

Math 2650

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```
[> restart:with(DEtools):with(plots):
```

▼ Euler's method

We now look at a very simple numerical method for approximating solutions of differential equations. It is essentially based on the direction field, and an approximation of the derivative as a difference quotient.

Consider the equation

$$\frac{dx}{dt} = x^2 - t^2 \quad x(0) = \frac{1}{2}$$

Lets try and solve the equation

```
> dsolve({diff(x(t),t)=x(t)^2-t^2,x(0)=1/2},x(t));
```

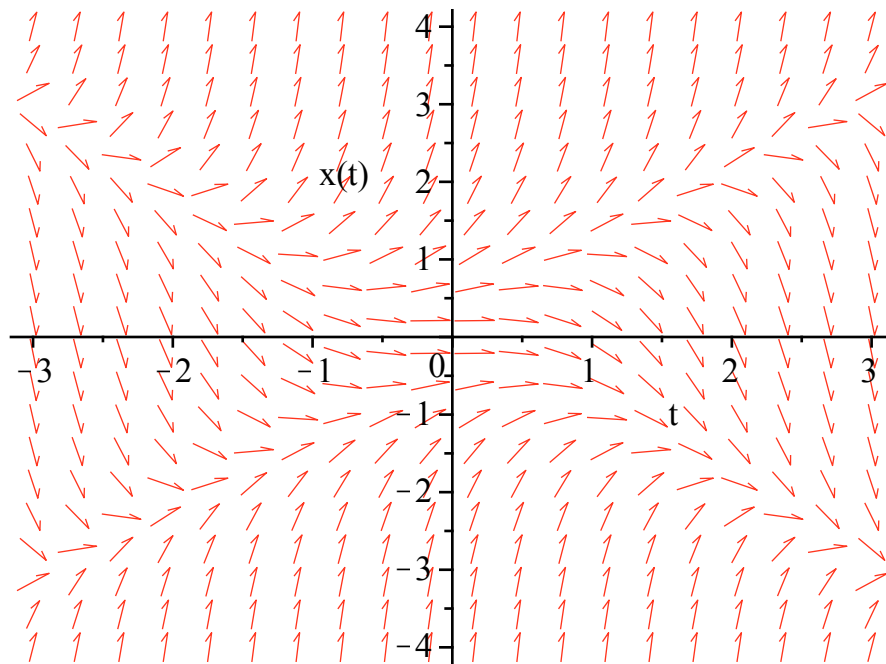
$$x(t) = \begin{cases} - \left(t \left(\frac{1}{4} \frac{\pi \left(2 \Gamma\left(\frac{3}{4}\right)^2 \sqrt{2} + \pi \right) \text{BesselI}\left(-\frac{3}{4}, \frac{1}{2} t^2\right)}{\Gamma\left(\frac{3}{4}\right)^2} - \text{BesselK}\left(\frac{3}{4}, \frac{1}{2} t^2\right) \right) \right) & t < 0 \\ \left(\frac{1}{4} \frac{\pi \left(2 \Gamma\left(\frac{3}{4}\right)^2 \sqrt{2} + \pi \right) \text{BesselI}\left(\frac{1}{4}, \frac{1}{2} t^2\right)}{\Gamma\left(\frac{3}{4}\right)^2} + \text{BesselK}\left(\frac{1}{4}, \frac{1}{2} t^2\right) \right) & \\ \frac{1}{2} & t = 0 \\ - \left(t \left(-\frac{1}{4} \frac{\pi \left(-2 \Gamma\left(\frac{3}{4}\right)^2 \sqrt{2} + \pi \right) \text{BesselI}\left(-\frac{3}{4}, \frac{1}{2} t^2\right)}{\Gamma\left(\frac{3}{4}\right)^2} - \text{BesselK}\left(\frac{3}{4}, \frac{1}{2} t^2\right) \right) \right) & \\ - \frac{1}{4} \frac{\pi \left(-2 \Gamma\left(\frac{3}{4}\right)^2 \sqrt{2} + \pi \right) \text{BesselI}\left(\frac{1}{4}, \frac{1}{2} t^2\right)}{\Gamma\left(\frac{3}{4}\right)^2} + \text{BesselK}\left(\frac{1}{4}, \frac{1}{2} t^2\right) & 0 < t \end{cases}$$

This is a mess so lets look at the direction field to get some insight into the problem

