ANALYSIS AND FINITE ELEMENT SIMULATION OF MHD FLOWS, WITH AN APPLICATION TO SEAWATER DRAG REDUCTION [1]

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Abstract. Much research effort has recently been devoted to the electromagnetic control of saltwater flows, exploiting the macroscopic interaction of saltwater with electric currents and magnetic fields. This interaction is governed by the equations of viscous incompressible MHD, essentially, the Navier-Stokes equations coupled to Maxwell's equations. A major problem in the analysis and numerical solution of these equations is the fact that while the Navier-Stokes equations are posed in the fluid domain, Maxwell's equations are generally posed on all of space. Consequently, electric and magnetic fields do not satisfy standard boundary conditions, but jump or continuity relations on the surface of the fluid domain (and other interfaces). Frequently the resulting difficulties are circumvented by prescribing more or less artificial boundary conditions.

In this paper we present a novel formulation of the MHD equations that avoids some inherent difficulties of more traditional approaches by employing the electric current density rather than the magnetic field as the primary electromagnetic variable. This formulation leads to initial-boundary value problems for a system of integro-differential equations in the fluid domain and lends itself naturally to the use of finite-element based discretization techniques. As a first application we describe a mixed finite-element method for the numerical solution of a class of stationary MHD flow problems and report on the computational simulation of a simple drag reduction experiment.

I. INTRODUCTION

It has long been known that the flow of an electrically conducting fluid, such as seawater, is affected by Lorentz forces, induced by the interaction of electric currents and magnetic fields in the fluid. Only recently has it been demonstrated that such Lorentz forces can be used to control the flow and to attain specific engineering design goals such as flow stabilization, suppression or delay of flow separation, reduction of near-wall turbulence and skin friction, drag reduction and thrust generation (see, for example, [4, 9, 10] and the references cited therein).

The theory that describes the macroscopic interaction of an electrically conducting fluid with electric currents and magnetic fields is magnetohydrodynamics (or MHD). Assuming the fluid to be viscous, incompressible, and finitely conducting, the governing equations are the Navier-Stokes and pre-Maxwell equations, coupled via the Lorentz force and Ohm's law. While the Navier-Stokes equations are posed in the fluid domain, Maxwell's equations are generally posed on all of space, and typically both interior and exterior fields must be determined. Only under special circumstances, most notably in the presence of perfectly conducting walls, is it legitimate to confine attention to the body of conducting fluid and to neglect its electromagnetic interaction with the outside world. In general this interaction is of critical importance; in fact, it constitutes what mostly distinguishes MHD from ordinary hydrodynamics and is a source of challenging mathematical and computational problems.

Traditionally, the MHD equations are formulated as a system of evolution equations for the fluid velocity and the magnetic field, along with an auxiliary equation for the electric field outside the fluid region. The fact that the magnetic field extends to all of space and may exhibit jump discontinuities across interfaces separating media with different electromagnetic properties causes analytical as well as computational difficulties, which are frequently circumvented by prescribing more or less artificial boundary conditions. In [5–8] and [12] we developed a novel approach to viscous incompressible MHD that avoids some intrinsic difficulties of the traditional method by employing fluid velocity and electric current density (rather than fluid velocity and magnetic field) as the primary variables. This "velocity-current formulation" exploits the fact that while magnetic fields may extend throughout space, the unknown currents inducing those fields are typically carried by conductors of finite extent. If we consider, for example, a single body of conducting fluid and assume all external field sources to be known, the only unknown current flows in the fluid region itself. In this case, the velocity-current formulation allows us to perform all computations on the fluid domain while still accounting exactly for the effects of the universal electromagnetic field. In general, the velocity-current formulation leads to a system of evolution equations for the fluid velocity and the unknown current density in the fluids and adjacent solid conductors, along with an auxiliary linear div-curl system, which can usually be solved analytically in terms of singular integrals.

The velocity-current formulation lends itself naturally to the use of finite-element based discretization techniques and provides a theoretical framework for the development of efficient computational tools for the simulation of a wide variety of MHD flow problems, including the electromagnetic control of seawater flow. While the method has not yet been applied on an industrial scale, it has been shown to be effective in the analysis and numerical solution of a class of stationary MHD flow problems (see [8]). In the following we describe the general approach (Section II), derive a mixed variational formulation for the stationary case (Section III), discuss a finite-element method based on this formulation (Section IV), and report on the computational simulation of a simple drag reduction experiment (Section V). Despite the academic nature of this simulation, it illustrates the potential usefulness of our approach in solving a variety of MHD flow control and design problems.

