Continuous Time Random Walk Limits: Governing Equations and Fractal Dimensions.

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Outline

Probability and transforms

If the random variable Y has density $f(x)$ so that

$$\mathbb{P}(a \leq Y \leq b) = \int_a^b f(x)dx$$

then $f(x)$ has Fourier transform

$$\begin{aligned}\hat{f}(k) = \mathbb{E}(e^{-ikY}) &= \int_{-\infty}^{\infty} e^{-ikx} f(x)dx \\ &= \int_{-\infty}^{\infty} (1 - ikx + \frac{1}{2!}(ikx)^2 + \cdots) f(x)dx \\ &= 1 - ik\mu_1 - \frac{1}{2!}k^2\mu_2 + \cdots\end{aligned}$$

where the lth moment is $\mu_l = \int_{-\infty}^{\infty} x^l f(x)dx$

Central limit theorem

If $\mu_1 = 0$ and $\mu_2 = 2$ then $\hat{f}(k) = 1 - k^2 + \cdots$
The IID sum $S(n) = Y_1 + \cdots + Y_n$ has FT $\hat{f}(k)^n$ and the normalized sum $S(n)/\sqrt{n}$ has FT

$$\begin{aligned}\left(\hat{f}(k/\sqrt{n})\right)^n =& (1 - (k/\sqrt{n})^2 + \cdots)^n \\ =& \left(1 - \frac{k^2}{n} + \cdots\right)^n \\ \to& e^{-k^2} \equiv \hat{g}(k) \quad \text{as} \quad n \to \infty.\end{aligned}$$

Inverting the Fourier transform reveals a Gaussian(Normal) density

$$g(x) = \frac{1}{\sqrt{4\pi}} e^{-x^2/4}$$

Brownian motion

If Y_n represents a particle jump at time n then $S(n) = Y_1 + \cdots + Y_n$ is the location of the particle at time n. Expanding the time scale by a factor of $c > 0$ and taking limits as $c \to \infty$ shows that $c^{-1/2} S([ct]) \implies W(t)$ since

$$\hat{f}(c^{-1/2}k)^{[ct]} = \left(1 - \frac{k^2}{c} + \cdots\right)^{[ct]}$$
$$\to e^{-k^2 t} \equiv \hat{p}(t,k) \quad \text{as} \quad c \to \infty$$

for all $t > 0$. Inverting the FT shows that the density of the limiting Brownian motion process $W(t)$ is Gaussian (Normal)

$$p(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.$$

Brownian motion

Classical random walk

$$S(t) = Y_1 + \cdots + Y_{[t]}$$

A particle takes a random jump Y_n at time $t = n$. Particle location at time t is a simple random walk $S(t)$ and scaling limit is a Brownian motion.

$$c^{-1/2} S(ct) \Rightarrow W(t) \approx \underbrace{N(0, \sigma^2 t)} \qquad (c \to \infty)$$

