

# Math 2650

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We will now look at (linearity and at) linear second order differential equations.

[> **restart:with(DEtools):with(plots):**

Consider the equation

$$\frac{d^2 x}{dt^2} + a(t) \frac{dx}{dt} + b(t)x = g(t) \quad x(0) = x_0, \quad \frac{dx}{dt}(0) = v_0.$$

This is the prototype linear, second order, differential equation and initial conditions. Together they form an initial value problem (i.v.p.).

We consider the differential equation (d.e.):

$$(*) \quad \frac{d^2 x}{dt^2} + a(t) \frac{dx}{dt} + b(t)x = g(t)$$

and the corresponding homogeneous d.e.

$$(**) \quad \frac{d^2 x}{dt^2} + a(t) \frac{dx}{dt} + b(t)x = 0$$

(note the zero r.h.s.).

Due to linearity if  $\phi_1$  and  $\phi_2$  are two solutions of the homogeneous d.e. (\*\*), then also so is every linear combination of  $\phi_1$  and  $\phi_2$ , i.e.,  $\phi(t) = c_1 \phi_1(t) + c_2 \phi_2(t)$ , where  $c_1$  and  $c_2$  are arbitrary constants. That is for any constants  $c_1$  and  $c_2$ ,  $\phi(t) = c_1 \phi_1(t) + c_2 \phi_2(t)$  is a solution of (\*\*).

Also assume that  $\phi_g$  is a solution of (\*), (a particular solution), then due to linearity, for any constants  $c_1$  and  $c_2$ ,  $\phi(t) = \phi_g(t) + c_1 \phi_1(t) + c_2 \phi_2(t)$  is a solution of (\*).

Furthermore if  $\phi_f$  is a solution of (\*) with the r.h.s.  $g(t)$  replaced by  $f(t)$ , then (due to linearity)

$\phi(t) = \phi_g(t) + \phi_f(t)$  is a solution of (\*) with the r.h.s.  $g(t)$  replaced by  $g(t) + f(t)$  and also

for any constants  $c_1$  and  $c_2$ ,  $\phi(t) = \phi_g(t) + \phi_f(t) + c_1 \phi_1(t) + c_2 \phi_2(t)$  is a solution of (\*) with the r.h.s.  $g(t)$  replaced by  $g(t) + f(t)$ .

Examples:

Lets define the differential equation:

$$\left[ \begin{array}{l} > \mathbf{de := D(D(x))(t) + a * D(x)(t) + b * x(t) = g(t);} \\ \quad \mathbf{de := D^{(2)}(x)(t) + a D(x)(t) + b x(t) = g(t)} \end{array} \right.$$

```
> ic:=x(0)=x_0,D(x)(0)=v_0;
      ic:=x(0)=x_0,D(x)(0)=v_0
```

