Math 2650 - Qualatative Analysis

Direction Fields

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Qualitative analysis

One of the emphases of the class is qualitative analysis.

This is important since most differential equations cannot be solved explicitly, although in many cases we can still say someting about the solution to the equation (existence, uniqueness, and qualatative behavior).

Even thogh we may not be able to find an explicit representation for the solution we may be able to show the solution exists and that it is unique, furthermore we may be able to answer the questions:

Does the solution tend to 0 or ∞ as t tends to ∞?

Is the solution periodic, and if so with what period?

Does the solution approach 0 , or some other value only as t tends to ∞, or for some finite time t?

Does the solution "converege" to some other function as t tends to ∞?

Direction Fields

Starting the DE tools.

The basic Maple command for the graphical representation of direction fields and approximate solution curves is DEplot ; it is part of a package of routines called DEtools, which must be explicitly loaded with the command.

```
> with(DEtools):
```

What are direction fields?

The direction field is comprised of slope lines of the differential equation, for example,

$$\frac{dx}{dt} = f(t, x)$$

then $f(t, x)$ is the slope of the graph of $x(t)$ the solution at t.

A solution of the differential equation is a function whose graph is consistent with the direction field (that is the tangent to the graph at each point must coincide with the slope line at that point).

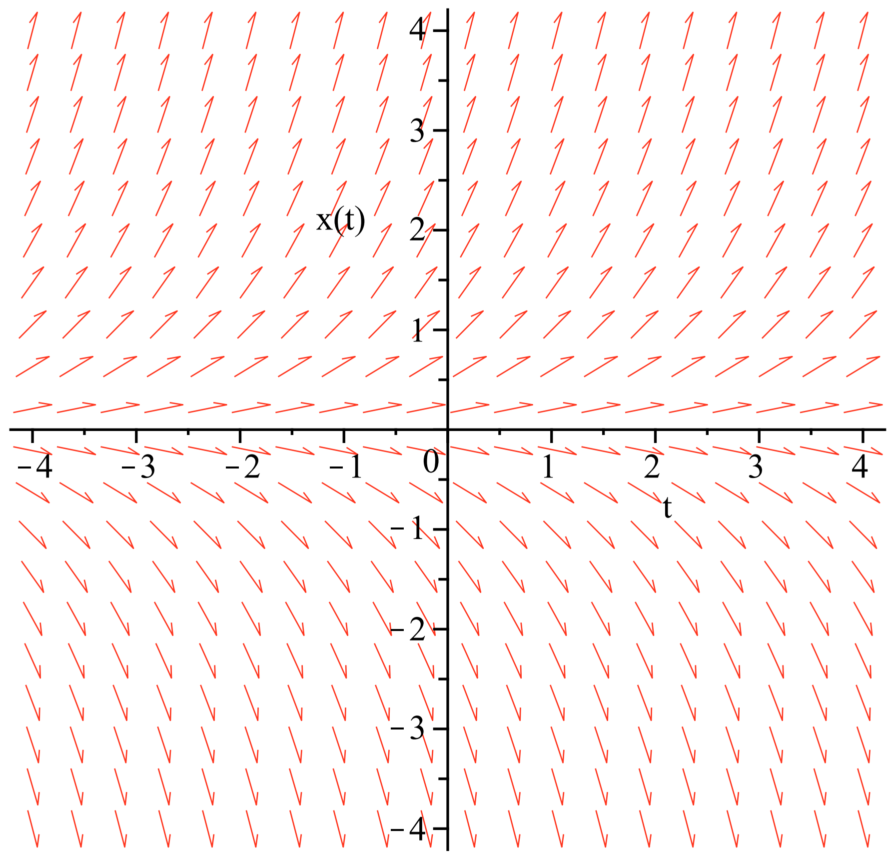
Isoclines are curves along which the slope lines have constant slope (they are given by $f(t, x) = c$, that is isoclines are level curves of the function f).

Some examples

$$\frac{dx}{dt} = k\,x$$

We will set k=1.

