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INTERSECTIONS OF CONTINUOUS, LIPSCHITZ, HÖLDER CLASS, AND SMOOTH FUNCTIONS

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ABSTRACT. We present some improvements of known theorems and examples concerning intersections of continuous or Lipschitz functions with smooth functions or intersections of smooth functions or Hölder class functions with smoother functions. We are particularly concerned with our ability to force the projection of the intersection to be uncountable within a given set M which is either large in measure or in category (or both).

1. INTRODUCTION

We shall restrict ourselves to the consideration of functions $f : [0, 1] \to \mathbb{R}$ which belong to the function classes indicated in the following diagram of implications:

$$\begin{array}{ccccccc} & C^{1\oplus 1} & & \mathrm{Lip}^1 & & & \\ \nearrow & & \searrow \;\; \nearrow & & \searrow & & \\ C^2 & & C^1 & & C & & \\ \searrow & & \nearrow \;\; \searrow & & \nearrow & & \\ & D^2 & & D^1 & & & \end{array}$$

C will denote the class of continuous functions and C^1 the class of continuously differentiable functions. Lip^1 will denote the class of functions such that $\{|f(x) - f(y)|/|x - y|: x, y \in [0, 1]\}$ is bounded, and $C^{1\oplus 1}$ denotes the "Hölder class" of C^1 functions f such that $f' \in \mathrm{Lip}^1$. For each $n \in \mathbb{N}$, D^n denotes the class of n-times differentiable functions and C^n denotes the class of n-times continuously differentiable functions. $\mathscr{L}$ denotes the Lebesgue measurable subsets of $[0, 1]$, and $\mathscr{L}_0$ denotes the measure zero sets in $\mathscr{L}$. Thus, $\mathscr{L}\backslash\mathscr{L}_0$ denotes the sets of positive measure and co-$\mathscr{L}_0$ denotes the sets of full

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measure. The statement that a subset M of $[0, 1]$ is measure dense in a subinterval I of $[0, 1]$ means that $\lambda(M \cap J) > 0$ for every subinterval J of I. B_w denotes the sets with the Baire property (wide sense) [4], FC denotes the first category sets, and co-FC denotes the residual sets.

The first theorem of the type mentioned in the abstract which we consider is the following theorem due to Laczkovich [5] and Agronski, Bruckner, Laczkovich, and Preiss [1].

Theorem 1. *If $M \in \mathscr{L}\backslash\mathscr{L}_0$, then for every $f \in C$ there exists a $g \in C^1$ such that $\{x \in M: f(x) = g(x)\}$ is uncountable.*

Thus, continuous functions have uncountable intersections with C^1 functions and you can make the projection of this intersection be uncountable inside any given set of positive measure. On the other hand, it is fairly easy to see that you cannot necessarily make this intersection be uncountable inside any given $B_w\backslash FC$ set because of the following partial converse to Theorem 1.

Theorem 2. *If $M \in \mathscr{L}_0$, there is an $f \in C$ such that $\{x \in M: f(x) = g(x)\}$ is countable for every $g \in D^1 \cup \mathrm{Lip}^1$.*

Proof. Let $M \in \mathscr{L}_0$. It follows that there is a sequence $C_1, C_2, \ldots$ of pairwise disjoint Cantor subsets of $[0, 1]$ such that $\lambda(C_1 \cup C_2 \cup \cdots) = 1$ and $M \subseteq G = [0, 1]\backslash(C_1 \cup C_2 \cup \cdots)$. The C_i can be chosen so that $1\lambda(C_1) + 2\lambda(C_2) + 3\lambda(C_3) + \cdots$ converges. Let $h: [0, 1] \to \mathbb{R}$ be such that $h(x) = i$ if $x \in C_i$ for some $i \in \mathbb{N}$, and $h(x) = 0$ otherwise. Define $f(x) = \int_0^x h(t)\,dt$. It follows that f is differentiable in the "extended sense" on G, with $f'(x) = +\infty$ for every $x \in G$. Thus, $\{x \in M: f(x) = g(x)\}$ is countable for every $g \in D^1 \cup \mathrm{Lip}^1$. □

2. Intersection of Lipschitz, Hölder class, and smooth functions

In 1944 Federer [2] (also see [3, Theorem 3.1.15]) proved a theorem from which it follows that functions in $D^1 \cup \mathrm{Lip}^1$ intersect C^1 functions and that you can make the projection of the intersection be of positive measure inside any given $\mathscr{L}\backslash\mathscr{L}_0$ set M. We now prove a category version of this result.