```

> DF:=DEplot(diff(x(t),t)=x(t)^2-t^2,x(t),
t=-3..3,x=-4..4):
> display(DF);

```



Lets try and use Euler's method to construct an approximate solution

Below we write a little Maple procedure for the forward Euler's method.

Euler's Method

```
> FEuler:=proc(f,t0,x0,h,N)
  local t,x,n,sol:
  t:=evalf(t0):
  x:=evalf(x0):
  sol:=[[t,x]]:
  for n from 1 to N do
    x:=x+h*f(t,x):
    t:=t+h:
    sol:=[op(sol),[t,x]]:
  od;
end;
```

```
FEuler:=proc(f,t0,x0,h,N)
```

```
  local t,x,n,sol;
```

```
  t:=evalf(t0);
```

```
  x:=evalf(x0);
```

```
  sol:=[[t,x]];
```

```
  for n to N do
```

```
    x:=x+h*f(t,x); t:=t+h; sol:=[op(sol),[t,x]]
```

```
  end do
```

```
end proc
```

```
> t0:=0;tf:=3;x0:=1/2;
```

```
t0:=0
```

```
tf:=3
```

$$x_0 := \frac{1}{2}$$

```
> N:=3;
```

```
N:=3
```

```
> h:=evalf((tf-t0)/N);
```

```
h:=1.
```

```
> f:=(t,x)->x^2-t^2;
```

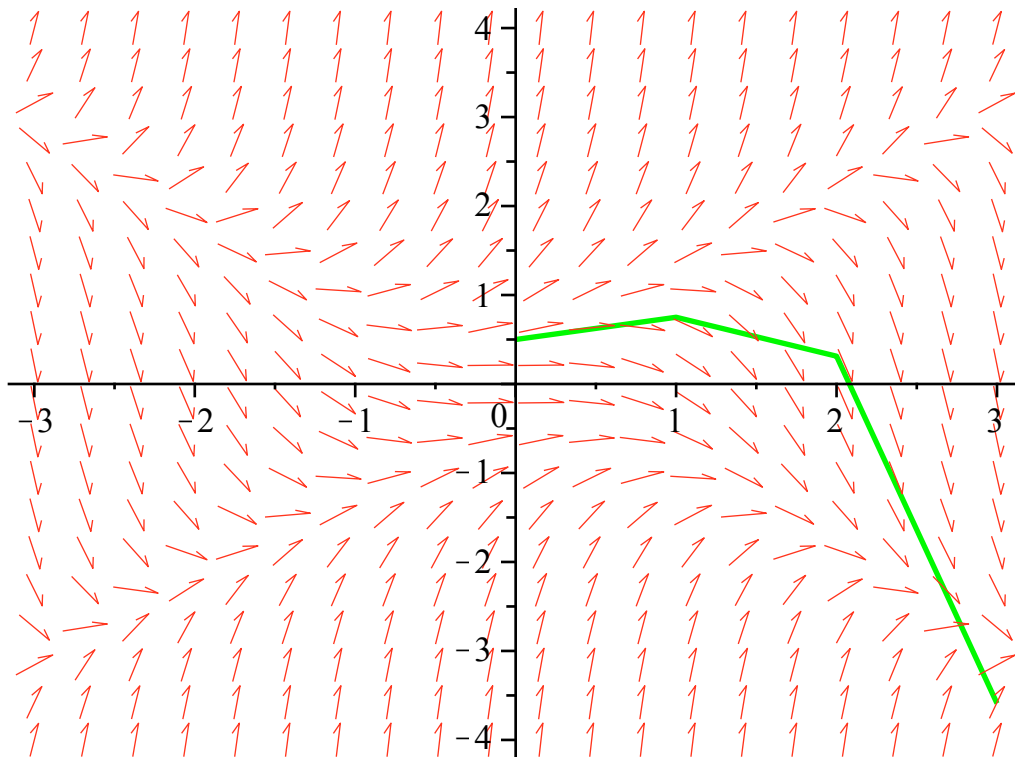
$$f := (t, x) \rightarrow x^2 - t^2$$