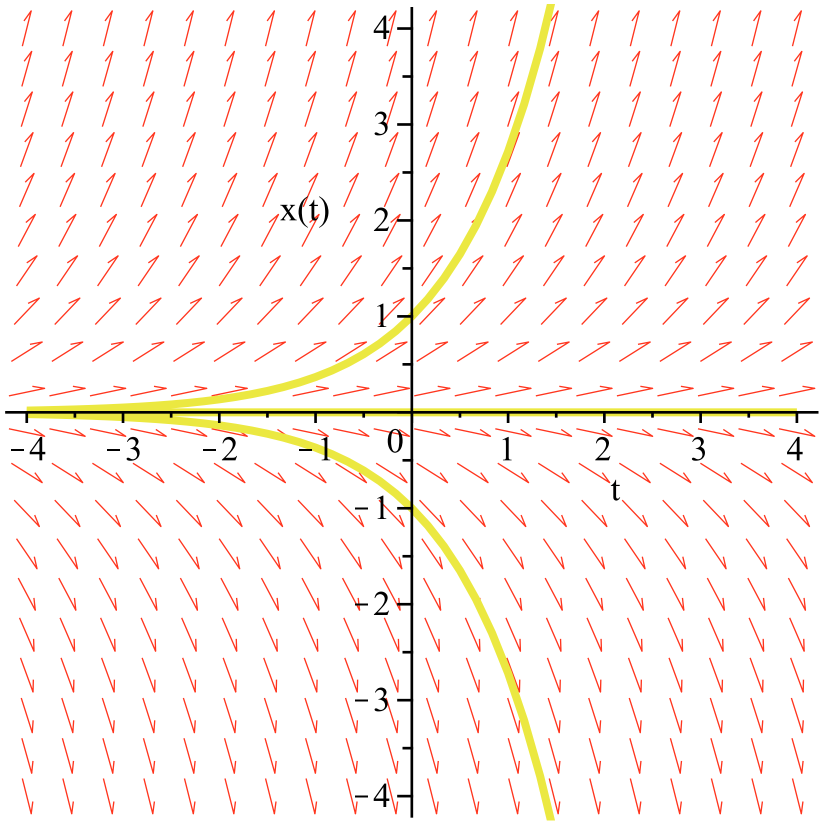
```
> k:=1:
> DEplot(D(x)(t)=k*x(t),x(t),t=-4..4,x=-4..4);
```



Properties:
For $0 < x(t)$, then x is increasing.
For $x(t) < 0$, then x is decreasing.
x cannot change sign.
$x = 0$ is a constant solution (corresponding to the initial condition $x(t_0) = 0$), it is the only constatnt solution.
x does not approach a constant value as t tends to ∞.

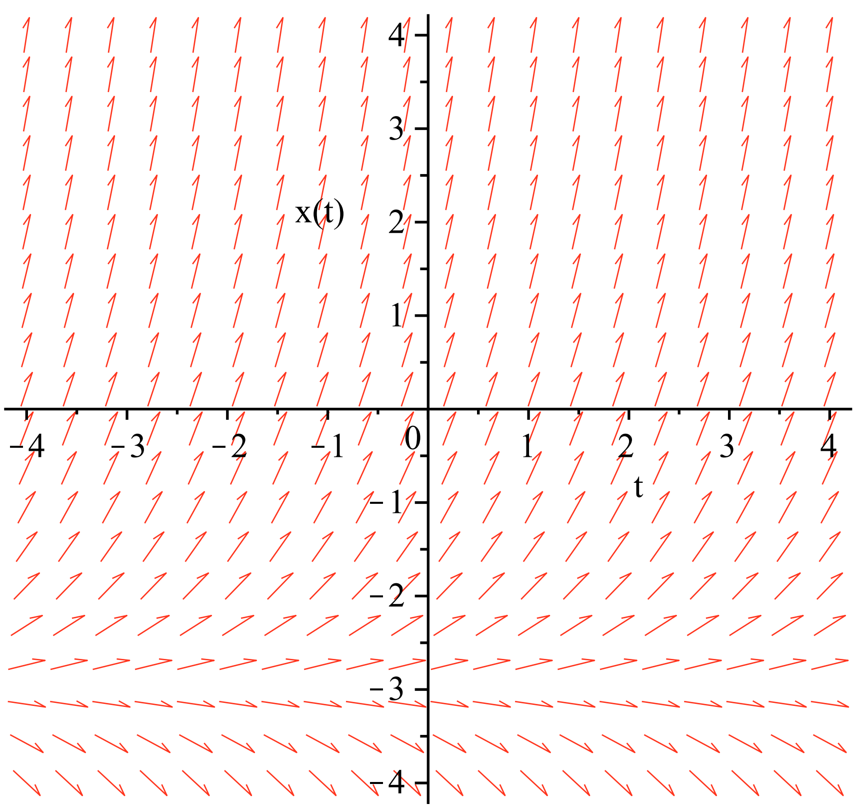
$$x(t) = c\,e^{kt}$$

```
> DEplot(D(x)(t)=k*x(t),x(t),t=-4..4,x=-4..4,
  {[x(0)=-1],[x(0)=0],[x(0)=1]});
```



$$\frac{dx}{dt} = k\,x + 3$$

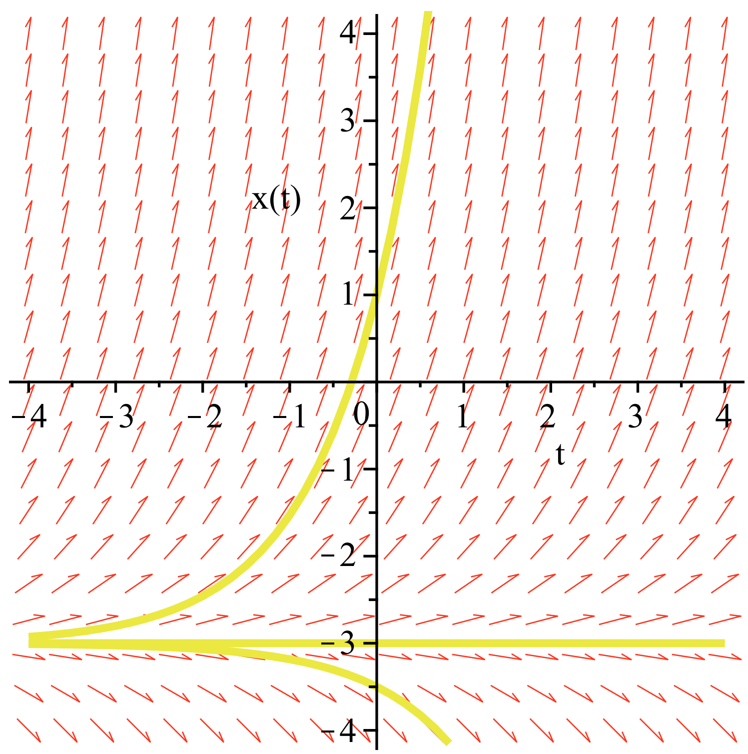
```
> DEplot(D(x)(t)=k*x(t)+3,x(t),t=-4..4,x=-4..4);
```