II. THE VELOCITY-CURRENT FORMULATION

We are concerned with the flow of a viscous, incompressible, electrically conducting fluid, confined to a bounded region of space and interacting with various body forces, electric currents, and electromagnetic fields. Under the assumptions of the MHD approximation, the flow is governed by the Navier-Stokes equations, posed in the fluid domain, and the pre-Maxwell equations, posed on all of space; both are coupled via the Lorentz force and Ohm's law. As discussed in the introduction, we seek to formulate the problem as a system of evolution equations for the fluid velocity $\mathbf{u}$ and the electric current density $\mathbf{J}$ in the fluid; both are solenoidal vector fields, depending on time t and position x.

The evolution of the velocity field is governed by the Navier-Stokes equations, that is, the momentum balance

$$\rho\mathbf{u}_t - \eta\Delta\mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mathbf{J}\times\mathbf{B} = \mathbf{F}_{\text{ext}} \tag{1}$$

along with the continuity equation

$$\nabla\cdot\mathbf{u} = 0\,, \tag{2}$$

reflecting the incompressibility of the fluid. Here ρ and η denote the (constant) density and viscosity of the fluid; $\mathbf{F}_{\text{ext}}$ is a given external body force; and p is the scalar pressure, an auxiliary unknown that plays the role of a Lagrange multiplier associated with the divergence constraint (2). Equations (1) and (2) are coupled to Maxwell's equations through the Lorentz force, $\mathbf{J}\times\mathbf{B}$, and Ohm's law,

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u}\times\mathbf{B})\,, \tag{3}$$

where $\mathbf{E}$ and $\mathbf{B}$ denote the (unknown) electric and magnetic fields; σ is the (constant) electric conductivity of the fluid. Additional currents $\mathbf{J}_{\text{ext}}$ may

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be flowing in external conductors, possibly connected to the fluid domain via electrodes on the surface. The total current distribution,

$$\tilde{\mathbf{J}} = \mathbf{J} + \mathbf{J}_{\text{ext}} = \begin{cases} \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) & \text{in the fluid,} \\ \mathbf{J}_{\text{ext}} & \text{in the exterior,} \end{cases}$$

must satisfy the continuity equation

$$\nabla \cdot \tilde{\mathbf{J}} = 0\,,$$

reflecting the conservation of charge.

In order to obtain an evolution equation for the current density, we need to represent $\mathbf{E}$ and $\mathbf{B}$ in terms of $\mathbf{J}$. To begin with, we write the magnetic field as

$$\mathbf{B} = \mathbf{B}_{\text{ext}} + \mathcal{B}(\tilde{\mathbf{J}})\,,$$

where $\mathbf{B}_{\text{ext}}$ is an applied field, possibly generated by permanent or electromagnets surrounding the fluid domain, while $\mathcal{B}(\tilde{\mathbf{J}})$ is the field induced by $\tilde{\mathbf{J}} = \mathbf{J}_{\text{ext}} + \mathbf{J}$. Adopting the quasi-stationary form of Maxwell's equations, as is the custom in MHD, we obtain $\mathcal{B}(\tilde{\mathbf{J}})$ as the solution of

$$\nabla \times \mu^{-1}\mathcal{B}(\tilde{\mathbf{J}}) = \tilde{\mathbf{J}} \quad \text{and} \quad \nabla \cdot \mathcal{B}(\tilde{\mathbf{J}}) = 0\,,$$

where μ denotes the magnetic permeability. For simplicity we assume the fluid as well as all materials outside to be nonmagnetic so that μ is the permeability of the vacuum.

