \mathbb{E}(e^{-\mu E(t)}) = \int_{0}^{\infty} e^{-\mu l} g_{E(t)}(l) dl = M_\beta(-\mu t^\beta). \]

Inverting now the \( \phi_n \)-transform, we get an \( L^2 \)-convergent solution of Equation (6) as (for each \( t \geq 0 \))

\[ u(t, x) = \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)M_\beta(-\mu_n t^\beta) \]  \hspace{1cm} (9)
To get the probabilistic form of the solution we proceed as

\[ u(t, x) = \sum_{n=1}^{\infty} \bar{f}(n) \phi_n(x) M_\beta(-\mu_n t^\beta) \]

\[ = \sum_{n=1}^{\infty} \bar{f}(n) \phi_n(x) \int_0^\infty e^{-\mu_n l} g_{E(t)}(l) dl \]

\[ = \int_0^\infty \left( \sum_{n=1}^{\infty} \bar{f}(n) e^{-\mu_n l} \phi_n(x) \right) g_{E(t)}(l) dl \]  \hspace{1cm} (10)

\[ = \int_0^\infty T_D(l) f(x) g_{E(t)}(l) dl \]

\[ = \int_0^\infty E_x[f(B(l))I(\tau_D > l)] g_{E(t)}(l) dl \]

\[ = E_x[f(B(E(t)))I(\tau_D(B) > E(t))] \]
The (classical) solution of
\begin{align}
\partial_t u(t, x) &= \sum_{j=1}^{2^n-1} \frac{t^{j/2^n-1}}{\Gamma(j/m)} \Delta_x^j f(x) + \Delta_x^{2^n} u(t, x), \quad x \in D, \ t > 0;
\end{align}
\begin{align}
u(t, x) &= \Delta_x^l u(t, x) = 0, \ t \geq 0, \ x \in \partial D, \ l = 1, \cdots 2^n - 1;
\end{align}
\begin{align}
u(0, x) &= f(x), \ x \in D
\end{align}
is given by (running
\begin{align}
l_{n+1}(t) = B_1(|B_2(\cdots |B_{n+1}(t)||)|) = B_1(|l_n(t)||)
\end{align}
\begin{align}
u(t, x) = E_x[f(l_{n+1}(t))1(\tau_D(B_1) > |l_n(t)||)]
\end{align}
\begin{align}
= 2 \int_0^{\infty} T_D(l)f(x)h(t, l)dl,
\end{align}
where $h(t, l)$ is the transition density of $\{l_n(t)\}$.
Proof: equivalence with fractional Cauchy problem for $\beta = 1/2^n$. 

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Corollary

\[ \|u(t, \cdot)\|_{L^2} \sim CM_\beta(-\mu_1 t^\beta) \sim \frac{C}{\mu_1 t^\beta} \]

In the Heat-equation case, since \( \beta = 1 \) we have
\[ M_\beta(-\mu_1 t^\beta) = e^{-\mu_1 t} \] so
\[ \|u(t, \cdot)\|_{L^2} \sim CM_1(-\mu_1 t) = Ce^{-\mu_1 t} \]
New time operators

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<th>Laplace symbol: $\psi(s)$</th>
<th>inverse subordinator</th>
<th>time operator</th>
</tr>
</thead>
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<tr>
<td>$\int_0^\infty (1 - e^{-sy})\nu(dy)$</td>
<td>$E_\psi(t)$</td>
<td>$\psi(\partial_t) - \delta(0)\nu(t, \infty)$</td>
</tr>
<tr>
<td>$s^{\beta}$</td>
<td>$E(t)$</td>
<td>$\partial_t^{\beta}$, Caputo</td>
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<tr>
<td>$\int_0^1 s^{\beta} \Gamma(1 - \beta)\mu(d\beta)$</td>
<td>$E_\mu(t)$</td>
<td>$\int_0^1 \partial_t^{\beta} \Gamma(1 - \beta)\mu(d\beta)$</td>
</tr>
<tr>
<td>$(s + \lambda)^{\beta} - \lambda^{\beta}$</td>
<td>$E_\lambda(t)$</td>
<td>$\partial_t^{\beta,\lambda}$ in (12)</td>
</tr>
</tbody>
</table>

$$\partial_t^{\beta,\lambda} g(t) = \psi_\lambda(\partial_t) g(t) - g(0) \phi_\lambda(t, \infty)$$

$$= e^{-\lambda t} \frac{1}{\Gamma(1 - \beta)} d_t \left[ \int_0^t e^{\lambda s} g(s) \, ds \right] - \lambda^{\beta} g(t) \quad (12)$$

$$- \frac{g(0)}{\Gamma(1 - \beta)} \int_t^\infty e^{-\lambda r} \beta r^{-\beta - 1} \, dr.$$
Subdiffusion: $0 < \beta < 1$, $\mathbb{E}_x(B(E(t)))^2 = \mathbb{E}(E(t)) \approx t^\beta$.