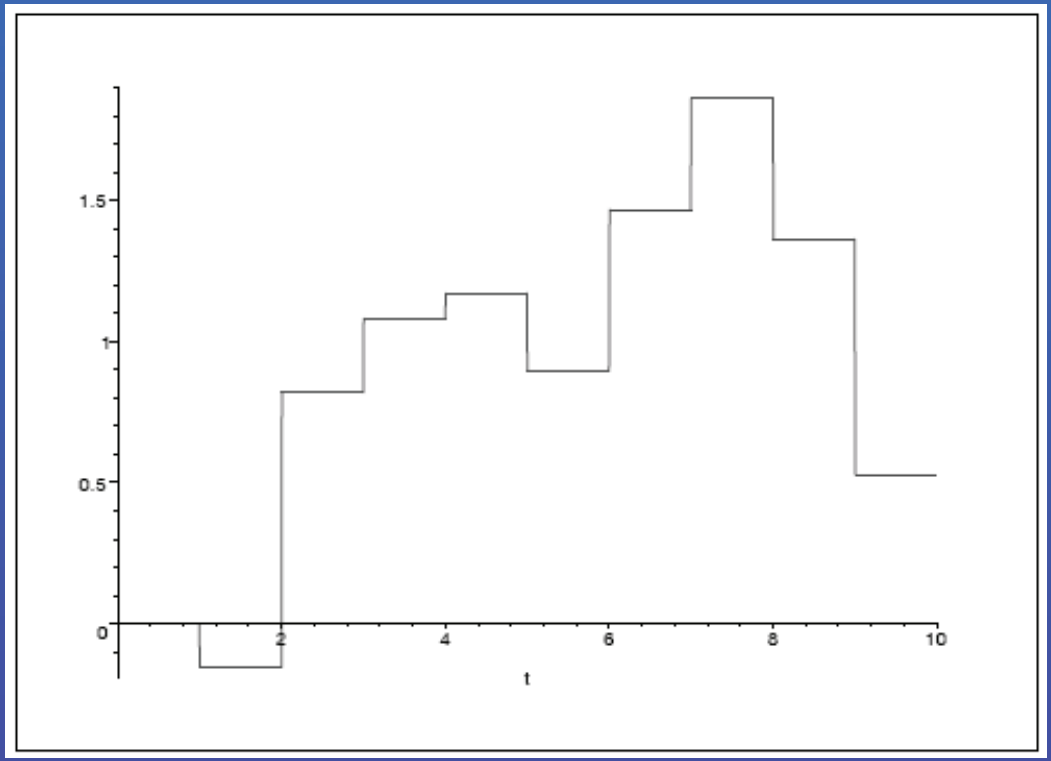
Contract spatial scale

Expand time scale

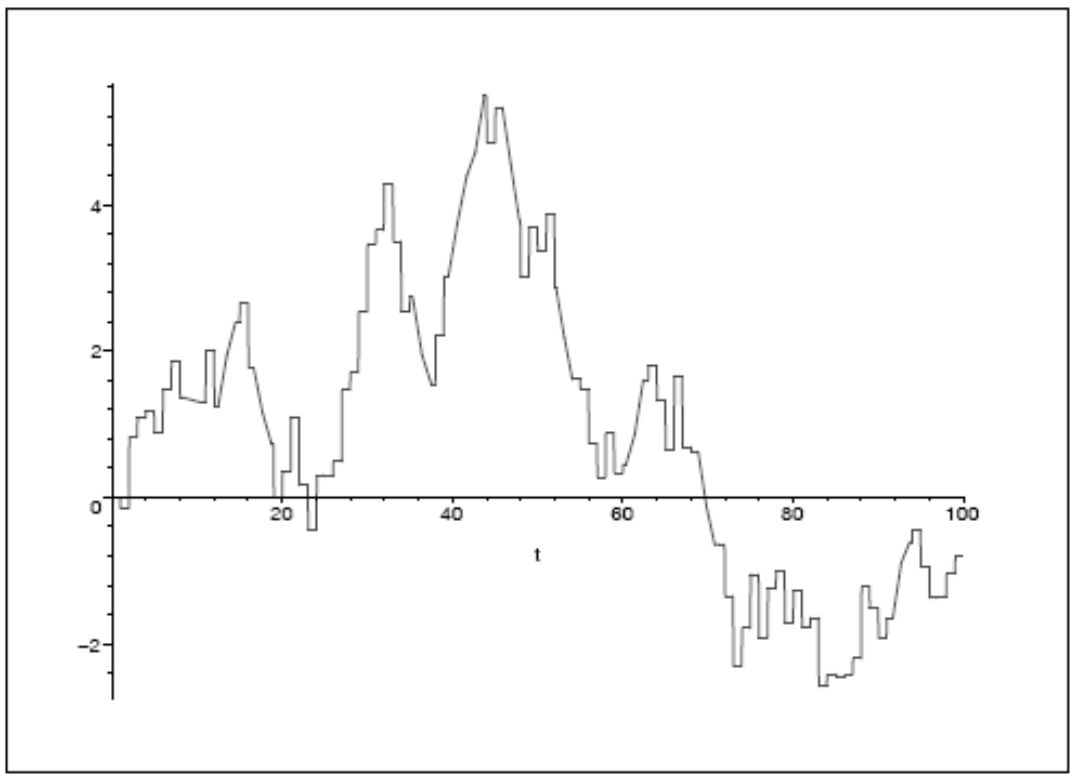
Normal limit density

Add an advective drift: $L(t) = vt + W(t) \approx N(vt, \sigma^2 t)$

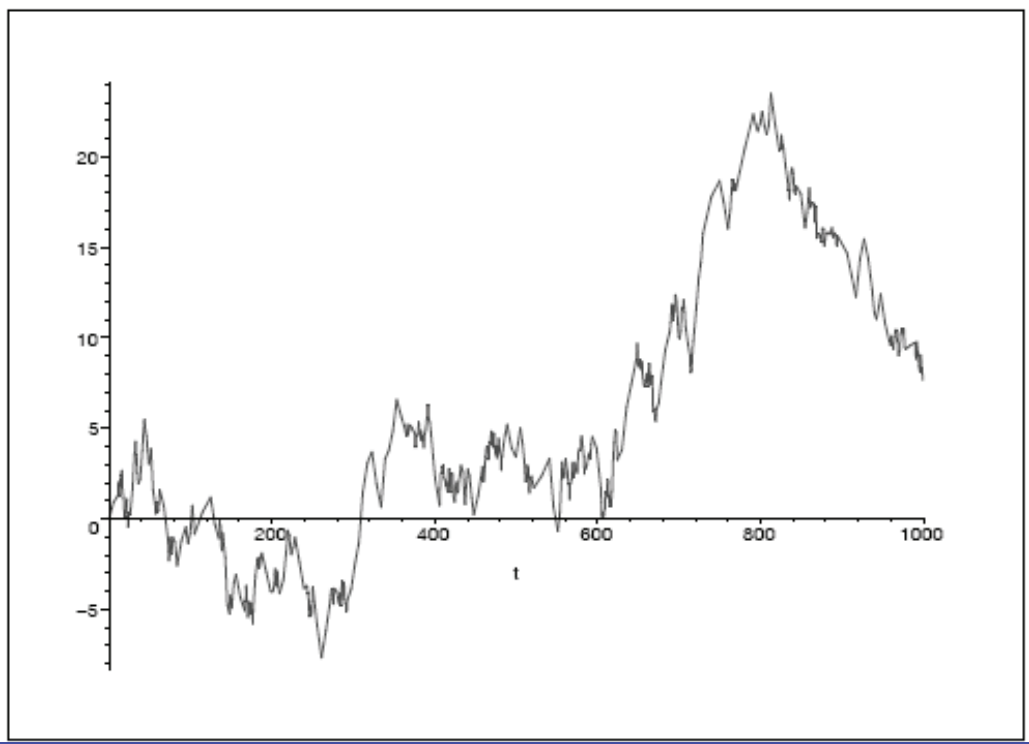
Random walk simulation



Longer time scale



Scaling limit: Brownian motion



Random graph of fractal dimension 1.5 and no jumps.

Most likely shape of a Brownian path.

Microsoft stock-the last two years

Some history of Brownian motion (BM)

- Robert Brown (1827), a Botanist: was first to observe that pollen grains in water move continuously and very erratically.
- Louis Bachelier (1900): presented a stochastic analysis of the stock and option markets using BM
- Albert Einstein (1905): used BM to determine the law of the position of the particle...
- Norbert Wiener (1923): Mathematical foundations of BM
- Doob (1956): connections to analysis, heat equation
- Kolmogorov, Lévy, Khintchine, ...

Derivatives and transforms

- If the Laplace transform of $f(t)$ is defined for $s > 0$ by

$$\tilde{f}(s) = \int_0^\infty e^{-st} f(t) dt$$

then $d_t f(t)$ has Laplace transform $s\tilde{f}(s) - f(0)$.

- If the Fourier transform of $f(x)$ is defined for $k \in \mathbb{R}$ by

$$\hat{f}(k) = \int_{\mathbb{R}} e^{-ikx} f(x) dx$$

then $d_x f(x)$ has Fourier transform $ik\hat{f}(k)$.

The diffusion (heat) equation

Taking Fourier transforms in the classical diffusion equation

$$\partial_t p(t,x) = \partial_x^2 p(t,x)$$

yields

$$\partial_t \hat{p}(t,k) = (ik)^2 \hat{p}(t,k) = -k^2 \hat{p}(t,k)$$

whose solution

$$\hat{p}(t,k) = e^{-k^2 t}$$

inverts to the same limit density for the Brownian motion $W(t)$. For a cloud of diffusing particles $p(t,x)$ is the particle density.