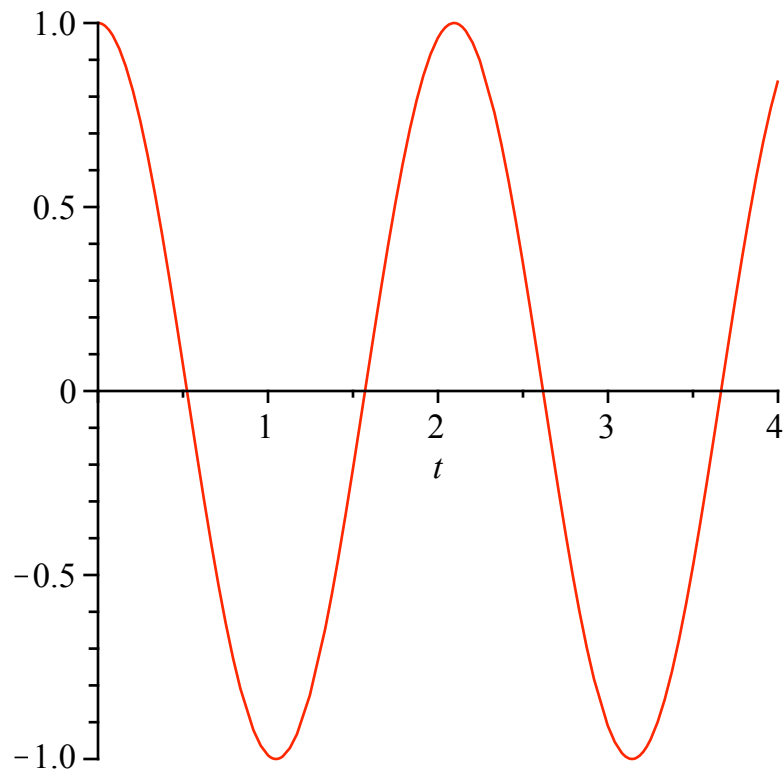
We will now consider homogeneous equations (with  $g(t) = 0$ ) and with constant coefficients ( $a$  and  $b$  constants). We will choose some values for  $a$  and  $b$  and solve the equation using the `dsolve` command. You can also try solving the equation by hand.

```
> g(t):=0;a:=0;b:=9;
      g(t):=0
      a:=0
      b:=9
> sol:=dsolve(de,x(t));
      sol:=x(t)=_C1 sin(3 t) + _C2 cos(3 t)
> g(t):=cos(t);
      g(t):=cos(t)
> sol:=dsolve(de,x(t));
      sol:=x(t)=sin(3 t)_C2 + cos(3 t)_C1 + 1/8 cos(t)
```

Observe the homogeneous solution is  $x_h := c_1 \sin(3t) + c_2 \cos(3t)$  and the particular solution is

$$x_p = \frac{\cos(t)}{8}.$$

```
> g(t):=0;
      g(t):=0
> x_0:=1;v_0:=0;
      x_0:=1
      v_0:=0
> de;ic;
      D(2)(x)(t) + 9 x(t) = 0
      x(0) = 1, D(x)(0) = 0
> soll:=dsolve({de,ic},x(t));
      soll:=x(t)=cos(3 t)
> soll:=rhs(soll);
      soll:=cos(3 t)
> plot(soll,t=0..4);
```



```
> g(t):=sin(2*t);
```

```
g(t) := sin(2 t)
```

```
> de;ic;
```