Properties:
For $-3 < x(t)$, then x is increasing.
For $x(t) < -3$, then x is decreasing.
x cannot cross the line $x(t) = -3$.
$x = -3$ is a constant solution (corresponding to the initial condition $x(t\_0) = -3$), it is the only constatnt solution.
x does not approach a constant value as t tends to ∞.

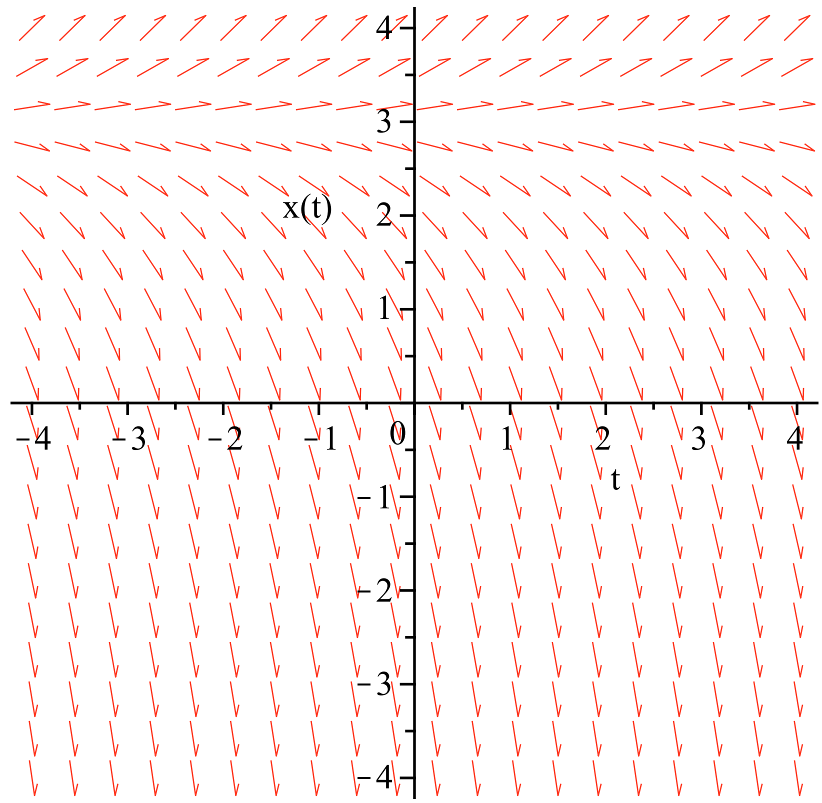
$$x(t) = c\,e^{kt} - \frac{3}{k}$$

```
> DEplot(D(x)(t)=k*x(t)+3,x(t),t=-4..4,x=-4..4,
{[x(0)=-7/2],[x(0)=-3],[x(0)=1]});
```



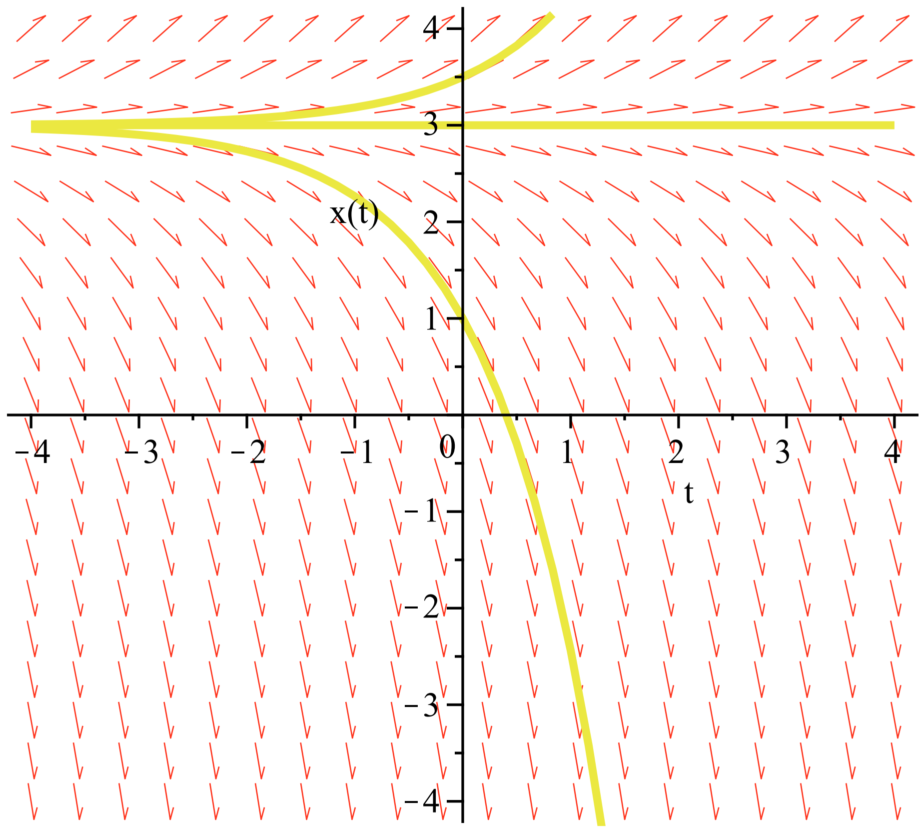
$$\frac{dx}{dt} = k\,x - 3$$

```
> DEplot(D(x)(t)=k*x(t)-3,x(t),t=-4..4,x=-4..4);
```