Let $W_t \in \mathbb{R}$ be Brownian motion started at x. Then the function (convolution of f and $p(t,x)$)

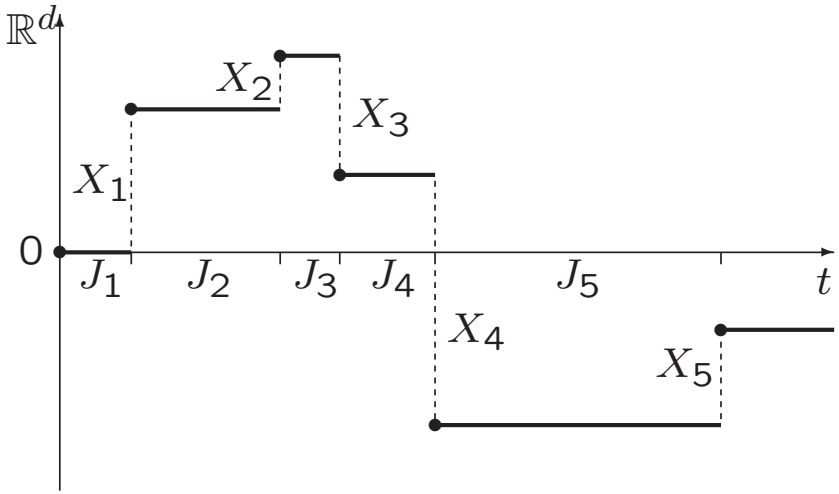
$$u(t,x) = \mathbb{E}_x[f(W(t))] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} f(y)dy$$

solves the heat equation

$$\begin{aligned} \partial_t u(t,x) &= \partial_x^2 u(t,x), \qquad t > 0, \quad x \in \mathbb{R} \\ u(0,x) &= f(x), \qquad x \in \mathbb{R}. \end{aligned}$$

This is due to J.L. Doob (1956).
In this case we say, Brownian motion $W(t)$ is a stochastic solution of the heat equation.

Continuous time random walks



The CTRW is a random walk with jumps X_n separated by random waiting times J_n. The random vectors (X_n, J_n) are i.i.d.

Heavy tailed waiting times

Random wait J_n between jumps, nth jump time given by a random walk

$$T(n) = J_1 + \cdots + J_n$$

Number of jumps by time t is inverse $N(t) \geq n \longleftrightarrow T(n) \leq t$

For heavy tail waiting times $P(J_n > t) \approx Ct^{-\beta} \quad (0 < \beta < 1)$

$$c^{-1/\beta} T(ct) \Rightarrow P(t) \quad \Longleftrightarrow \quad c^{-\beta} N(ct) \Rightarrow Q(t)$$

Inverse processes have inverse scaling

$$P(ct) \approx c^{1/\beta} P(t) \quad \Longleftrightarrow \quad Q(ct) \approx c^{\beta} Q(t)$$

Continuous time random walks (CTRW)

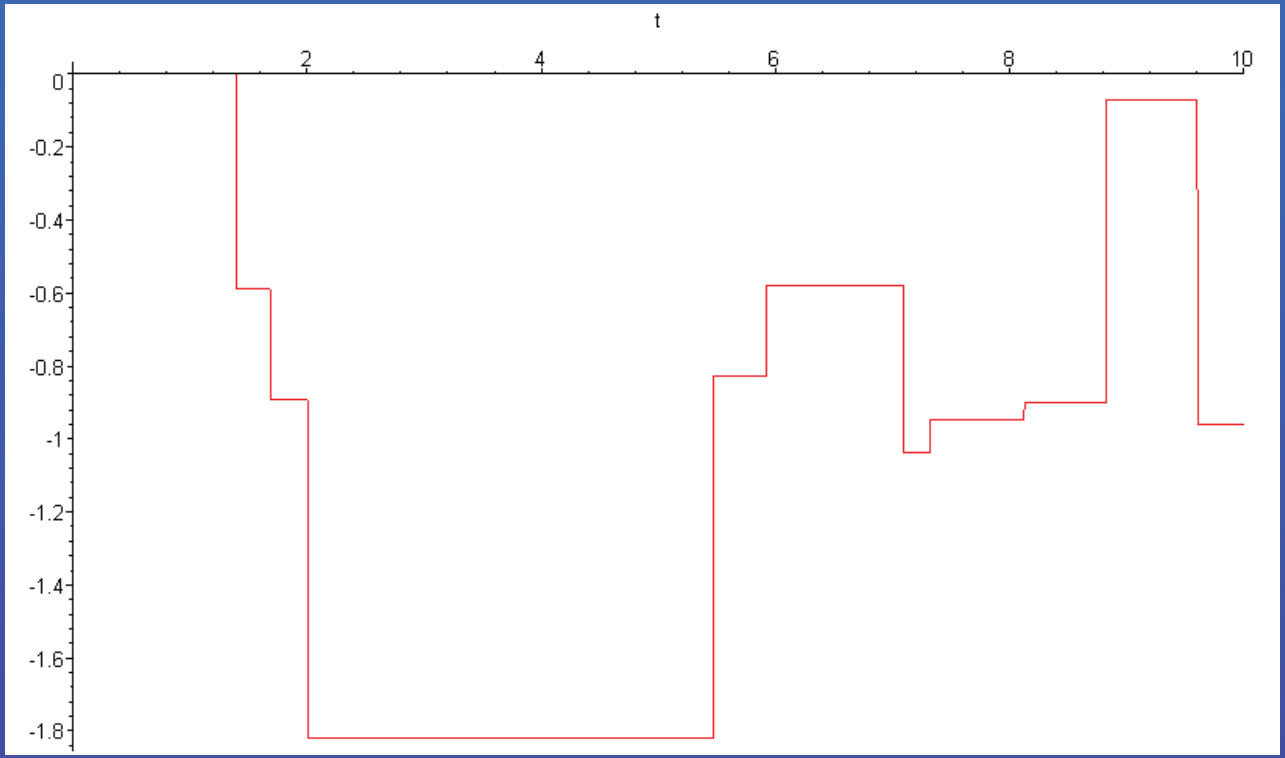
Particle jump random walk has scaling limit
$c^{-1/2}S([ct]) \implies W(t)$.
Number of jumps has scaling limit $c^{-\beta}N(ct) \implies Q(t)$.
CTRW is a random walk subordinated to (a renewal process) $N(t)$