$$D^{(2)}(x)(t) + 9x(t) = \sin(2t)$$

$$x(0) = 1, D(x)(0) = 0$$

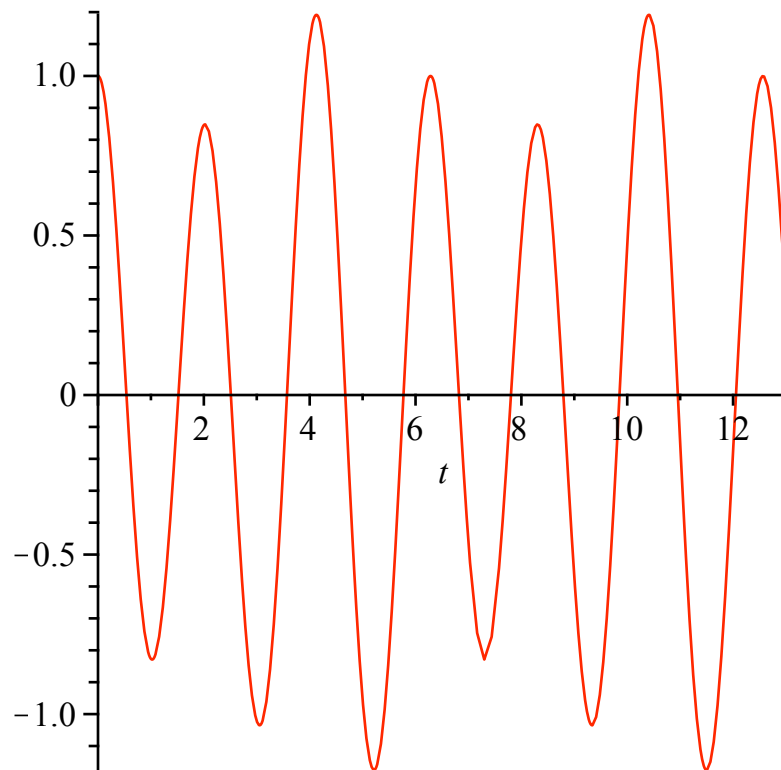
```
> sol2:=dsolve({de,ic},x(t));
```

$$\text{sol2} := x(t) = -\frac{2}{15} \sin(3t) + \cos(3t) + \frac{1}{5} \sin(2t)$$

```
> sol2r:=rhs(sol2);
```

$$\text{sol2r} := -\frac{2}{15} \sin(3t) + \cos(3t) + \frac{1}{5} \sin(2t)$$

```
> plot(sol2r,t=0..13);
```



```
> xh:=select(has,sol2r,3*t);
```

$$xh := -\frac{2}{15} \sin(3t) + \cos(3t)$$

```
> xp:=select(has,sol2r,2*t);
```

$$xp := \frac{1}{5} \sin(2t)$$

```
> odetest(x(t)=cos(3*t)-2/15*sin(3*t),de);  
-sin(2t)
```

```
> odetest(x(t)=1/5*sin(2*t),de);  
0
```

```
> g(t):=sin(3*t);
```