Theorem 3. *If $M \in B_w\backslash FC$, then for every $f \in D^1 \cup \mathrm{Lip}^1$ there exists a $g \in C^1$ such that $\{x \in M: f(x) = g(x)\}$ is uncountable.*

Proof. M will be residual in some subinterval of $[0, 1]$. We assume without loss of generality that M is co-FC on $[0, 1]$ and let $G_1 \supseteq G_2 \supseteq \cdots$ be a sequence of dense open subsets of $[0, 1]$ such that $G_1 \cap G_2 \cap \cdots \subseteq M$.

We first consider the case where $f \in \mathrm{Lip}^1$ (the D^1 case follows easily from this case). We know that f is a.e. differentiable, that f' is bounded, and that $f(x) = \int_0^x f'(t)\,dt$ for every $x \in [0, 1]$. Let M and m be the supremum and infimum, respectively, of the range of f'. We assume without loss of generality that $0 \le m < M \le 1$. Otherwise, we could work with the function $F(x) = \int_0^x F'(t)\,dt$, where $F'(x) = (f'(x) - m)/(M - m)$, and transform back to f at the end of the argument.

Case I. We assume there is a number $0 < t < 1$ and an interval $I = [a, b] \subseteq [0, 1]$ such that the sets $U_t = \{x: f'(x) > t\}$ and $V_t = \{x: f'(x) < t\}$ are both measure dense in I. It follows that the function $h(x) = f(x) - f(a) - t(x - a)$ is "nowhere monotone" on I (i.e., monotone on no subinterval of I). It follows

[6] that there is a number s such that the set $T = \{x \in I : h(x) = s\}$ is uncountable. Let x_0 and x_1 be elements of T. Using continuity of h, we can find short pairwise disjoint closed intervals I_0 and I_1 located close to x_0 and x_1, respectively, lying interior to $G_1 \cap I$, having common length d_1, and such that if $x \in I_0$ and $y \in I_1$, then $|x-y| > d_1$ and $|h(y)-h(x)|/|y-x| < 1/1$. This completes stage 1 of the construction.

Now, suppose n is a positive integer and that the intervals $\{I_b : b \in \{0, 1\}^n\}$, all having common length d_n, have been constructed. Let $b \in \{0, 1\}^n$. There is a number s_b such that the set $T_b = \{x \in \dot{I}_b : h(x) = s_b\}$ is uncountable. Let x_{b0} and x_{b1} be two elements of T_b. We can find short pairwise disjoint closed intervals I_{b0} and I_{b1} close to x_{b0} and x_{b1}, respectively, lying interior to $G_{n+1} \cap I_b$, having length $d_{n+1} < d_n/2$ (same d_{n+1} for all $b \in \{0, 1\}^n$), and such that if $x \in I_{b0}$ and $y \in I_{b1}$, then $|x-y| > d_{n+1}$ and $|h(y)-h(x)|/|y-x| < 1/(n+1)$.

Now, let $P = \bigcap_{n\in\mathbb{N}} \bigcup\{I_b : b \in \{0, 1\}^n\}$. P is a perfect subset of $G_1 \cap G_2 \cap \cdots \subseteq M$ and $h|P$ satisfies the "uniform differentiability" requirements of the Whitney Extension Theorem [8; 3, Theorem 3.1.14] which guarantee that $h|P$ can be extended to a C^1 function H. Then, the C^1 function g defined by $g(x) = H(x)+f(a)+t(x-a)$ agrees with f over P, and the theorem is proved in this case.