$$S(N(t)) = X_1 + X_2 + \cdots + X_{N(t)}$$

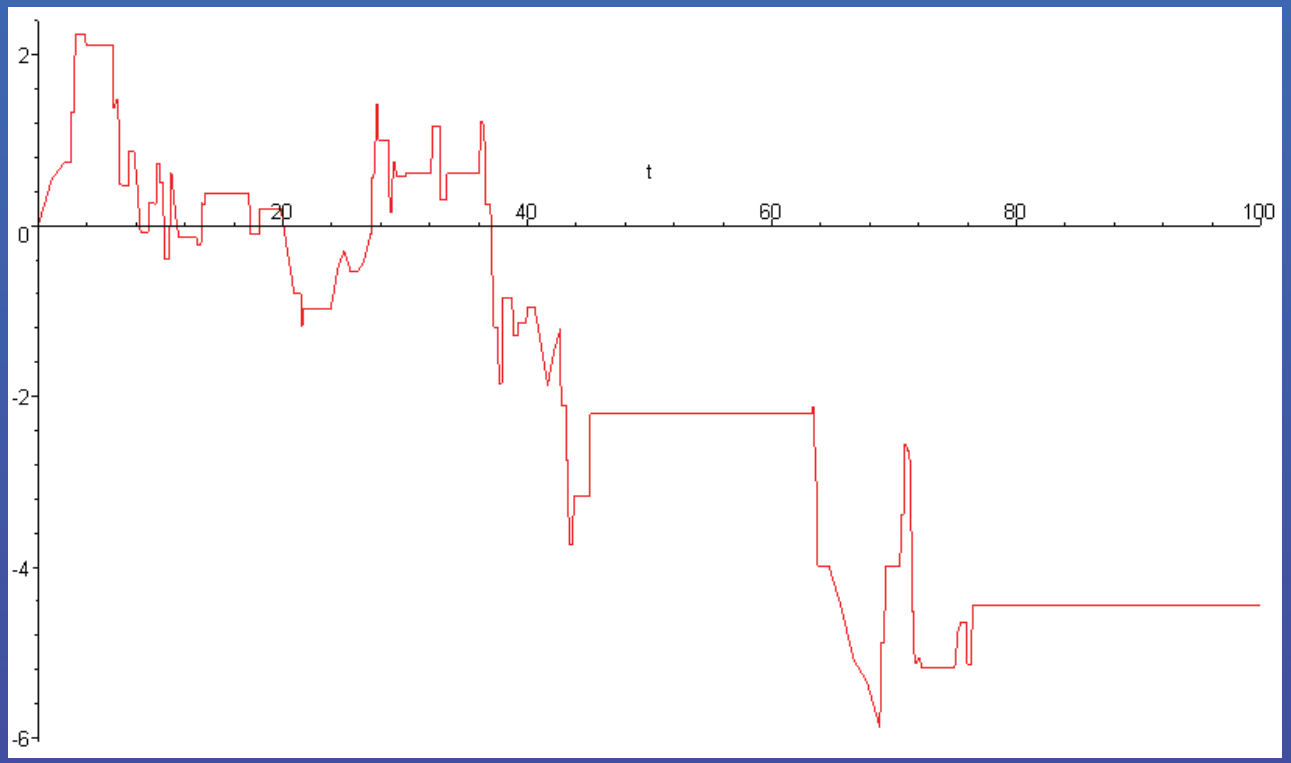
CTRW scaling limit is a subordinated process:

$$\begin{aligned} c^{-\beta/2}S(N(ct)) &= (c^{\beta})^{-1/2}S(c^{\beta} \cdot c^{-\beta}N(ct)) \\ &\approx (c^{\beta})^{-1/2}S(c^{\beta}Q(t)) \implies W(Q(t)). \end{aligned}$$

CTRW simulation with heavy tail waiting times



Longer time scale



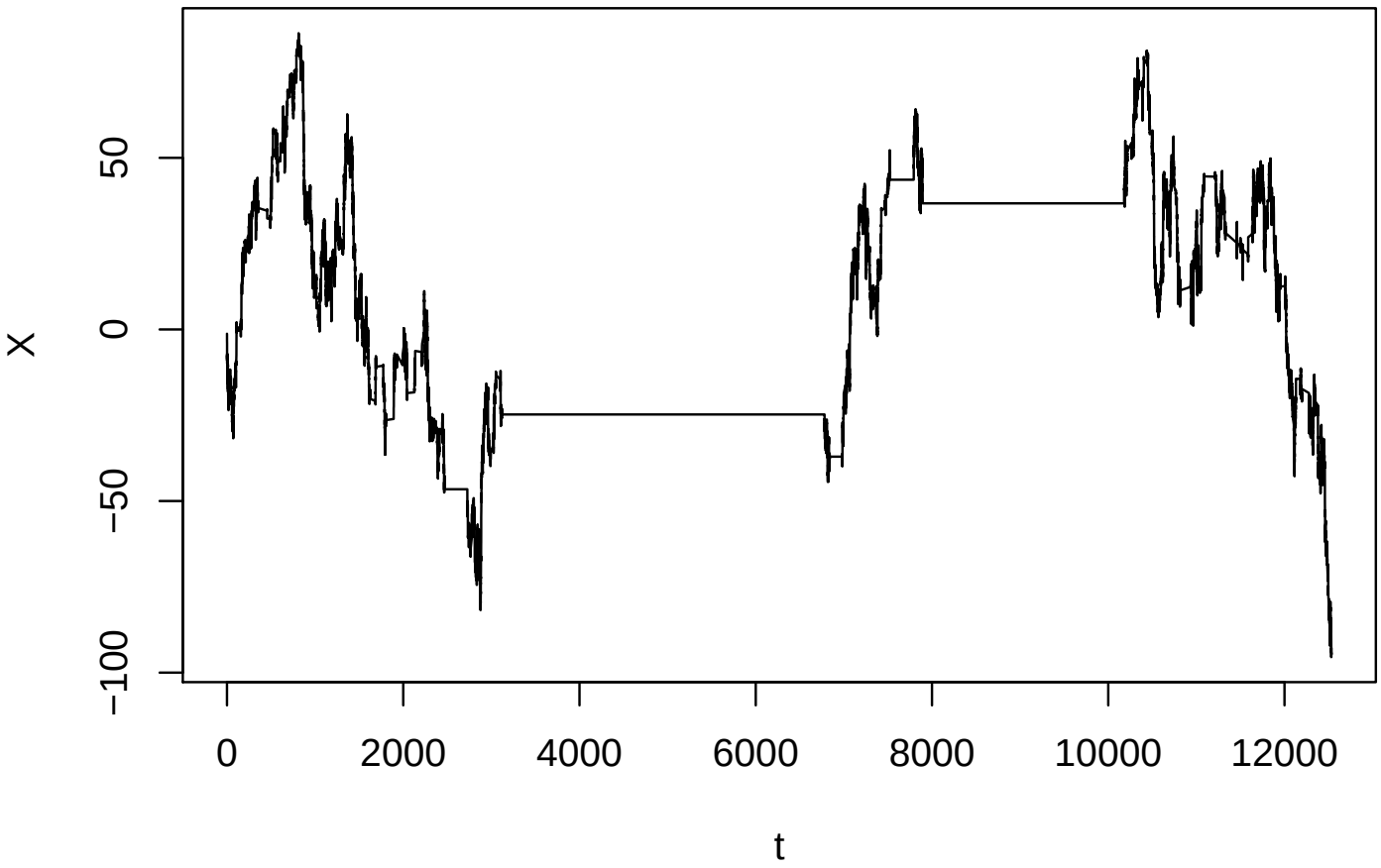


Figure: Typical sample path of the iterated process $W(Q(t))$. Here $W(t)$ is a Brownian motion and $Q(t)$ is the inverse of a $\beta = 0.8$-stable subordinator. Graph has dimension $1 + \beta/2 = 1 + 0.4$. The limit process retains long resting times

"Locally constant" property has also been observed by Davydov (2011) in his studies of the convex hull of the sample path of fractional Brownian motion.