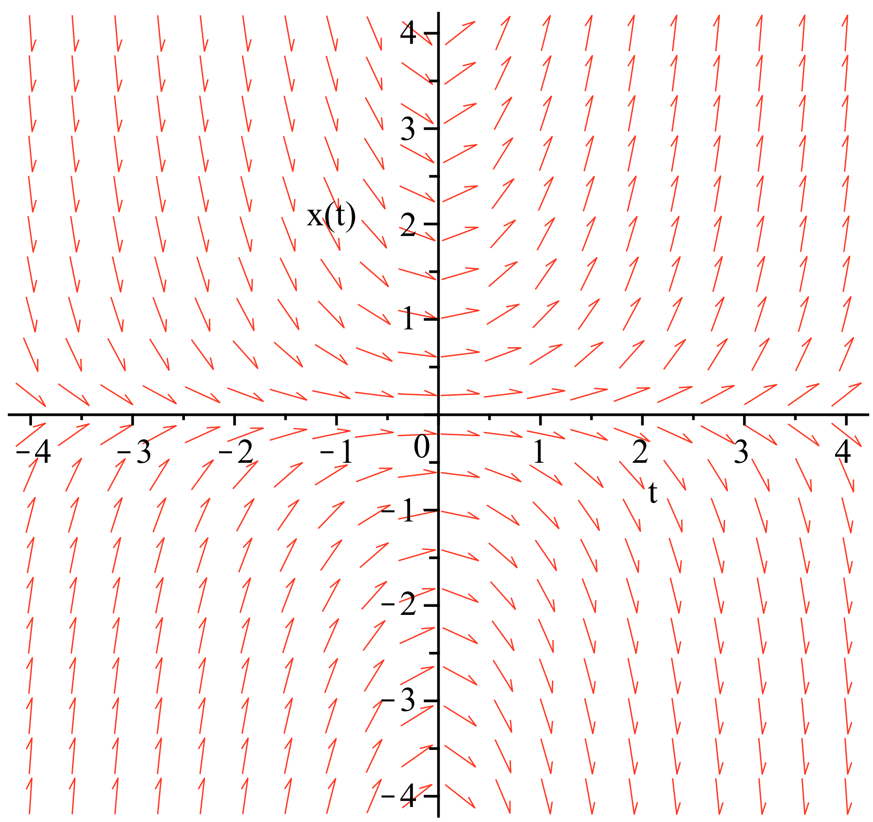
$$x(t) = c\, e^{kt} + \frac{3}{k}$$

```
> DEplot(D(x)(t)=k*x(t)-3,x(t),t=-4..4,x=-4..4,
  {[x(0)=1],[x(0)=3],[x(0)=7/2]});
```