Case II. We assume the bare denial of Case I (i.e., for every number $0 < t < 1$ and every subinterval I of $[0, 1]$, there is a subinterval J of I such that either $\lambda(U_t \cap J) = 0$ or $\lambda(V_t \cap J) = 0$). Let $n \in \mathbb{N}$. It follows from the assumptions that there will exist an interval $I \subseteq [0, 1]$ and an integer i, $1 \le i \le n$, such that $\lambda(\{x \in I : f'(x) > i/n \text{ or } f'(x) < (i-1)/n\}) = 0$. To see this, first start with $i = n$ and $t = i/n$. There is a subinterval J_n of $[0, 1]$ such that either $\lambda(U_t \cap J_n) = 0$ or $\lambda(V_t \cap J_n) = 0$. If it is the latter case set $I = J_n$ and we are through. Otherwise, set $i = n-1$ and repeat to get $J_{n-1} \subseteq J_n$. If $\lambda(V_t \cap J_{n-1}) = 0$, we are through. Otherwise, if necessary, continue the process until we get to $J_1 \subseteq J_2$, which must suffice for I.

Consequently, there will exist a collection H_n of such intervals I such that the intervals in H_n are pairwise disjoint, the closure of each interval in H_n lies in the interior of an interval of H_{n-1}, and $\bigcup H_n$ is dense in $[0, 1]$. Let $H = \bigcap_{n\in\mathbb{N}} \bigcup H_n$.

It follows that for every $x \in H$, $f'(x)$ exists. To see this, let $I_1, I_2, \ldots$ be the sequence of intervals from $H_1, H_2, \ldots$, respectively, containing x. For each n, let i_n be the integer i described above for which

$$\lambda(\{y \in I_n : f'(y) > i/n \text{ or } f'(y) < (i-1)/n\}) = 0.$$

Notice that the sequences $\{(i_n - 1)/n\}$ and $\{i_n/n\}$ both converge to some common number m. $f'(x)$ must exist and equal m. To see this suppose $\varepsilon > 0$ and let n be such that $1/n < \varepsilon$. Consider any y in I_n. $[f(y)-f(x)]/(y-x) = \int_x^y f'(t)\,dt/(y-x)$, which lies between $(i_n-1)/n$ and i_n/n, which in turn differ by less than ε and also have m between them.

It is clear that for every $x \in H$, f' is "essentially continuous" at x (i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\{y : |x-y| < \delta,\ f'(y) \text{ exists, and } |f'(y) - f'(x)| \ge \varepsilon\}$ is $\mathscr{L}_0$).

We will now describe a special Cantor subset P of $G = G_1 \cap G_2 \cap \cdots$ (defined at beginning of proof). Choose two points x_0 and x_1 of $G \cap H$ and two disjoint closed neighborhoods I_0 and I_1 of x_0 and x_1, respectively, lying in G_1, lying inside intervals of H_1, and having length $< 1/2$ and two open intervals J_0 and J_1 of length less than $1/2$ such that $\lambda(\{x \in I_i: f'(x)$ exists and $f'(x) \notin J_i\}) = 0$ for $i = 0, 1$. Let K be the closed interval lying between I_0 and I_1.

Now suppose n is a positive integer and $b \in \{0, 1\}^n$. Choose two points x_{b0} and x_{b1} of $G \cap H$ interior to I_b and two disjoint closed neighborhoods I_{b0} and I_{b1} of x_{b0} and x_{b1}, respectively, lying in $G_{n+1} \cap I_b$, lying inside intervals of H_{n+1}, and having length $< 1/2^{n+1}$ and two open intervals J_{b0} and J_{b1} of length less than $1/2^{n+1}$, lying inside J_b, and such that

$$\lambda(\{x \in I_{bi}: f'(x) \text{ exists and } f'(x) \notin J_{bi}\}) = 0$$

for $i = 0, 1$. Let K_b be the closed interval lying between I_{b0} and I_{b1}.