Davydov defines a **Cantor–staircase function** $f(t)$ as a continuous, nondecreasing function such that for almost every t, there exists an $\varepsilon > 0$ such that $f(s) = f(t)$ for all $s \in (t - \varepsilon, t + \varepsilon)$.

It can be proven using Theorem 4 in Chapter III of Bertoin (1996) that, with probability one, the sample paths of $Q(t)$ are Cantor–staircase functions.

Power law waiting times

- Wait between solar flares $1 < \beta < 2$
- Wait between raindrops $\beta = 0.68$
- Wait between money transactions $\beta = 0.6$
- Wait between emails $\beta \approx 1.0$
- Wait between doctor visits $\beta \approx 1.4$
- Wait between earthquakes $\beta = 1.6$
- Wait between trades of German bond futures $\beta \approx 0.95$
- Wait between Irish stock trades $\beta = 0.4$ (truncated)

Hausdorff dimension

For any $\alpha > 0$, the α-dimensional Hausdorff measure of $F \subseteq \mathbb{R}^d$ is defined by

$$s^\alpha\text{-}m(F) = \lim_{\epsilon \to 0} \inf \left\{ \sum_i (2r_i)^\alpha : F \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i < \epsilon \right\}, \quad (1)$$

where $B(x, r)$ denotes the open ball of radius r centered at x.
It is well-known that s^α-m is a metric outer measure and every Borel set in $\mathbb{R}^d$ is s^α-m measurable.
The Hausdorff dimension of F is defined by

$$\dim_{\mathrm{H}} F = \inf \left\{ \alpha > 0 :\ s^\alpha\text{-}m(F) = 0 \right\} = \sup \left\{ \alpha > 0 :\ s^\alpha\text{-}m(F) = \infty \right\}.$$

It is easily verified that $\dim_{\mathrm{H}}$ satisfies the σ-stability property: For any $F_n \subseteq \mathbb{R}^d$, one has

$$\dim_{\mathrm{H}}\left(\bigcup_{n=1}^{\infty} F_n\right) = \sup_{n\geq 1} \dim_{\mathrm{H}} F_n. \tag{2}$$

- The Euclidean space $\mathbb{R}^d$ has Hausdorff dimension d.
- The circle S1 has Hausdorff dimension 1.
- Line segments have Hausdorff dimension 1.
- Countable sets have Hausdorff dimension 0.
- The Cantor set (a zero-dimensional topological space) has Hausdorff dimension $\ln 2/\ln 3$

Hausdorff dimension of image

Let $Z = \{Z(t), t \geq 0\}$ be the iterated process with values in $\mathbb{R}^d$ defined by $Z(t) = Y(Q(t))$, where the processes Y and Q independent and $Q(t)$ is a nondecreasing continuous process. If $Q(1) > 0$ a.s. and there exist constants c_1 and c_2 such that for all constants $0 < a < \infty$

$$\dim_{\mathrm{H}} Y([0,a]) = c_1, \ \ a.s. \tag{3}$$

then almost surely

$$\dim_{\mathrm{H}} Z([0,1]) = c_1. \tag{4}$$

Applying this result to the space-time process $x \mapsto (x, Y(x))$ with values in $\mathbb{R}^{d+1}$, one obtains immediately the following corollary. If $Q(1) > 0$ a.s. and there exist constants c_2 such that for all constants $0 < a < \infty$, $\dim_{\mathrm{H}} \mathrm{Gr}\, Y([0,a]) = c_2 \quad a.s.$, then $\dim_{\mathrm{H}} \{(Q(t), Y(Q(t))) : t \in [0,1]\} = \dim_{\mathrm{H}} \mathrm{Gr}\, Y([0,1]), \quad a.s.$

Hausdorff dimension of graph

Let

$$A(x) = (P(x), Y(x)), \qquad \forall x \geq 0, \tag{5}$$

where $P = \{P(x), x \geq 0\}$ is defined by

$$P(x) = \inf\left\{t > 0 : Q(t) > x\right\}. \tag{6}$$

If $Q(1) > 0$ a.s. and there exist a constant c_3 such that for all constants $0 < a < \infty$

$$\dim_{\mathrm{H}} A([0, a]) = c_3 \tag{7}$$

then

$$\begin{aligned} \dim_{\mathrm{H}} \mathrm{Gr} Z([0,1]) &= \dim_{\mathrm{H}} \{(t, Z(t)) : t \in [0,1]\} \\ &= \max\left\{1, \dim_{\mathrm{H}} A([0,1])\right\}, \qquad \text{a.s.} \end{aligned} \tag{8}$$

Due to Meerschaert, Nane and Xiao (2011).

Sketch of proof

The sample function $x \mapsto P(x)$ is a.s. strictly increasing and we can write the unit interval [0, 1] in the state space of P as

$$[0,1] = P([0, Q(1))) \cup \bigcup_{i=1}^{\infty} I_i, \tag{9}$$

where for each $i \geq 1$, I_i is a subinterval on which $Q(t)$ is a constant. Using P we can express $I_i = [P(x_i-), P(x_i))$, which is the gap corresponding to the jumping site x_i of P, except in the case when $x_i = Q(1)$. In the latter case, $I_i = [P(x_i-), 1]$. Notice that I_i $(i \geq 1)$ are disjoint intervals and

$$Q(t) = Q(s) \text{ if and only if } s, t \in I_i \text{ for some } i \geq 1.$$

Thus, over each interval I_i, the graph of X is a horizontal line segment.

Thus, over each interval I_i, the graph of Z is a horizontal line segment. More precisely, we can decompose the graph set of Z as

$$\begin{aligned}\mathrm{Gr}Z([0,1]) &= \big\{(t, Y(Q(t))) : t \in [0,1]\big\} \\ &= \big\{(t, Y(Q(t))) : t \in P([0, Q(1)))\big\} \\ &\cup \bigcup_{i=1}^{\infty} \big\{(t, Y(Q(t))) : t \in I_i\big\}.\end{aligned} \tag{10}$$

Hence, by the σ-stability of $\dim_{\mathrm{H}}$, we have

$$\dim_{\mathrm{H}} \mathrm{Gr}Z([0,1]) = \max\big\{1, \dim_{\mathrm{H}}\big\{(t, Y(Q(t))) : t \in P([0, Q(1)))\big\}\big\}. \tag{11}$$

On the other hand, every $t \in P([0, Q(1)))$ can be written as $t = P(x)$ for some $0 \le x < Q(1)$ and $Q(t) = Q(P(x)) = x$, we see that a.s.

$$\big\{(t, Y(Q(t))) : t \in P([0, Q(1)))\big\} = \big\{(P(x), Y(x)) : x \in [0, Q(1)))\big\}, \tag{12}$$