$$\frac{dx}{dt} = k\,x\,t$$

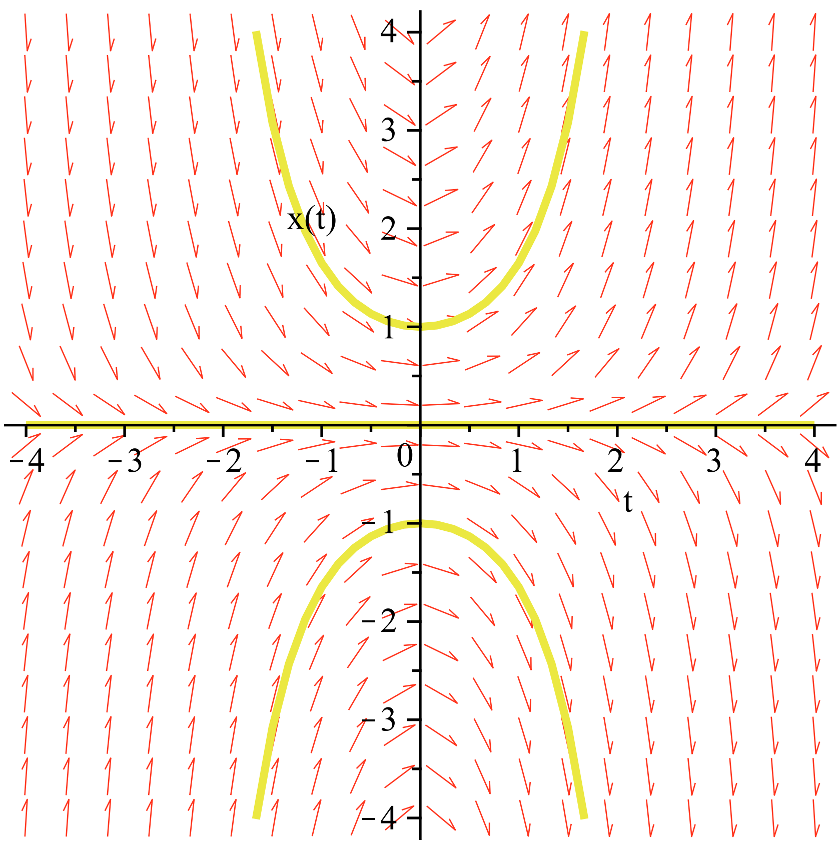
```
> DEplot(D(x)(t)=k*x(t)*t,x(t),t=-4..4,x=-4..4);
```



Properties:
For $0 < x\,t$, then x is increasing.
For $x\,t < 0$, then x is decreasing.
$x(t) = 0$ is the only constant solution (it correspond to the initial condition $x(t\_0) = 0$).
if the initial condition is not zero (that is if $x(t\_0) \neq 0$), then x is never 0.
$\lim_{t \to \infty} x = \infty$ if the initial value is positive and $\lim_{t \to \infty} x = -\infty$ if the initial value is negative.

$$x(t) = c\,e^{\frac{1\,k\,t^2}{2}}$$

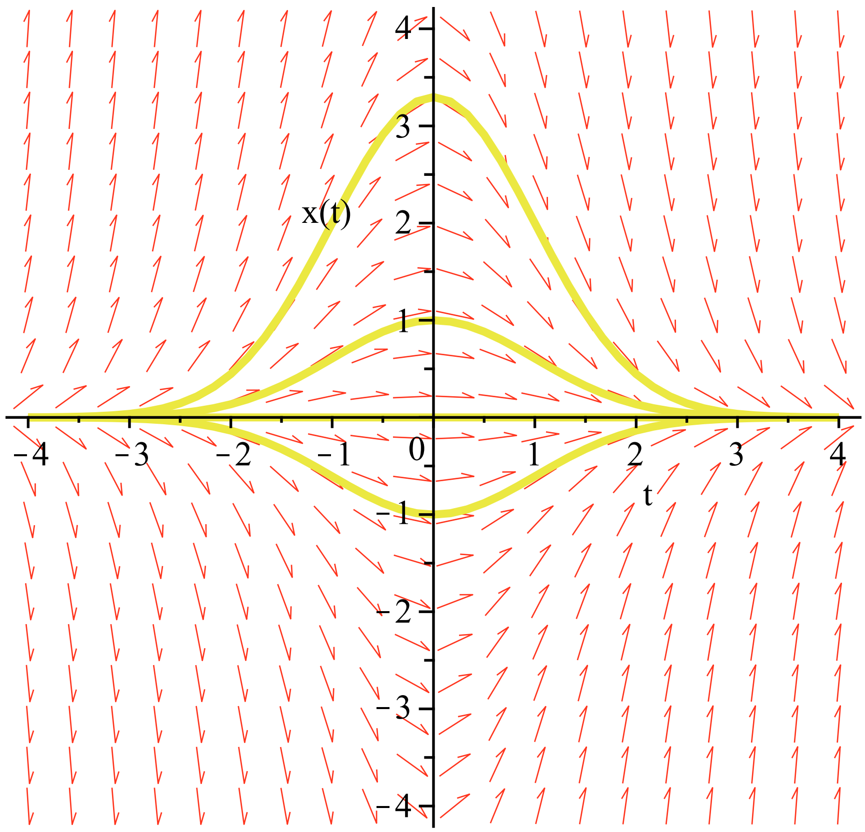
```
> DEplot(D(x)(t)=k*x(t)*t,x(t),t=-4..4,x=-4..4,
  {[x(0)=-1],[x(0)=0],[x(0)=1]});
```



Try the previous example with k=-1.