The set $P = \bigcap_{n \in \mathbb{N}} \bigcup \{I_b: b \in \{0, 1\}^n\}$ is a perfect subset of G and $f'|P$ is continuous. We now extend $f'|P$ in a special way to a continuous function h on $[0, 1]$. First let (c, d) be the interval contiguous to P which contains the interval K. Notice that $f'(x) \in (-0.5, 1.5)$ for a.e. $x \in (c, d)$, so h can be defined continuously on $[c, d]$ so that (1) $h(c) = f'(c)$ and $h(d) = f'(d)$, (2) $h(x) \in (-0.5, 1.5)$ for every $x \in (c, d)$, and (3) $\int_c^d h(t)\,dt = \int_c^d f'(t)\,dt$. This can be accomplished in two steps. First choose c' close to c and d' close to d with $c < c' < d' < d$. Let $h_0(x) = f'(c)$ on $[c, c')$ and $h_0(x) = f'(d)$ on $(d', d]$. For $c' - c$ and $d - d'$ small enough we can set $h_0(x)$ equal to some constant $k \in (-0.5, 1.5)$ for $x \in [c', d']$ so that (3) holds for h_0 in place of h. Then for some small $\delta > 0$, we can let $h(x) = h_0(x)$ on $[c, c' - \delta]$, $[c'+\delta, d'-\delta]$, and $[d'+\delta, d]$, and let h be linear across the gaps $[c'-\delta, c'+\delta]$ and $[d' - \delta, d' + \delta]$, and we will have the desired h.

Suppose n is a positive integer and $b \in \{0, 1\}^n$. Let $(c(b), d(b))$ be the interval contiguous to P which contains the interval K_b. Notice that $f'(x) \in J_b$ for a.e. $x \in (c(b), d(b))$; so using the method described above, h can be defined continuously on $[c(b), d(b)]$ so that (1) $h(c(b)) = f'(c(b))$ and $h(d(b)) = f'(d(b))$, (2) $h(x) \in J_b$ for every $x \in (c(b), d(b))$, and (3) $\int_{c(b)}^{d(b)} h(t)\,dt = \int_{c(b)}^{d(b)} f'(t)\,dt$.

Set $h|P = f'|P$, and extend h continuously to the left and right of P. Let a be the least element of P. Now, if g is defined by $g(x) = f(a) + \int_a^x h(t)\,dt$, it follows that $f|P = g|P$, and this completes the proof for the case $f \in \mathrm{Lip}^1$.

If $f \in D^1$, then there exists an interval $I \subseteq [0, 1]$ relative to which f is Lip^1, and the argument given above will apply relative to I. □

Remark. Federer's original theorem [2] does not just apply to functions in $D^1 \cap \mathrm{Lip}^1$, it actually yields the desired conclusion for functions which are continuous and which are what would be called "a.e. pointwise Lip^1". This class of functions is much larger than $D^1 \cup \mathrm{Lip}^1$ and, in fact, includes the absolutely continuous (and even the CBV) functions. We did not state our Theorem 3 for this larger class because the conclusion of our Theorem 3 does not even hold for the absolutely continuous functions. The function constructed in the proof of Theorem 2 shows this.

Of course, we cannot get the exact category analog of Federer's intersection theorem, obtaining $\{x \in M: f(x) = g(x)\} \in \mathrm{B_w}\backslash\mathrm{FC}$ at the end, because of the following.

Example 4. There exists a Lip^1 function f such that for every $D^1 g$, the set $\{x: f(x) = g(x)\}$ is nowhere dense.