```
> out:=FEuler(f,t0,x0,h,N);
```

```
out := [[0., 0.5000000000], [1., 0.7500000000], [2., 0.3125000000], [3., -3.589843750]]
```

```
> EulerApprox:=plot(out,color=green,thickness=2):
```

```
> display(EulerApprox,DF);
```



```
> N:=6;
```

```
> h:=evalf((tf-t0)/N);
```

```
> out:=FEuler(f,t0,x0,h,N);
```

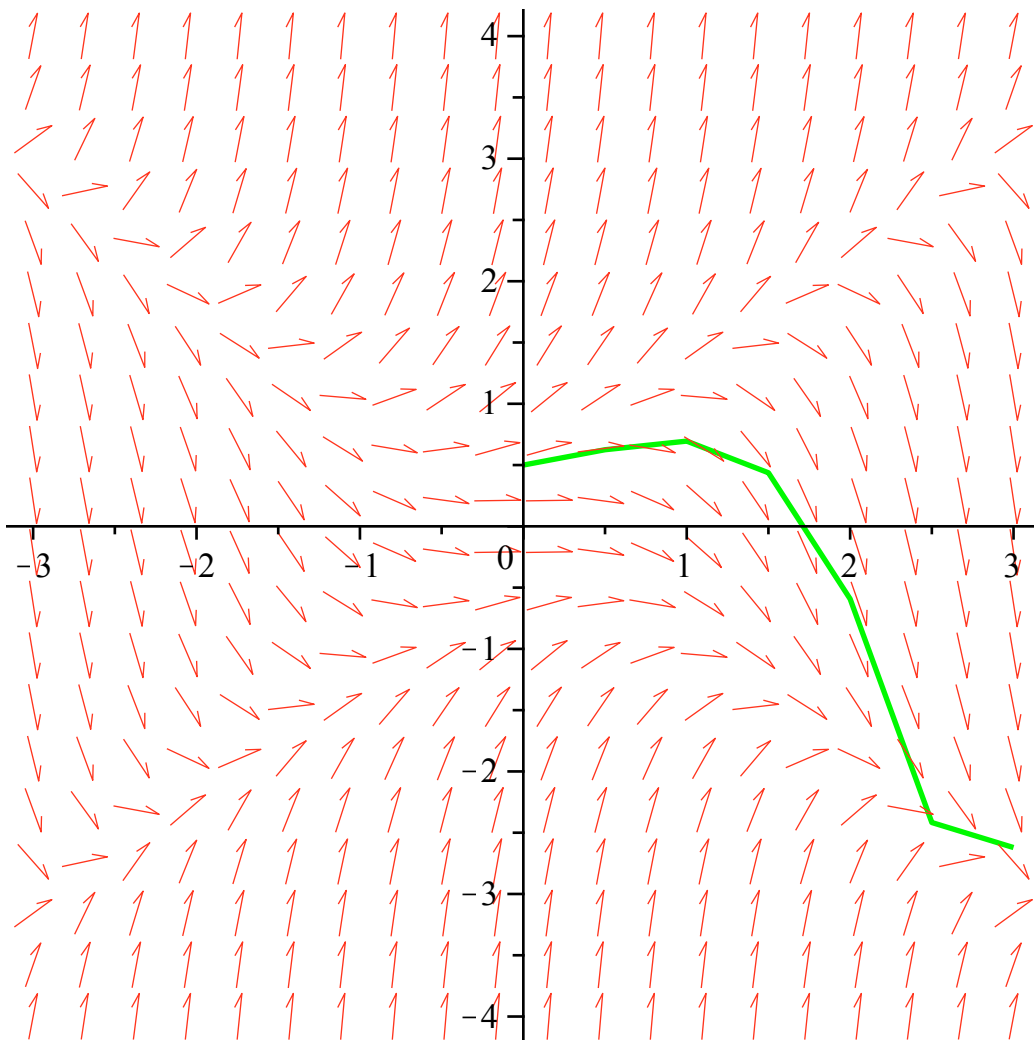
```
> EulerApprox:=plot(out,color=green,thickness=2):
```