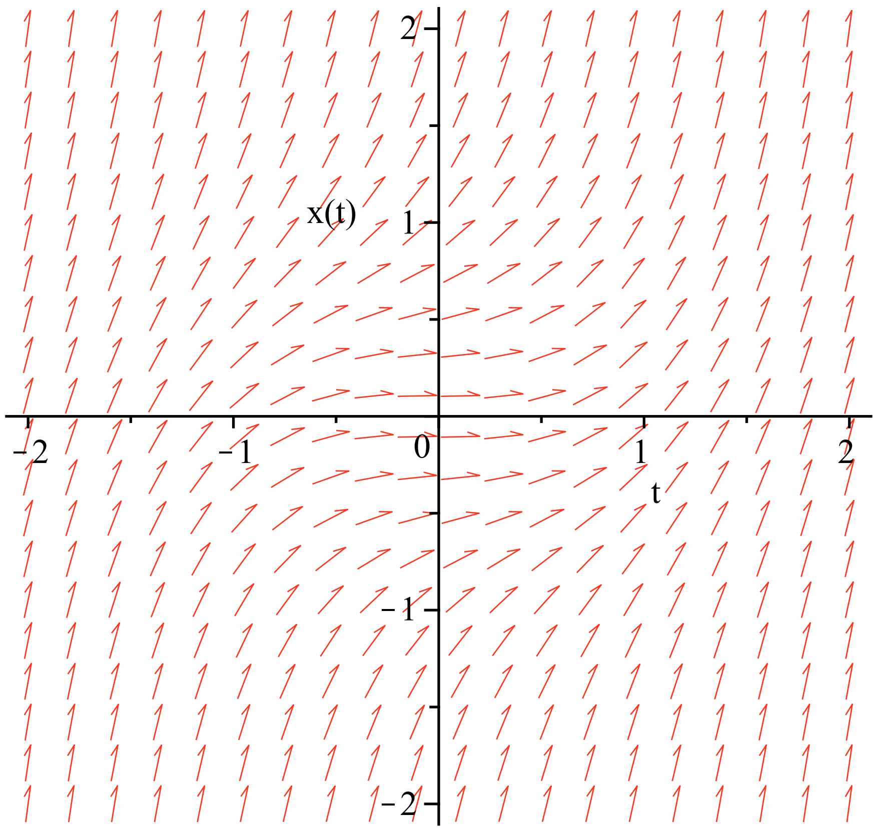
```
> DEplot(D(x)(t)=-k*x(t)*t,x(t),t=-4..4,x=-4..4,
  {[x(-1)=2],[x(0)=-1],[x(0)=0],[x(0)=1]});
```

Note: now $\lim_{t \to \infty} x = 0$ and also $\lim_{t \to -\infty} x = 0$.



$$\frac{dx}{dt} = x^2 + t^2$$

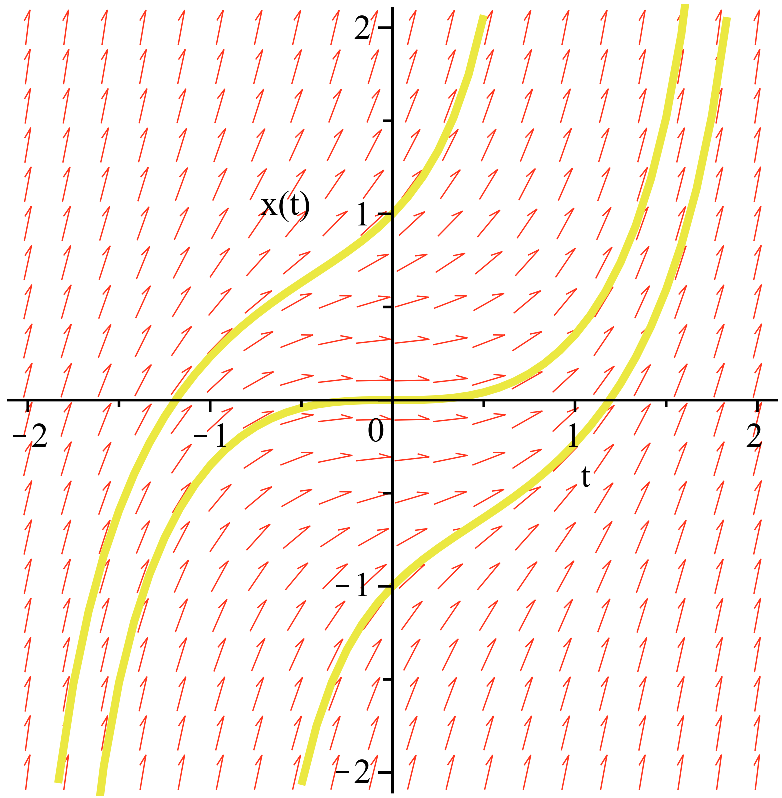
```
> DEplot(D(x)(t)=x(t)^2+t^2,x(t),t=-2..2,x=-2..2);
```



Properties:
x is alway increasing.
There are no constant solutions.