Ultraslow diffusion: For special $\mu \in RV_0(\theta - 1)$ for some $\theta > 0$:

$\mathbb{E}_x(B(E_\mu(t)))^2 = \mathbb{E}(E_\mu(t)) \approx (\log t)^\theta$.

Intermediate between subdiffusion and diffusion: Tempered fractional diffusion

$$\mathbb{E}_x(B(E_\lambda(t)))^2 \approx \begin{cases} 
  t^\beta / \Gamma(1 + \beta), & t << 1 \\
  t/\beta, & t >> 1.
\end{cases}$$

$B(E_\lambda(t))$ occupies an intermediate place between subdiffusion and diffusion (Stanislavsky et al., 2008)
New space operators

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<th>Laplace exp.: $\psi(s)$</th>
<th>subord.</th>
<th>process</th>
<th>Generator</th>
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<td>$\int_0^\infty (1 - e^{-sy})\nu(dy)$</td>
<td>$D_\psi(t)$</td>
<td>$B(D_\psi(t))$</td>
<td>$-\psi(-\Delta)$</td>
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<tr>
<td>$s^\beta$</td>
<td>$D(t)$</td>
<td>$B(D(t))$</td>
<td>$-(-\Delta)^\beta$</td>
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<tr>
<td>$(s + m^{1/\beta})^\beta - m$</td>
<td>$T_\beta(t, m)$</td>
<td>$B(T_\beta(t, m))$</td>
<td>$-[(\Delta + m^{1/\beta})^\beta - m]$</td>
</tr>
<tr>
<td>$\log(1 + s^\beta)$</td>
<td>$D_{\log}(t)$</td>
<td>$B(D_{\log}(t))$</td>
<td>$-\log(1 + (-\Delta)^\beta)$</td>
</tr>
</tbody>
</table>

$B(t)$, Brownian motion
$B(D(t))$, symmetric stable process
$B(T_\beta(t, m))$, relativistic stable process
$B(D_{\log}(t))$, geometric stable process
Space-time fractional diffusion in bounded domains

\[ [\psi_1(\partial_t) - \delta(0)\nu(t, \infty)]u(t, x) = -\psi_2(-\Delta_x)u(t, x); \quad x \in D, \ t > 0 \]

\[ u(t, x) = 0, \ x \in \partial D \ (\text{or } x \in D^C), \ t > 0; \]

\[ u(0, x) = f(x), \ x \in D. \]

has the unique (classical) solution

\[ u(t, x) = \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)h_{\psi_1}(t, \lambda_n) \]

\[ = \mathbb{E}_x[f(B(D_{\psi_2}(E_{\psi_1}(t))))I(\tau_D(B(D_{\psi_2}(E_{\psi_1})))) > t)] \]

\[ h_{\psi_1}(t, \lambda) = \mathbb{E}(e^{-\lambda E_{\psi_1}(t)}) \] is the solution of

\[ [\psi_1(\partial_t) - \delta(0)\nu(t, \infty)]h_{\psi_1}(t, \lambda) = -\lambda h_{\psi_1}(t, \lambda); \quad h_{\psi_1}(0, \lambda) = 1. \]

and
\[-\psi_2(-\Delta_x)\phi_n(x) = -\lambda_n\phi_n(x); \quad \phi_n(x) = 0, \quad x \in \partial D \ (\text{or} \ x \in D^C).\]

\[\psi_2(s) = s \text{ with} \]
- \[\psi_1(s) = s^\beta: \text{subdiffusion} \]
- \[\psi_1(s) = \int_0^1 s^\beta \Gamma(1 - \beta)\mu(d\beta): \text{Ultraslow diffusion} \]
- \[\psi_1(s) = (s + \lambda)^\beta - \lambda^\beta: \text{tempered fractional diffusion}. \]

\[\psi_2(s) = s^{\beta_2} \text{ with the three } \psi_1 \text{s: space-time fractional diffusion,} \]
\[(-\Delta)^{\beta_2} \]

Further research

- Work completed recently for the symmetric stable process as the outer process. The corresponding space operator is \((-\Delta)^{\alpha/2}\) for \(0 < \alpha \leq 2\).
- Work in progress for other Subordinated Brownian motions, e.g. relativistic stable process as the outer process. Corresponding to the operator \((-\Delta + m^{1/\beta})^\beta - m\) for \(0 < \alpha \leq 2, m \geq 0\).
- Extension to Neumann boundary conditions...
- Fractal properties of \(B(E(t))\) and other time-changed processes
- Applications-interdisciplinary research
Thank You!