Periodic functions

\[
\text{restart: with(plots):

Warning, the name changecoords has been redefined

\[ f1(t) := \sin(2 \cdot t) \]

\[ f2(t) := \cos(3 \cdot t) \]

\[ f3(t) := \cos(5 \cdot t + 2) \]

\[ f4(t) := \sin(2^{1/2} \cdot t) \]

\[ f5(t) := \cos(\pi \cdot t) \]

\[ f1(t); f2(t); f3(t); f4(t); f5(t); \]

\( \sin(2 \cdot t) \)

\( \cos(3 \cdot t) \)

\( \cos(5 \cdot t + 2) \)

\( \sin(\sqrt{2} \cdot t) \)

\( \cos(\pi \cdot t) \)

\[ \text{plot(f1(t)+f2(t),t=0..8);} \]
Phase amplitude form

In order to better see the behavior of solutions of second order equations we can use the phase amplitude form of periodic functions. That is given a periodic function of the form

\[ f(t) = a \cos(\omega t) + b \sin(\omega t) \]

we can write it as

\[ f(t) = A \cos(\omega t - \delta) \]

where

\[ A = \sqrt{a^2 + b^2} \]

and

\[ \delta = \arctan\left(\frac{b}{a}\right) \]

(note this last equation has two solutions which differ by \( \pi \), you must choose the correct solution).

Also recall the two trigonometric identities

\[ \cos(\alpha) - \cos(\beta) = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right) \]

and

\[ \sin(\alpha) - \sin(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right) \]

using these identities we can often rewrite periodic functions to better see the phenomenon of beats.

For example, consider the equation \( D^{(2)}(x)(t) + 9x(t) = \sin(2t) \) with initial conditions?

\[ > \text{restart:with(DEtools):} \]
\[ > \text{sol:=rhs(dsolve}\{D(D(x))(t)+9\times(t) = \sin(2\times(t)),x(0) = 1,D(x)(0)=0\},x(t))\}; \]
\[ sol := -\frac{2}{15} \sin(3\ t) + \cos(3\ t) + \frac{1}{5} \sin(2\ t) \]

Using the above identities we have that the solution can be written as
Free oscillation

undamped oscillations

Consider the differential equation \( \frac{d^2x}{dt^2} + 16x = 0 \). With initial conditions \( x(0) = 1 \) and \( \frac{dx}{dt}(0) = 1 \).

The characteristic equation is \( r^2 + 16 = 0 \). This equation has the solutions:

\[
16 > \text{restart;}
16 > \text{solve}(r^2+16=0); \quad 4 \, \text{i}, \quad -4 \, \text{i}
\]

As you know, the homogeneous problem has the general solution

\[
x_h = c_1 \cos(4t) + c_2 \sin(4t).
\]

We now find \( c_1 \) and \( c_2 \)

\[
16 > \text{c1c2:= \{c1,c2\};}
16 > \text{sol:=c1*cos(4*t)+c2*sin(4*t);} \quad \text{sol := c1 cos(4 t) + c2 sin(4 t)}
16 > \text{eq1:=subs(t=0,sol);} \quad \text{eq1 := c1 cos(0) + c2 sin(0)}
16 > \text{eq2:=subs(t=0,diff(sol,t));}\quad \text{eq2 := -4 c1 sin(0) + 4 c2 cos(0)}
16 > \text{solve}\{\text{eq1=1,eq2=1}\,\text{,c1c2};}\quad \left\{ c2 = \frac{1}{4}, c1 = 1 \right\}
16 > \text{sol:=cos(4*t)+(1/4)*sin(4*t);} \quad \text{sol := cos(4 t) + \frac{1}{4} sin(4 t)}
16 > \text{plot(sol,t=0..4);}
underdamped oscillations

Consider the differential equation \( \frac{d^2x}{dt^2} + 2 \frac{dx}{dt} + 16 x = 0 \). With initial conditions \( x(0) = 1 \) and \( \frac{dx}{dt}(0) = 1 \).