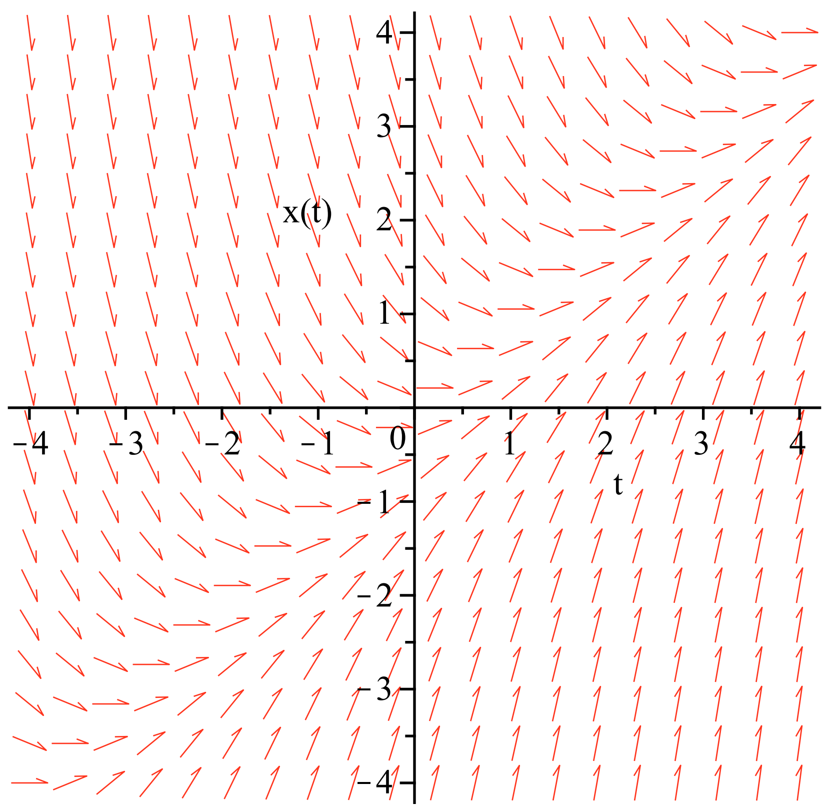
Solution curves

```
> DEplot(D(x)(t)=x(t)^2+t^2,x(t),t=-2..2,x=-2..2,
  {[x(0)=-1],[x(0)=0],[x(0)=1]});
```



$$\frac{dx}{dt} = t - x$$

```
> DEplot(D(x)(t)=t-x(t),x(t),t=-4..4,x=-4..4);
```



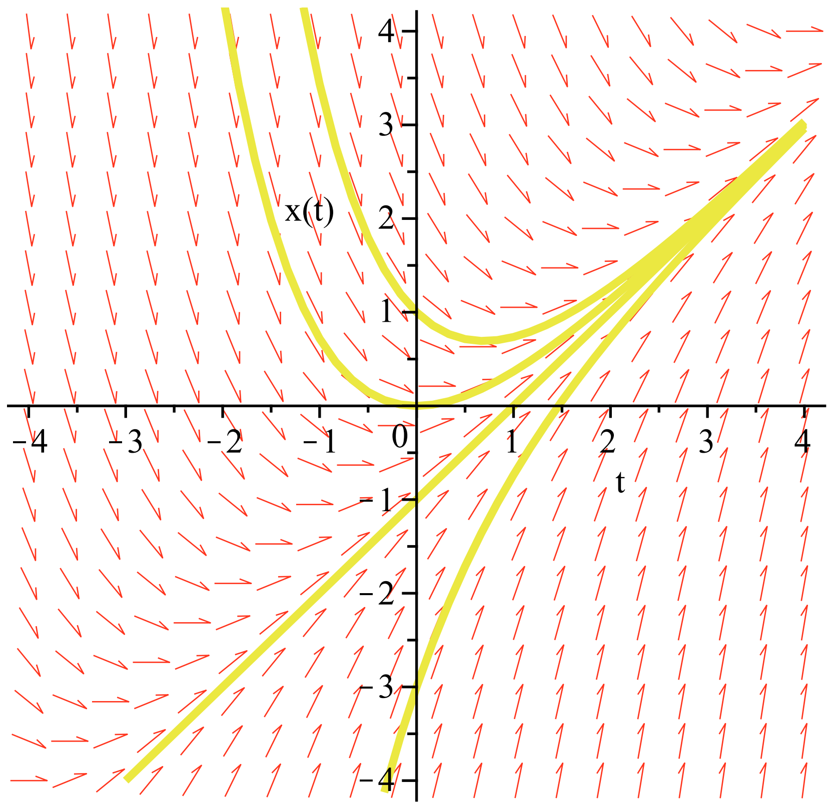
Solution curves

```
> DEplot(D(x)(t)=t-x(t),x(t),t=-4..4,x=-4..4,
  {[x(0)=-1],[x(0)=0],[x(0)=1],[x(0)=-3]});
```

Note: solutions "converge" to the function $x(t) = t - 1$.