Proof. Let $C_1, C_2, \dots$ be a sequence of pairwise disjoint Cantor Subsets of $[0, 1]$ such that $\lambda(C_1 \cup C_2 \cup \cdots) = 1$ and such that if $x \in C_n$ for some n, then C_n has one-sided (left or right) upper density equal to 1 at x. Let $M = [0, 1]\backslash(C_1 \cup C_2 \cup \cdots)$. Define h so that $h(x) = 1/n$ if $x \in C_n$ and $h(x) = 0$, otherwise. Let $f(x) = \int_0^x h(t)\,dt$. It is easily seen that f is differentiable on M and that $f'(x) = 0$ for every $x \in M$; and it follows from the density requirements on the C_n that if f is differentiable at some $x \in C_n$, then $f'(x) = 1/n$. Suppose g is a D^1 function such that the set $A = \{x: f(x) = g(x)\}$ is dense in some interval I. Since A is closed, it follows that $f(x) = g(x)$ for every $x \in I$. This would imply that the range of $g'|I$ is a disconnected set, which is a contradiction. □

In proving Theorems 6 and 9, we will need the following lemma, which was essentially proved by Olevskii [7] without the extra stipulations involving the set M.

Lemma 5. *If $M \in B_w\backslash FC$, $f \in C^1$, and f' is nowhere monotone, then there exists a $g \in C^2$ such that $\{x \in M: f(x) = g(x)\}$ is uncountable.*

Proof. We will make some slight modifications of Olevskii's proof of Theorem 2 of [7] to account for the set M (we will also be assuming Olevskii's Lemmas 2 and 3). If $M \in \mathrm{B_w}\backslash\mathrm{FC}$, M is residual in some interval I (assume without loss of generality that $I = [0, 1]$), and there will exist a sequence $G_1 \supseteq G_2 \supseteq \cdots$ of dense open subsets of I such that $G_1 \cap G_2 \cap \cdots \subseteq M$. f' is nowhere monotone, so pick the points a_1 and a_2 in Olevskii's argument for which $f'(a_1) = f'(a_2) = [f(a_2) - f(a_1)]/(a_2 - a_1)$ to lie inside a common component of G_1. Then, pick the δ_i $(i = 1, 2)$ in his argument small enough so that the closed neighborhoods Δ_i $(i = 1, 2)$ lie interior to G_1, in addition to satisfying Olevskii's other requirements. Likewise, pick the points a_{ij} $(i, j = 1, 2)$ and the δ_2 small enough that the closed neighborhoods Δ_{ij} $(i, j = 1, 2)$ lie interior to G_2 and satisfy Olevskii's other requirements. Continuing this process, we will force Oleveskii's resulting set E (on which f agrees with some $g \in C^2$) to lie inside $G_1 \cap G_2 \cap \cdots \subseteq M$. □

Olevskii [7] proved that C^1 functions necessarily have uncountable intersections with C^2 functions, but he did not address the question of making the projection of the intersection be uncountable within some given large set M. We do this in the following theorem.

Theorem 6. *If $M \in B_w \cap \mathscr{L}$ and M is measure dense in some interval I in which M is residual, then for every $f \in C^1$ there exists a $g \in C^2$ such that $\{x \in M: f(x) = g(x)\}$ is uncountable.*

Proof. If there exists an interval $B \subseteq I$ on which f' is monotone, then f'' exists a.e. on B and there will exist a set $A \subseteq B \cap M$, $A \in \mathscr{L}\backslash\mathscr{L}_0$, such that $f''|A$ is bounded. It follows from [3, Theorem 3.1.15] that there is a C^2 g such that $\{x \in A: f(x) = g(x)\}$ is of positive measure and therefore uncountable.

If, on the other hand, f' is nowhere monotone on I, the result follows from Lemma 5. □

The requirements on M in the hypothesis of Theorem 6 seem to be quite strong at first glance. Still, the author actually believes that the converse (for $M \in \mathrm{B_w} \cap \mathscr{L}$) of the theorem holds, but he has been unable to prove this. In any case, the following two examples show that weaker requirements would be insufficient.

Example 7. There exists an F_σ, co-$\mathscr{L}_0$ set M and an $F \in C^1$ such that for every $g \in D^2 \cup C^{1\oplus 1}$, $\{x \in M: F(x) = g(x)\}$ is countable.

