**Exact Differential Equations**

Consider the differential equation

$$M(t, x) + N(t, x) \frac{dx}{dt} = 0$$

or (equivalently) written in differential form the equation

$$M(t, x) \, dt + N(t, x) \, dx = 0.$$

**Theorem.** Let the functions $M, N, M_x, N_t$ (where subscripts denote partial derivatives) be continuous in a simply connected region $R$ (of the $t - x$ plane). The above differential equation is an exact differential equation in $R$ if and only if

$$M_x(t, x) = N_t(t, x)$$

at each point of $R$. That is, there exists a function $\Phi$ satisfying

$$\Phi_t(t, x) = M(t, x) \quad \text{and} \quad \Phi_x = N(t, x)$$

if and only if $M$ and $N$ satisfy $M_x(t, x) = N_t(t, x)$.

If the equation

$$M(t, x) \, dt + N(t, x) \, dx = 0$$

is exact then an explicit solution is given by

$$\Phi(t, x) = c$$

where $\Phi$ satisfies $\Phi_t(t, x) = M(t, x)$ and $\Phi_x = N(t, x)$.

This implicit solution may be constructed as follows:

$$\Phi(t, x) = \int M(t, x) \, dt + h(x)$$

and

$$\Phi_x(t, x) = \frac{\partial}{\partial x} \left( \int M(t, x) \, dt \right) + \frac{d}{dx} h(x)$$

$$= M_x(t, x) \, dt + \frac{d}{dx} h(x)$$

which yields

$$\frac{d}{dx} h(x) = N(t, x) - \left( \int M_x(t, x) \, dt \right)$$

hence

$$\Phi(t, x) = \int M(t, x) \, dt + \int \left( N(t, x) - \left( \int M_x(t, x) \, dt \right) \right) \, dx.$$
Example

1. Consider

\[(x \cos(t) + 2 \, t \, e^x) + (\sin(t) + t^2 \, e^x - 1) \frac{dx}{dt} = 0\]

we first check that this is in fact an exact differential equation

\[
> \text{diff}(x \cos(t) + 2 \, t \, e^x, x);
\]

\[\cos(t) + 2 \, t \, e^x\]

\[
> \text{diff}(\sin(t) + t^2 \, e^x - 1, t);
\]

\[\cos(t) + 2 \, t \, e^x\]

so it is exact. Now lets calculate the solution

\[
> \text{int}(x \cos(t) + 2 \, t \, e^x, t);
\]

\[x \sin(t) + t^2 \, e^x\]

so

\[\Phi(t, x) = x \sin(t) + t^2 \, e^x + h(x)\]

and

\[
\frac{d}{dx} h(x) = \sin(t) + t^2 \, e^x - 1 - (\sin(t) + t^2 \, e^x)
\]

so

\[
\frac{d}{dx} h(x) = -1
\]

and

\[h(x) = -x.\]

Therefore an implicit solution is given by

\[x \sin(t) + t^2 \, e^x - x = c.\]

Problems

Determine whether each of the equations is exact, if it is find the solution.

1. \((2 \, x + 3) + (2 \, y - 2) \frac{dy}{dx} = 0\)

2. \((2 \, x + 4 \, y) + (2 \, x - 2 \, y) \frac{dy}{dx} = 0\)

3. \((3 \, t^2 - 2 \, t \, x + 2) \ dt + (6 \, x^2 - t^2 + 3) \ dx = 0\)

4. \((2 \, t \, x^2 + 2 \, x) \ dt + (2 \, t^2 \, x + 2 \, t) \ dx = 0\)