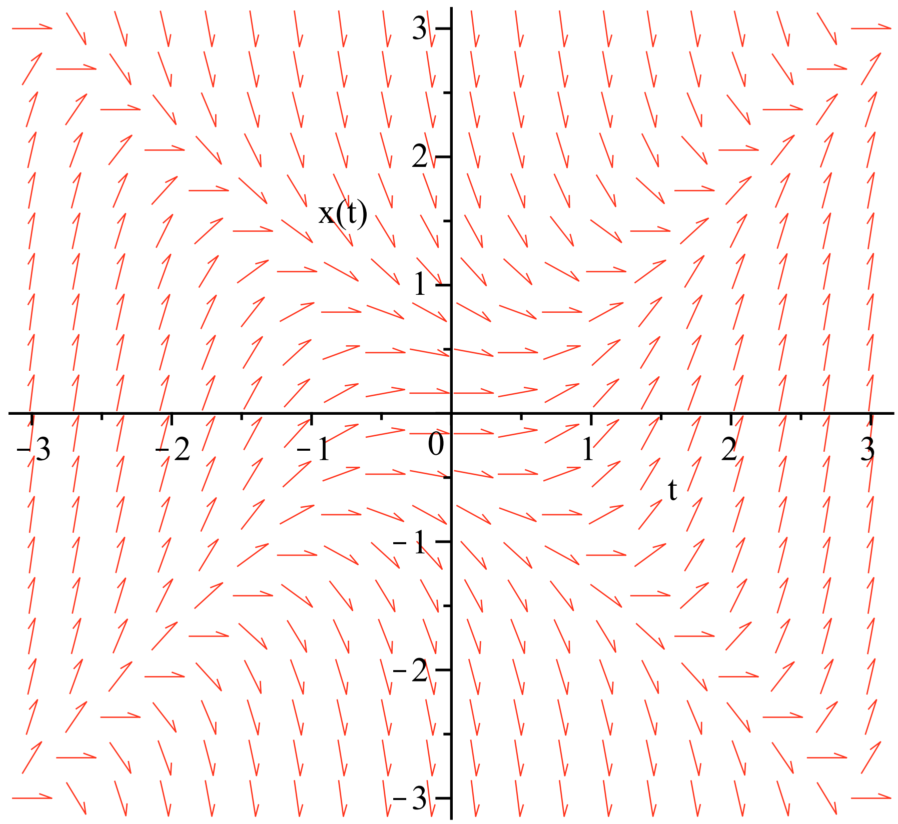
This becomes obvious if you know the solution is:

$$x(t) = t - 1 + c\,e^{-t}$$



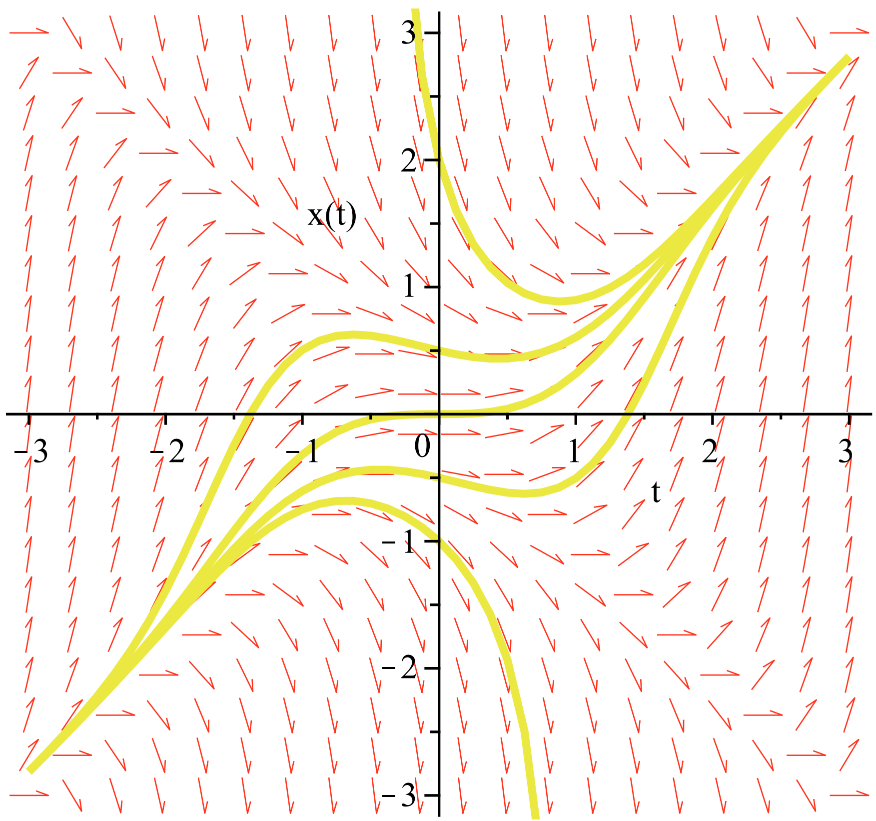
$$\frac{dx}{dt} = t^2 - x^2$$

```
> DEplot(D(x)(t)=t^2-x(t)^2,x(t),t=-3..3,x=-3..3);
```



Solution curves

```
> DEplot(D(x)(t)=t^2-x(t)^2,x(t),t=-3..3,x=-3..3,
  {[x(0)=-1],[x(0)=-1/2],[x(0)=0],[x(0)=1/2],[x(0)=2]});
```



Project 1

Study the problems:

$$\text{i.} \quad \frac{dx}{dt} = x^3 \qquad\qquad \text{ii.} \quad \frac{dx}{dt} = t^2 - x^2 + 1$$

a. Plot their direction field.
b. When is x(t) increasing, decreasing, constant, positive, and negative?
c. Is the solution valid for all t?
d. What are $\lim_{t \to \infty} x(t)$, $\lim_{t \to -\infty} x(t)$?