Proof. Let P_1 be the Cantor set P, h_1 be the function h, and f_1 be the function f constructed in the proof of Theorem 22 (with $\varepsilon = 0.5$) of [1]. If $[a, b]$ is a subinterval of $[0, 1]$, let

$$P_{[a,b]} = \{x: x = a + (b-a)y \text{ for some } y \in P_1\},$$

and let $h_{[a,b]}$ be the function defined by

$$h_{[a,b]}(x) = (b-a)h_1((x-a)/(b-a)) \quad \text{for each } x \in [a, b].$$

Let $P_2 = \bigcup\{P_{[a,b]} : (a, b) \text{ is an interval contiguous to } P_1\} \cup P_1$, let h_2 be the function such that $h_2(x) = h_{[a,b]}(x)$ if x is in an interval (a, b) contiguous to P_1 and $h_2(x) = 0$ if $x \in P_1$, and define $f_2(x) = \int_0^x h_2(t)\,dt$. For each n, let $P_n = \bigcup\{P_{[a,b]}: (a, b) \text{ is an interval contiguous to } P_{n-1}\} \cup P_{n-1}$, let h_n be the function such that $h_n(x) = h_{[a,b]}(x)$ if x is in an interval (a, b) contiguous to P_{n-1}, let $h_n(x) = 0$ if $x \in P_{n-1}$, and define $f_n(x) = \int_0^x h_n(t)\,dt$. Each h_n is continuous, and $h_1 + h_2 + \cdots$ converges uniformly to a continuous function that we denote by H. We denote the C^1 limit of $f_1 + f_2 + \cdots$ by F. Let $M = P_1 \cup P_2 \cup \cdots$, which is co-$\mathscr{L}_0$.

Suppose there is a $g \in D^2 \cup C^{1\oplus 1}$ such that $M' = \{x \in M: F(x) = g(x)\}$ is uncountable. Either $M' \cap P_1$ is uncountable or there is a first $n \geq 2$ such that $M' \cap P_{[a,b]}$ is uncountable for some interval (a, b) contiguous to P_{n-1}. If $M' \cap P_1$ is uncountable, we can achieve a contradiction similar to the contradiction obtained in [1] and similar to the contradiction we obtain in the more complicated latter case, so we assume the latter is the case. It follows that there is a decreasing sequence $x_1, x_2, \ldots$ of elements of $M' \cap P_{[a,b]}$ converging to an element x_0 of $M' \cap P_{[a,b]}$ such that each x_i is a limit point of M'. We may also assume that all of the x_i lie in a neighborhood N of x_0 which contains no midpoint of any interval (a_j, b_j) containing (a, b) and contiguous to P_j for any $j \leq n-1$. It follows that each h_j $(j = 1, 2, \ldots, n-1)$ is continuously differentiable on N and that $E_{n-1} = f_1 + f_2 + \cdots + f_{n-1}$ is twice continuously differentiable on N. Let $F_n = F - E_{n-1}$, $G = g - E_{n-1}$, and $H_n = h_n + h_{n+1} + \cdots$.

In the case that $g \in D^2$, it follows that G is twice differentiable on N and $G'(x_0) = F_n'(x_0) = H_n(x_0) = 0$, so that L'Hôpital's rule can be applied as it was in [1] to conclude that

$$\lim_{i\to\infty} [G(x_i) - G(x_0)]/(x_i - x_0)^2 = G''(x_0)/2.$$

On the other hand, $G(x_i) = F_n(x_i)$ for $i = 0, 1, 2, \ldots$, and the geometry of the construction of $h_{[a,b]}$ is similar enough to that in [1] to conclude that since

$$\lim_{i\to\infty} [f_n(x_i) - f_n(x_0)]/(x_i - x_0)^2 = +\infty,$$

the larger difference quotients will do the same and

$$\lim_{i\to\infty}[F_n(x_i) - F_n(x_0)]/(x_i - x_0)^2 = +\infty .$$

This provides the desired contradiction for the case $g \in D^2$.