```
> display(EulerApprox,DF);
```

```
N:=6
```

```
h:=0.5000000000
```

```
out := [[0., 0.5000000000], [0.5000000000, 0.6250000000], [1.0000000000, 0.6953125000], [1.5000000000, 0.4370422364], [2.0000000000, -0.5924548056], [2.5000000000, -2.416953458], [3.0000000000, -2.621121449]]
```

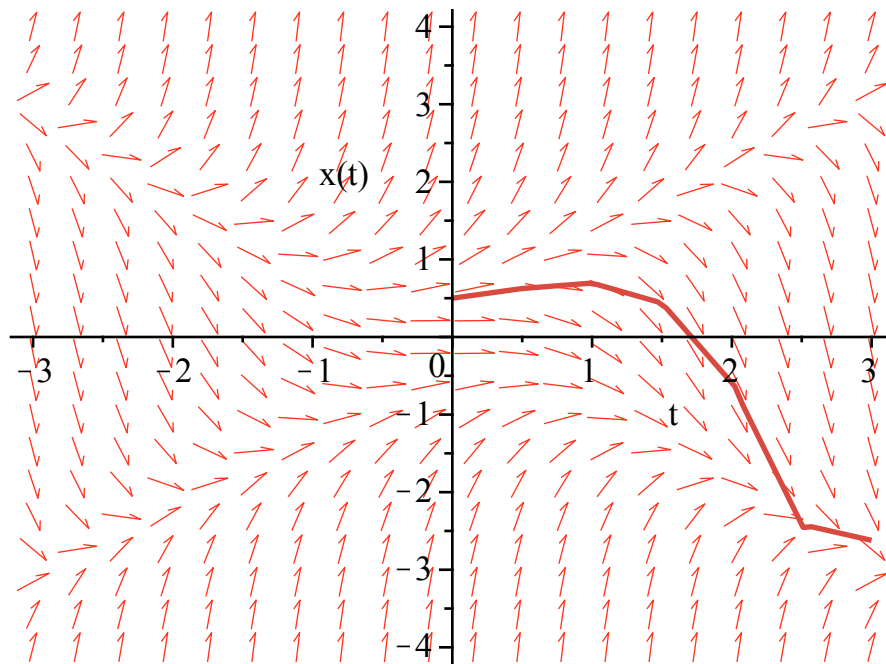


We can also use Maple's built in numerical o.d.e. "solvers", but then we may not have precise control over the step size.

```
> ans:=dsolve({diff(x(t),t)=x(t)^2-t^2,x(0)=1/2},x(t),numeric,
method=classical[foreuler],stepsize=h);

           ans := proc(x_classical) ... end proc

> approx:=odeplot(ans,[t,x(t)],0..3,color=orange,
thickness=2):
> display(DF,approx);
```



We now look at a very simple stiff o.d.e. Consider

$$\frac{dx}{dt} = -10x \quad x(0) = 10$$