Examples

Let $Z = \{Y(Q(t)), t \geq 0\}$, where $Y = \{Y(x) : \ x \geq 0\}$ is a **stable Lévy motion** of index $\alpha \in (0, 2]$ with values in $\mathbb{R}^d$ and $Q(t)$ is the **inverse of a stable subordinator** of index $0 < \beta < 1$, independent of Y. Then

$$\dim_{\mathrm{H}} Z([0,1]) = \dim_{\mathrm{H}} Y([0,1]) = \min\{d, \alpha\}, \qquad \text{a.s.} \tag{13}$$

$$\dim_{\mathrm{H}} \mathrm{Gr} Z([0,1]) = \begin{cases} \max\{1, \alpha\} & \text{if } \alpha \leq d, \\ 1 + \beta(1 - \frac{1}{\alpha}) & \text{if } \alpha > d = 1, \end{cases} \qquad \text{a.s.} \tag{14}$$

compare to

$$\begin{aligned} \dim_{\mathrm{H}} \mathrm{Gr}\, Y([0,1]) &= \dim_{\mathrm{H}} \{(Q(t), Y(Q(t))) : t \in [0,1]\} \\ &= \begin{cases} \max\{1, \alpha\} & \text{if } \alpha \leq d, \\ 2 - \frac{1}{\alpha} & \text{if } \alpha > d = 1. \end{cases} \qquad \text{a.s.} \end{aligned} \tag{15}$$

Let $Z = \{Y(Q(t)), t \geq 0\}$, where Y is a **fractional Brownian motion** with values in $\mathbb{R}^d$ of index $H \in (0,1)$ and $Q(t)$ is the the **inverse of a β-stable subordinator** P which is independent of Y. Then

$$\dim_{\mathrm{H}} Z([0,1]) = \dim_{\mathrm{H}} Y([0,1]) = \min\left\{d, \frac{1}{H}\right\}, \qquad \text{a.s.} \tag{16}$$

$$\dim_{\mathrm{H}} \mathrm{Gr} Z([0,1]) = \begin{cases} \frac{1}{H} & \text{if } 1 \leq Hd, \\ \beta + (1 - H\beta)d = d + \beta(1 - Hd) & \text{if } 1 > Hd, \end{cases} \tag{17}$$

compare to

$$\dim_{\mathrm{H}} \{(Q(t), Y(Q(t))) : t \in [0,1]\} = \dim_{\mathrm{H}} \mathrm{Gr}\, Y([0,1])$$
$$= \begin{cases} \frac{1}{H} & \text{if } 1 \leq Hd, \\ 1 + (1 - H)d = d + (1 - Hd) & \text{if } 1 > Hd, \end{cases} \qquad \text{a.s.} \tag{18}$$

Fractional derivatives: An old idea gets new life

- Fractional derivatives $D^\beta f(x)$ for any $\beta > 0$ were invented by Leibniz (1695) soon after the more familiar integer derivatives.
- The Caputo fractional derivative of order $0 < \beta < 1$ defined by

$$D_t^\beta g(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t d_s g(s) \frac{ds}{(t-s)^\beta} \tag{19}$$

  was invented to properly handle initial values (Caputo 1967).
- Laplace transform of $D_t^\beta g(t)$ is $s^\beta \tilde{g}(s) - s^{\beta-1} g(0)$ incorporates the initial value in the same way as the first derivative.

examples

- $$D_t^\beta(t^p) = \frac{\Gamma(1+p)}{\Gamma(p+1-\beta)} t^{p-\beta}$$
- $$D_t^\beta(e^{\lambda t}) = \lambda^\beta e^{\lambda t} - \frac{t^{-\beta}}{\Gamma(1-\beta)}?$$
- $$D_t^\beta(\sin t) = \sin(t + \frac{\pi\beta}{2})$$

Time-fractional model for Anamolous sub-diffusion

Let $0 < \beta < 1$. Nigmatullin (1986) gave a physical derivation of fractional diffusion

$$\partial_t^\beta u(t,x) = \partial_x^2 u(t,x); \quad u(0,x) = f(x) \tag{20}$$

Zaslavsky (1994) used this to model Hamiltonian chaos.
(20) has the unique solution

$$u(t,x) = \mathbb{E}_x[f(W(Q(t)))] = \int_0^\infty p(s,x) g_{Q(t)}(s) ds$$

where $p(t,x) = \mathbb{E}_x[f(W(t))]$ and $Q(t) = \inf\{\tau > 0 : \ P(\tau) > t\}$, $P(t)$ is a stable subordinator with index β and $\mathbb{E}(e^{-sP(t)}) = e^{-ts^\beta}$ (Baeumer and Meerschaert, 2002).
$\mathbb{E}_x(W(Q(t)))^2 = \mathbb{E}(Q(t)) \approx t^\beta$.

Taking Fourier-Laplace transform of the Equation (20) gives

$$\begin{aligned}\bar{u}(s,k) &= \frac{s^{\beta-1}\hat{f}(k)}{s^{\beta}+k^2} \\ &= s^{\beta-1}\int_0^{\infty}\exp(-[s^{\beta}+k^2]l)\hat{f}(k)dl\end{aligned} \tag{21}$$

The next step is to invert this Fourier-Laplace transform using the fact that $Q(t)$ has density

$$f_{Q(t)}(s) = \frac{t}{\beta}g_{\beta}(\frac{t}{s^{1/\beta}})s^{-1/\beta-1}, \text{ and } \int_0^{\infty} e^{-su}g_{\beta}(u) = e^{-s^{\beta}}.$$

In the case $\beta = 1/2$,

$$f_{Q(t)}(s) = \frac{2}{\sqrt{4\pi t}}e^{-s^2/4t} = f_{|W(t)|}(s)$$

This proof is due to Meerschaert, Benson, Scheffler and Baeumer (2002)

Equivalence to Higher order PDE's

Let $\Delta f = \sum_{k=1}^{d} \partial_{x_k}^2 f(x)$, Laplacian of f.

- For any $m = 2, 3, 4, \ldots$ both the Cauchy problem

$$\partial_t u(t,x) = \sum_{j=1}^{m-1} \frac{t^{j/m-1}}{\Gamma(j/m)} \Delta^j f(x) + \Delta^m u(t,x); \quad u(0,x) = f(x) \tag{22}$$

  and the fractional Cauchy problem:

$$\partial_t^{1/m} u(t,x) = \Delta u(t,x); \quad u(0,x) = f(x), \tag{23}$$

  have the same unique solution given by

$$u(t,x) = \int_0^\infty p((t/s)^{1/m}, x) g_{1/m}(s)\, ds = \mathbb{E}_x(f(W(Q(t))))$$

- Due to Baeumer, Meerschaert, and Nane TAMS(2009).