The characteristic equation is \( r^2 + 2r + 16 = 0 \). This equation has the solutions:

\[
\text{restart:} \\
\text{solve}(r^2+2*r+16=0); \\
K1 \text{-I15, } K1 + \text{I15}
\]

As you know, the homogeneous problem has the general solution
\( x_h = c_1 e^{-t} \cos(\sqrt{15} t) + c_2 e^{-t} \sin(\sqrt{15} t) \).

We now find \( c_1 \) and \( c_2 \):

\[
\text{clc2:= \{c1,c2\}; } \\
\text{clc2:= \{c1, c2\}} \\
\text{sol:=c1*exp(-t)*cos(sqrt(15)*t)+c2*exp(-t)*sin(sqrt(15)*t);} \\
sol := c1 e^{-t} \cos(\sqrt{15} t) + c2 e^{-t} \sin(\sqrt{15} t) \\
\text{eq1:=subs(t=0,sol); } \\
eq 1 := c1 e^0 \cos(0) + c2 e^0 \sin(0) \\
\text{eq2:=subs(t=0,diff(sol,t)); } \\
eq 2 := -c1 e^0 \cos(0) - c1 e^0 \sin(0) \sqrt{15} - c2 e^0 \sin(0) + c2 e^0 \cos(0) \sqrt{15} \\
\text{solve({eq1=1,eq2=1},clc2);} \\
solve({eq1=1, eq2=1}, clc2);
\]
The characteristic equation is $r^2 + 8r + 16 = 0$. This equation has the solutions:

$$r_1 = r_2 = -4$$

As you know, the homogeneous problem has the general solution $x_h = c_1 e^{-4t} + c_2 t e^{-4t}$.

We now find $c_1$ and $c_2$

$$c_1 c_2 := \{c_1, c_2\}$$

$$c_1 c_2 := \{c_2, c_1\}$$

$$sol := c_1 \exp(-4t) + c_2 t \exp(-4t)$$

$$sol := c_1 e^{-4t} + c_2 t e^{-4t}$$

$$eq1 := \text{subs}(t=0, sol)$$
overdamped oscillations

Consider the differential equation $\frac{d^2 x}{dt^2} + 10 \frac{dx}{dt} + 16 x = 0$. With initial conditions $x(0) = 1$ and $\frac{dx}{dt}(0) = 1$.

The characteristic equation is $r^2 + 10r + 16 = 0$. This equation has the solutions:

```
> restart;
> solve(r^2+10*r+16=0);
```

```
-2, -8
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As you know, the homogeneous problem has the general solution $x_h = c_1 e^{-2t} + c_2 e^{-8t}$.

We now find $c_1$ and $c_2$

```
> clic2:= {c1,c2};
```

```
c1c2:= {c1, c2}
```

```
> sol:=c1*exp(-2*t)+c2*exp(-8*t);
```
\[
\text{sol} := c1 e^{-2t} + c2 e^{-8t}
\]

\[
\text{eq1} := \text{subs}(t=0, \text{sol});
\]

\[
\text{eq1} := c1 e^0 + c2 e^0
\]

\[
\text{eq2} := \text{subs}(t=0, \text{diff(sol,t)});
\]

\[
\text{eq2} := -2 c1 e^0 - 8 c2 e^0
\]

\[
\text{solve}\{\text{eq1}=1,\text{eq2}=1\}, \{c1, c2\};
\]

\[
\left\{c2 = -\frac{1}{2}, c1 = \frac{3}{2}\right\}
\]

\[
\text{sol} := (3/2) \exp(-2t) - (1/2) \exp(-8t);
\]

\[
\text{sol} := \frac{3}{2} e^{-2t} - \frac{1}{2} e^{-8t}
\]

\[
\text{plot(sol, t=0..2)};
\]