```
> t0:=0;tf:=3;x0:=10;
> N:=10;
> h:=evalf((tf-t0)/N);
> f:=(t,x)->-10*x;
> out:=FEuler(f,t0,x0,h,N);
> EulerApprox:=plot(out,color=green,thickness=2):
> DF:=DEplot(diff(x(t),t)=-10*x(t),x(t),
t=-1..3,x=-10..10):
> display(EulerApprox,DF);
```

$t0 := 0$

$tf := 3$

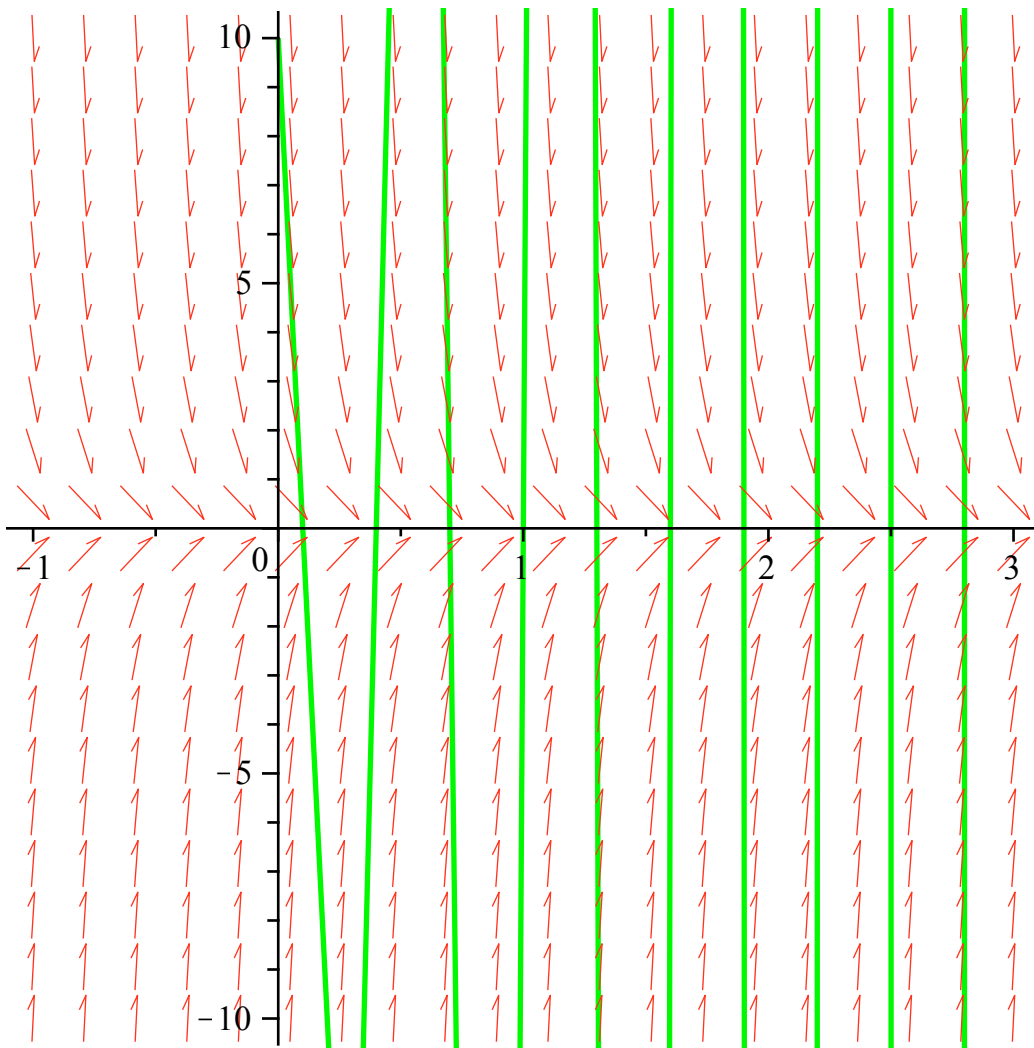
$x0 := 10$

$N := 10$

$h := 0.3000000000$

$f := (t, x) \rightarrow -10x$