Connections to iterated Brownian motions

- Orsingher and Benghin (2004) and (2008) show that for $\beta = 1/2^n$ the solution to

$$\partial_t^{1/2^n} u(t,x) = \Delta_x u(t,x); \quad u(0,x) = f(x), \tag{24}$$

is given by running

$$I_{n+1}(t) = W_1(|W_2(|W_3(|\cdots(W_{n+1}(t))\cdots|)|)|)$$

Where W_j's are independent Brownian motions, i.e., $u(t,x) = \mathbb{E}_x(f(I_{n+1}(t)))$ solves (24), and solves (22) for $m = 2^n$.

Corollary

- We obtain the equivalence of one dimensional distributions in the case $Q(t)$ is the inverse stable subordinator of index $\beta = 1/2^n$

$$I_{n+1}(t) = W_1(|W_2(|W_3(|\cdots(W_{n+1}(t))\cdots|)|)|) \overset{(d)}{=} W_1(Q(t))$$

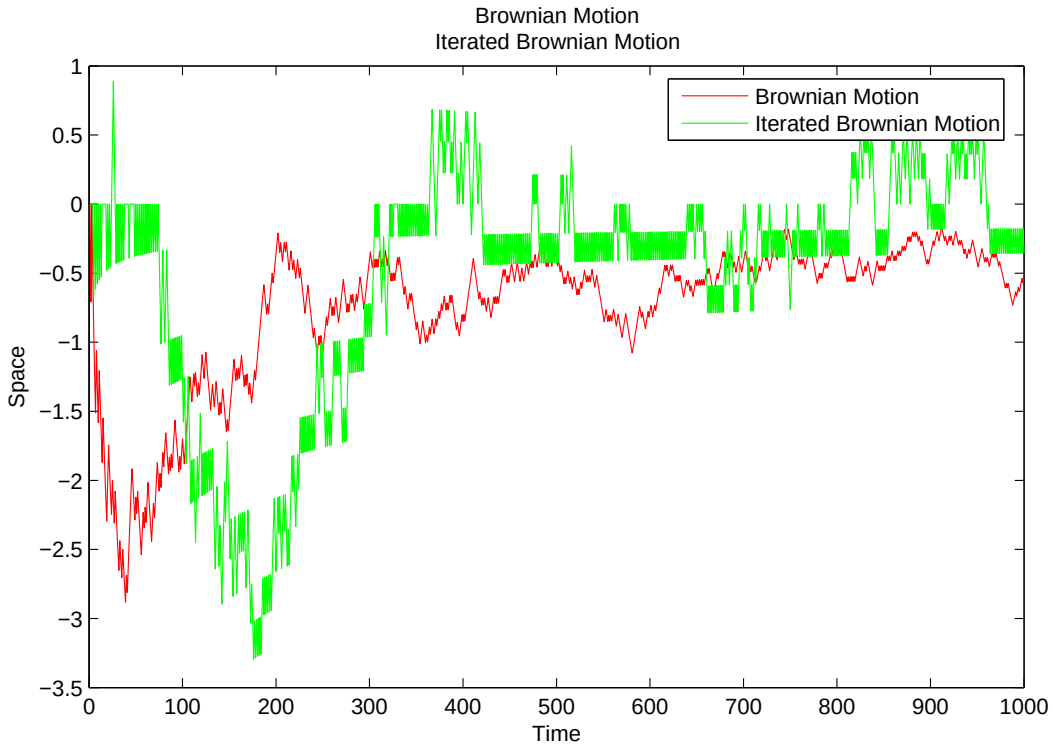


Figure: Simulations of iterated Brownian motions

Heat equation in bounded domains

Heat equation in D with Dirichlet boundary conditions:

$$\begin{aligned} \partial_t u(t,x) &= \Delta u(t,x), \quad x \in D, \ t > 0, \\ u(t,x) &= 0, \quad x \in \partial D; \quad u(0,x) = f(x), \quad x \in D. \end{aligned}$$

When $D = (0, M)$, the heat equation can be solved by separation of variables: set $u(t,x) = \phi(x)T(t)$. Hence $\phi(x)$ satisfies

$$\partial_x^2 \phi(x) = -\lambda \phi(x), \quad x \in (0, M), \lambda > 0; \quad \phi(0) = 0, \quad \phi(M) = 0$$

and $T(t)$ satisfies $\partial_t T(t) = -\lambda T(t); \quad T(0) = 1.$

$$\phi_n(x) = \left(\frac{2}{M}\right)^{1/2} \sin(n\pi x/M), \quad \lambda_n = \frac{\pi^2 n^2}{M^2}, T_n(t) = e^{-\lambda_n t}.$$

Same applies in any dimension $d \geq 1$.

Denote the eigenvalues and the eigenfunctions of Δ_D by $\{\lambda_n, \phi_n\}_{n=1}^{\infty}$, where $\phi_n \in C^{\infty}(D)$. The corresponding heat kernel is given by

$$p_D(t,x,y) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x)\phi_n(y).$$

The series converges absolutely and uniformly on $[t_0, \infty) \times D \times D$ for all $t_0 > 0$. In this case, the semigroup given by

$$\begin{aligned} T_D(t)f(x) &= \mathbb{E}_x[f(W(t))I(t < \tau_D(X))] = \int_D p_D(t,x,y)f(y)dy \\ &= \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x)\bar{f}(n) \end{aligned}$$

solves the Heat equation in D with Dirichlet boundary conditions.

Fractional diffusion in bounded domains

$$\begin{aligned} \partial_t^\beta u(t,x) &= \Delta u(t,x); \quad x \in D, \ t > 0 \\ u(t,x) &= 0, \ x \in \partial D, \ t > 0; \quad u(0,x) = f(x), \ x \in D. \end{aligned} \tag{25}$$

Separation of variables gives the unique (classical) solution as

$$\begin{aligned} u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x) M_\beta(-\lambda_n t^\beta) \\ &= \mathbb{E}_x[f(W(Q(t)))I(\tau_D(W) > Q(t))] \\ &= \mathbb{E}_x[f(W(Q(t)))I(\tau_D(W(Q)) > t)] \\ &= \frac{t}{\beta}\int_0^\infty T_D(l) f(x) g_\beta(t l^{-1/\beta}) l^{-1/\beta-1} dl \end{aligned}$$

Joint work with Meerschaert and Vellaisamy, AOP (2009).