In case $g \in C^{1\oplus 1}$, we still have that G is differentiable on N, $G'(x_0) = 0$, and G' is Lip^1 (relative to N). For each i, there will exist $x_0 < y_i < x_i$ such that

$$[G(x_i) - G(x_0)]/(x_i - x_0)^2 = G'(y_i)/(x_i - x_0) = [G'(y_i) - G'(x_0)]/(x_i - x_0)$$

so that, since G' is Lip^1 relative to N,

$$\limsup_{i\to\infty}[G(x_i) - G(x_0)]/(x_i - x_0)^2 < +\infty .$$

But this contradicts the fact that

$$\lim_{i\to\infty}[F_n(x_i) - F_n(x_0)]/(x_i - x_0)^2 = +\infty ,$$

which still holds. □

Example 8. There exists an $N \in (\mathscr{L}\backslash\mathscr{L}_0) \cap (\text{co-FC})$ and an $f \in C^1$ such that $\{x \in M \colon f(x) = g(x)\}$ is countable for every $g \in D^2 \cup C^{1\oplus 1}$.

Proof. Again, let $\varepsilon = 1/2$, P_1 be the set P with $\lambda(P) = 1 - \varepsilon$, and f_1 be the function f described in the proof of Theorem 22 of [1]. Let $P_2, P_3, \ldots$ be a sequence of pairwise disjoint Cantor subsets of $[0, 1]\backslash P_1$ such that $\lambda(P_i) = 1/2^i$ for $i = 2, 3, \ldots$. Let k be the function f described in Theorem 2 with $M = [0, 1]\backslash(P_1 \cup P_2 \cup \cdots)$, and let $f_2(x) = \int_0^x k(t)\,dt$. The function $f = f_1 + f_2$ is C^1 and $N = P_1 \cup M$ is residual and of positive measure.

Suppose $g \in D^2 \cup C^{1\oplus 1}$ and that the set $A = \{x \in N \colon f(x) = g(x)\}$ is uncountable.

If $A \cap M$ is uncountable, then $A \cap M$ contains a condensation point x of itself such that x is not a midpoint of any interval contiguous to P_1. It follows that $f_1''(x) = 0$ but $f_2''(x) = +\infty$, so that $f''(x) = D^+g(x) = +\infty$, which is a contradiction. D^+ and D_+ are the notation for the upper and lower right Dini derivatives.

If $A \cap P_1$ is uncountable, it follows that there is a convergent decreasing sequence $x_k \to x_0$, where $x_k \in A \cap P_1$, for $k = 0, 1, 2, \ldots$, and each x_k is a bilateral condensation point of $A \cap P_1$. It follows, as in [1], that $D^+f_1'(x_0) = +\infty$. Since $f_2' = k$ is strictly increasing, $D_+f_2'(x_0) \geq 0$. Thus, $D^+f'(x_0) = D^+g'(x_0) = +\infty$, which is a contradiction. □

In 1951 Whitney [9] (also see [3, Theorem 3.1.15]) proved a theorem from which it follows that functions in $D^2 \cup C^{1\oplus 1}$ intersect C^2 functions and that the projection of the intersection could be made of positive measure inside any given $\mathscr{L}\backslash\mathscr{L}_0$ set. We now prove the following category analog of that result.

Theorem 9. *If $M \in B_w\backslash FC$, then for every $f \in D^2 \cup C^{1\oplus 1}$ there exists a $g \in C^2$ such that $\{x \in M \colon f(x) = g(x)\}$ is uncountable.*

Proof. The proof is essentially similar to the proof of Theorem 3. We start with the same sets $G_1, G_2, \ldots$ and proceed by replacing f in that proof with f' in this argument and replacing f' in that proof with f'' in this argument. We first assume $f \in C^{1\oplus 1}$.

Case I. Because the sets $U_t = \{x: f''(x) < t\}$ and $V_t = \{x: f''(t) > t\}$ are both measure dense in I, it follows that the function $h(x) = f'(x) - f'(a) - t(x-a)$ is nowhere monotone on I. We then set

$$H(x) = f(x) - f'(a)(x-a) - t(x-a)^2/2,$$

call on Lemma 5 as it applies to the function H, and use this to finish the proof for this case.