$$g(t) := \sin(3t)$$

```
> x_0:=1;v_0:=0;
```

$$x_0 := 1$$

$$v_0 := 0$$

```
> de;ic;
```

$$D^{(2)}(x)(t) + 9x(t) = \sin(3t)$$

$$x(0) = 1, D(x)(0) = 0$$

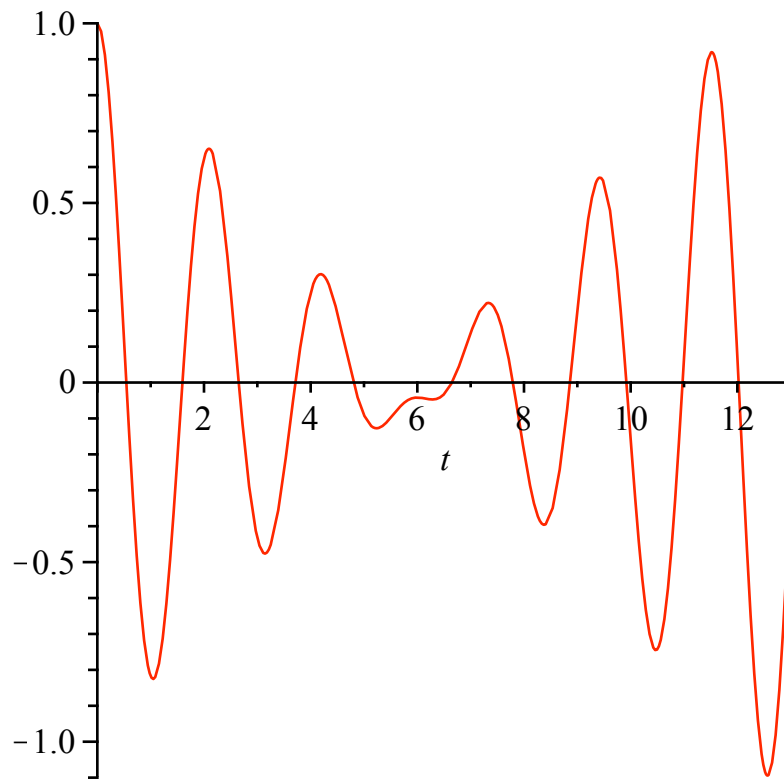
```
> sol3:=dsolve({de,ic},x(t));
```

$$sol3 := x(t) = \frac{1}{18} \sin(3t) + \cos(3t) - \frac{1}{6} \cos(3t)t$$

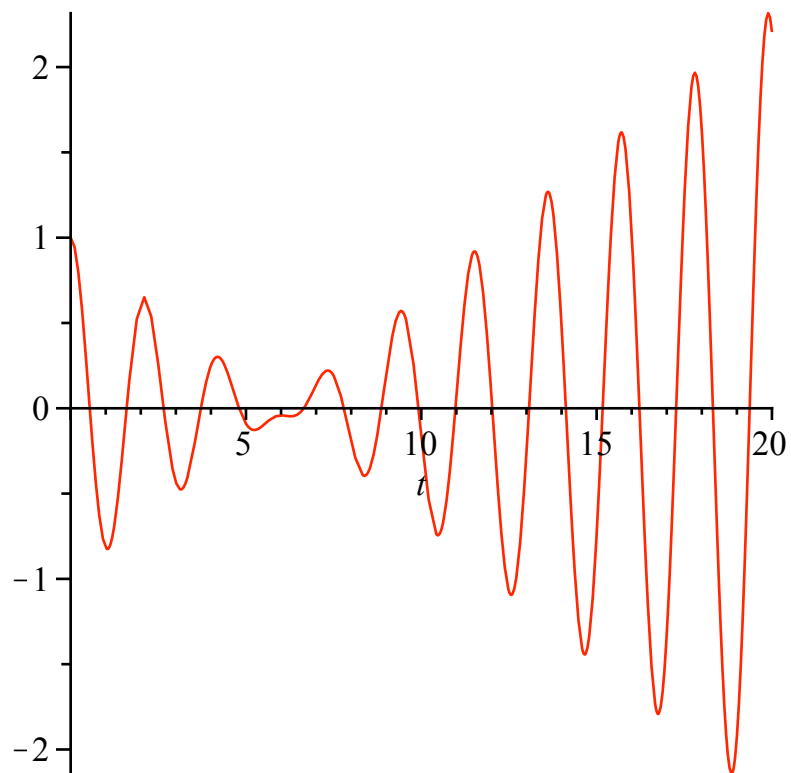
```
> sol3r:=rhs(sol3);
```

$$\text{sol3r} := \frac{1}{18} \sin(3t) + \cos(3t) - \frac{1}{6} \cos(3t)t$$

```
> plot(sol3r,t=0..13);
```



```
> plot(sol3r,t=0..20);
```



```

> g(t):=sin(2*t);
                                g(t) := sin(2 t)
=
> sol4:=dsolve(de,x(t));
                                sol4 := x(t) = sin(3 t) _C2 + cos(3 t) _C1 + 1/5 sin(2 t)
=
> sol4:=rhs(sol4);
                                sol4 := sin(3 t) _C2 + cos(3 t) _C1 + 1/5 sin(2 t)
=
> xh:=select(has,sol4,3*t);
                                xh := sin(3 t) _C2 + cos(3 t) _C1
=
> xp:=select(has,sol4,2*t);
                                xp := 1/5 sin(2 t)
=
> x_0:='x_0';v_0:='v_0';
                                x_0 := x_0
                                v_0 := v_0
=
> eq1:=subs(t=0,xh+xp)=x_0;
                                eq1 := sin(0) _C2 + cos(0) _C1 + 1/5 sin(0) = x_0
=
> eq2:=subs(t=0,diff((xh+xp),t))=v_0;
                                eq2 := 3 _C2 cos(0) - 3 _C1 sin(0) + 2/5 cos(0) = v_0
=
> eval(eq1);
                                _C1 = x_0
=
> eval(eq2);
                                3 _C2 + 2/5 = v_0

```

Exercise:

Solve the equation:

$$\frac{d^2 x}{dt^2} + 16x = \sin(2t) \quad x(0) = 0, \quad \frac{dx}{dt}(0) = 1.$$

Identify the homogeneous solution and the particular solution.

Is the solution periodic, if it is what is the period?

Repeat the above for

$$\frac{d^2 x}{dt^2} + 16x = \sin(4t) \quad x(0) = 0, \quad \frac{dx}{dt}(0) = 1.$$