Analytic solution in intervals $(0, M) \subset \mathbb{R}$ was obtained by Agrawal (2002).
In this case, eigenfunctions and eigenvalues are

$$\phi_n(x) = \left(\frac{2}{M}\right)^{1/2} \sin(n\pi x/M), \quad \lambda_n = \frac{\pi^2 n^2}{M^2}$$

The time fractional diffusion on $(0, M)$ has the solution

$$u(t,x) \quad = \quad \sum_{n=1}^{\infty} \bar{f}(n) \left(\frac{2}{M}\right)^{1/2} \sin(n\pi x/M) M_\beta(-\lambda_n t^\beta)$$

here

$$M_\beta(z) = \sum_{k=0}^{\infty} \frac{z^n}{\Gamma(1+\beta n)}$$

For $\beta = 1$, $M_1(-z) = e^{-z}$, and u coincides with the solution of the heat equation on $(0, M)$.

Extensions

Uniformly elliptic operator of divergence form is defined on C^2 functions by

$$Lu = \sum_{i,j=1}^{d} \partial_{x_j} \left(a_{ij}(x)(\partial_{x_i} u) \right) \tag{26}$$

with $a_{ij}(x) = a_{ji}(x)$ and, for some $\lambda > 0$,

$$\lambda \sum_{i=1}^{n} y_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) y_i y_j \leq \lambda^{-1} \sum_{i=1}^{n} y_i^2, \quad \forall y \in \mathbb{R}^d. \tag{27}$$

New time operators

Laplace symbol: $\psi(s)$	inverse subordinator	time operator
$\int_0^\infty (1-e^{-sy})\nu(dy)$	$Q_\psi(t)$	$\psi(\partial_t) - \delta(0)\nu(t,\infty)$
s^β	$Q(t)$	∂_t^β, Caputo
$\int_0^1 s^\beta \Gamma(1-\beta)\mu(d\beta)$	$Q_\mu(t)$	$\int_0^1 \partial_t^\beta \Gamma(1-\beta)\mu(d\beta)$
$(s+\lambda)^\beta - \lambda^\beta$	$Q_\lambda(t)$	$\partial_t^{\beta,\lambda}$ in (28)

$$\begin{aligned}\partial_t^{\beta,\lambda} g(t) &= \psi_\lambda(\partial_t) g(t) - g(0)\phi_\lambda(t,\infty) \\ &= e^{-\lambda t}\frac{1}{\Gamma(1-\beta)} d_t\left[\int_0^t \frac{e^{\lambda s} g(s)\, ds}{(t-s)^\beta}\right] - \lambda^\beta g(t) \\ &- \frac{g(0)}{\Gamma(1-\beta)}\int_t^\infty e^{-\lambda r}\beta r^{-\beta-1}\, dr.\end{aligned} \tag{28}$$

Subdiffusion: $0 < \beta < 1$, $\mathbb{E}_x(B(Q(t)))^2 = \mathbb{E}(Q(t)) \approx t^\beta$.
Ultraslow diffusion: For special $\mu \in RV_0(\theta - 1)$ for some $\theta > 0$: $\mathbb{E}_x(B(Q_\mu(t)))^2 = \mathbb{E}(Q_\mu(t)) \approx (\log t)^\theta$.
Intermediate between subdiffusion and diffusion: Tempered fractional diffusion

$$\mathbb{E}_x(B(Q_\lambda(t)))^2 \approx \begin{cases} t^\beta/\Gamma(1+\beta), & t << 1 \\ t/\beta, & t >> 1. \end{cases}$$

$B(Q_\lambda(t))$ occupies an intermediate place between subdiffusion and diffusion (Stanislavsky et al., 2008)

New space operators

Laplace exp.: $\psi(s)$	subord.	process	Generator
$\int_0^\infty (1-e^{-sy})\nu(dy)$	$D_\psi(t)$	$W(D_\psi(t))$	$\psi(-\Delta)$
s^β	$Q_\beta(t)$	$W(Q_\beta(t))$	$(-\Delta)^\beta$
$(s+m^{1/\beta})^\beta - m$	$T_\beta(t,m)$	$W(T_\beta(t,m))$	$(-\Delta+m^{1/\beta})^\beta - m$
$\log(1+s^\beta)$	$D_{\log}(t)$	$W(D_{\log}(t))$	$\log(1+(-\Delta)^\beta)$

$W(t)$, Brownian motion

$W(Q_\beta(t))$, symmetric stable process

$W(T_\beta(t,m))$, relativistic stable process

$W(D_{\log}(t))$, geometric stable process

Space-time fractional diffusion in bounded domains

$$
\begin{aligned}
[\psi_1(\partial_t) - \delta(0)\nu(t,\infty)]u(t,x) &= -\psi_2(-\Delta_x)u(t,x); \quad x \in D, \ t > 0 \\
u(t,x) &= 0, \ x \in \partial D \ (\text{or } x \in D^C), \ t > 0; \\
u(0,x) &= f(x), \ x \in D.
\end{aligned}
$$

has the unique (classical) solution

$$
\begin{aligned}
u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)h_{\psi_1}(t,\lambda_n) \\
&= \mathbb{E}_x[f(B(D_{\psi_2}(Q_{\psi_1}(t))))I(\tau_D(B(D_{\psi_2}(Q_{\psi_1}))) > t)]
\end{aligned}
$$

$h_{\psi_1}(t,\lambda) = \mathbb{E}(e^{-\lambda Q_{\psi_1}(t)})$ is the solution of

$$[\psi_1(\partial_t) - \delta(0)\nu(t,\infty)]h_{\psi_1}(t,\lambda) = -\lambda h_{\psi_1}(t,\lambda); \quad h_{\psi_1}(0,\lambda) = 1.$$

$$-\psi_2(-\Delta_x)\phi_n(x) = -\lambda_n\phi_n(x); \ \phi_n(x) = 0, \ x \in \partial D \ (\text{or } x \in D^C).$$

$\psi_2(s) = s$ with

- $\psi_1(s) = s^\beta$: subdiffusion
- $\psi_1(s) = \int_0^1 s^\beta \Gamma(1-\beta)\mu(d\beta)$: Ultraslow diffusion
- $\psi_1(s) = (s+\lambda)^\beta - \lambda^\beta$: tempered fractional diffusion.

$\psi_2(s) = s^{\beta_2}$ with the three ψ_1s: space-time fractional diffusion, $(-\Delta)^{\beta_2}$

Joint work with Meerschaert, Vellaisamy and Chen (2009,2010, 2012).

Further research

- Work completed recently for the symmetric stable process as the outer process. The corresponding space operator is $(-\Delta)^{\alpha/2}$ for $0 < \alpha \leq 2$.
- Work in progress for other Subordinated Brownian motions, e.g. relativistic stable process as the outer process.... corresponding to the operator $(-\Delta + m^{1/\beta})^{\beta} - m$ for $0 < \alpha \leq 2$, $m \geq 0$.
- Extension to Neumann boundary conditions...
- Fractal properties of $W(Q(t))$ and other subordinate processes
- Applications-interdisciplinary research

Thank You!