Case II. We proceed as we did in the proof of Theorem 3, using f'' in place of f', obtaining the similar sets H_n and the set H on which f'' is essentially continuous. We define similar points x_b and intervals I_b, J_b, and K_b for finite binary sequences b, using f'' in place of f', obtaining a similar perfect set P. We then need to extend $f''|P$ to a continuous function h on $[0, 1]$, but there is an extra requirement on the extension in this argument. At the point where $b \in [0, 1]^n$ and $(c(b), d(b))$ is the interval contiguous to P which contains the interval K_b we define h continuously on $[c(b), d(b)]$ so that (1) $h(c(b)) = f''(c(b))$ and $h(d(b)) = f''(d(b))$, (2) $h(x) \in J_b$ for every $x \in (c(b), d(b))$, and (3) $h_1(d(b)) - h_1(c(b)) = f'(d(b)) - f'(c(b))$ and (4) $h_2(d(b)) - h_2(c(b)) = f(d(b)) - f(c(b))$, where $h_1(x) = \int_{c(b)}^x h(t)\,dt$ and $h_2(x) = \int_{c(b)}^x h_1(t)\,dt$.

Construction of this function h is a bit more complicated than the construction of the function h in Case II of the proof of Theorem 3. To illustrate the method of construction, assume (c, d) is an interval such that $f''(c)$ and $f''(d)$ exist and lie in an interval $J = (u, v)$, which is such that $u < f''(x) < v$ for a.e. $x \in (c, d)$. Consider what the function f' must be like on $[c, d]$. Extend lines U_1 and V_1 with slopes u and v, respectively, to the right from $(c, f'(c))$ and lines U_2 and V_2 with slopes u and v, respectively, to the left from $(d, f'(d))$. Since $u < f''(x) < v$ a.e. on (c, d) and $f'(x) = f'(c) + \int_c^x f''(t)\,dt$ for $x \in [c, d]$, it follows that these four lines form a parallelogram Q with $(c, f'(c)$ and $(d, f'(d))$ at opposite corners. It is also the case that $f'|(c, d)$ lies interior to the parallelogram Q. Now, choose c' close to c and d' close to d with $c < c' < d' < d$. Let $h_0(x) = f'(c) + f''(c)(x-c)$ for $x \in [c, c']$, and let $h_0(x) = f'(d) + f''(d)(x-d)$ for $x \in [d', d]$. Assuming $c' - c$ and $d - d'$ are small enough, it will be possible to choose an $s \in (c', d')$ and extend h_0 continuously on $[c', d']$ to be linear on $[c', s]$ and linear on $[s, d']$, with the slopes of both extensions between u and v, and in such a way that $\int_c^d h_0(t)\,dt = \int_c^d f'(t)\,dt$. h_0 is polygonal line graph on $[c, d]$, and we now need to "round off" the corners which occur at $(c', h_0(c'))$, $(s, h_0(s))$, and $(d', h_0(d'))$ to make it C^1. In this smoothing operation, we must make sure that the resulting function h_1 still satisfies $u < h_1'(x) < v$ for $x \in [c, d]$ and $\int_c^d h_1(t)\,dt = \int_c^d f'(t)\,dt$.

Now, we define h_1 like this on each interval $(c, d) = (c(b), d(b))$ for $b \in \{0, 1\}^n$ and $J = J_b$, as described in the previous paragraph, and define $h_1|P = f'|P$. Then h_1' is the desired function h.

Then, if a is the least element of P and g is defined by $g(x) = f(a) + \int_a^x g_1(t)\,dt$, where g_1 is defined by $g_1(x) = \int_a^x h(t)\,dt$, it follows that $f|P = g|P$, and this completes the proof for the case where $f \in C^{1\oplus 1}$.

The case where $f \in D^2$ follows easily. □

Example 10. There exists a $C^{1\oplus 1}$ function f such that for every $D^2 g$ the set $\{x: f(x) = g(x)\}$ is nowhere dense.

Proof. To obtain this example, just take the indefinite integral of the function of Example 4.

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