```
out := [[0., 10.], [0.3000000000, -20.00000000], [0.6000000000, 40.00000000],
[0.9000000000, -80.00000000], [1.2000000000, 160.00000000], [1.5000000000,
-320.00000000], [1.8000000000, 640.00000000], [2.1000000000, -1280.00000000],
[2.4000000000, 2560.00000000], [2.7000000000, -5120.00000000], [3.0000000000,
10240.00000000]]
```

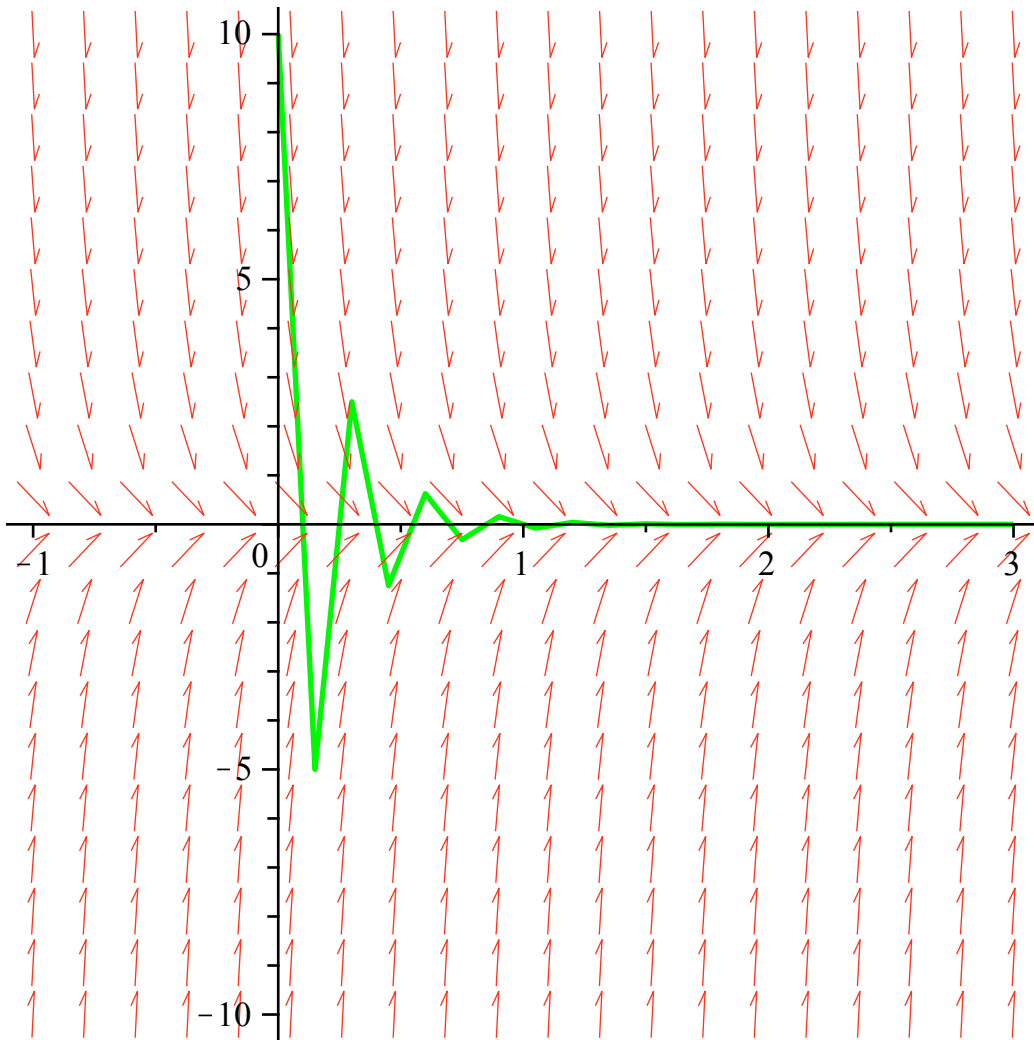


```
> N:=20;
> h:=evalf((tf-t0)/N);
> out:=FEuler(f,t0,x0,h,N);
> EulerApprox:=plot(out,color=green,thickness=2):
```

```
> DF:=DEplot(diff(x(t),t)=-10*x(t),x(t),
t=-1..3,x=-10..10):
> display(EulerApprox,DF);
```

$N := 20$

$h := 0.1500000000$



```
> N:=40;
> h:=evalf((tf-t0)/N);
> out:=FEuler(f,t0,x0,h,N);
> EulerApprox:=plot(out,color=green,thickness=2):
```

```
> DF:=DEplot(diff(x(t),t)=-10*x(t),x(t),
t=-1..3,x=-10..10):
> display(EulerApprox,DF);
```

$N := 40$

$h := 0.07500000000$