Next we introduce vector potentials for the (solenoidal) vector fields $\mathbf{B}_{\text{ext}}$ and $\mathcal{B}(\tilde{\mathbf{J}})$, that is, vector fields $\mathbf{A}_{\text{ext}}$ and $\mathcal{A}(\tilde{\mathbf{J}})$ satisfying

$$\nabla \times \mathbf{A}_{\text{ext}} = \mathbf{B}_{\text{ext}} \quad \text{and} \quad \nabla \cdot \mathbf{A}_{\text{ext}} = 0\,,$$

$$\nabla \times \mathcal{A}(\tilde{\mathbf{J}}) = \mathcal{B}(\tilde{\mathbf{J}}) \quad \text{and} \quad \nabla \cdot \mathcal{A}(\tilde{\mathbf{J}}) = 0\,.$$

Since we have $\nabla \times \mu^{-1}\mathcal{B}(\tilde{\mathbf{J}}) = \tilde{\mathbf{J}}$ and since μ is assumed to be constant, $\mathcal{A}(\tilde{\mathbf{J}})$ satisfies

$$\nabla \times \nabla \times \mathcal{A}(\tilde{\mathbf{J}}) = \mu\tilde{\mathbf{J}} \quad \text{and} \quad \nabla \cdot \mathcal{A}(\tilde{\mathbf{J}}) = 0\,,$$

or equivalently,

$$-\Delta\mathcal{A}(\tilde{\mathbf{J}}) = \mu\tilde{\mathbf{J}}\,.$$

Under a suitable radiation condition at infinity, this equation has a unique solution,

$$\mathcal{A}(\tilde{\mathbf{J}}) = \mu\mathcal{L}(\tilde{\mathbf{J}}) = \mu\mathcal{L}(\mathbf{J}_{\text{ext}}) + \mu\mathcal{L}(\mathbf{J})\,,$$

where (formally) $\mathcal{L} = (-\Delta)^{-1}$. Similarly,

$$\mathbf{A}_{\text{ext}} = \mathcal{L}(\nabla \times \mathbf{B}_{\text{ext}}) = \nabla \times \mathcal{L}(\mathbf{B}_{\text{ext}})\,.$$

We note that $\mathcal{L}$ is a weakly singular integral operator, given by

$$\mathcal{L}(\mathbf{f})(x) = \frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{\mathbf{f}(y)}{|x-y|}\,dy\,,$$

for any sufficiently regular vector field $\mathbf{f}$ with sufficiently fast decay at infinity, and that

$$\nabla \times \mathcal{L}(\mathbf{f})(x) = -\frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{x-y}{|x-y|^3} \times \mathbf{f}(y)\,dy\,.$$

The resulting representation of the magnetic field,

$$\mathbf{B} = \mathbf{B}_{\text{ext}} + \mathcal{B}(\tilde{\mathbf{J}}) = \mathbf{B}_{\text{ext}} + \mu\nabla \times \mathcal{L}(\mathbf{J}_{\text{ext}}) + \mu\nabla \times \mathcal{L}(\mathbf{J})\,, \tag{4}$$

is commonly called the Biot-Savart law.

Turning to the electric field $\mathbf{E}$, we observe that according to Faraday's law,

$$\nabla \times \mathbf{E} = -\mathbf{B}_t\,.$$

Since $\mathbf{B} = \nabla \times \mathbf{A}$ with $\mathbf{A} = \mathbf{A}_{\text{ext}} + \mathcal{A}(\tilde{\mathbf{J}})$, it follows that $\nabla \times (\mathbf{E} + \mathbf{A}_t) = 0$ and thus, $\mathbf{E} + \mathbf{A}_t = -\nabla\phi$ for some scalar potential ϕ. But

$$\begin{aligned} \mathbf{A}_t &= \mathbf{A}_{\text{ext},t} + \mathcal{A}(\tilde{\mathbf{J}})_t \\ &= \nabla \times \mathcal{L}(\mathbf{B}_{\text{ext},t}) + \mu\mathcal{L}(\mathbf{J}_{\text{ext},t}) + \mu\mathcal{L}(\mathbf{J}_t) \end{aligned}$$

and thus,

$$\mathbf{E} = \mathbf{E}_{\text{ext}} - \mu\mathcal{L}(\mathbf{J}_t) - \nabla\phi\,,$$

where

$$\mathbf{E}_{\text{ext}} = -\nabla \times \mathcal{L}(\mathbf{B}_{\text{ext},t}) - \mu\mathcal{L}(\mathbf{J}_{\text{ext},t})\,. \tag{5}$$

Substituting this into Ohm's law (3), we obtain

$$\mathbf{J} = \sigma(\mathbf{E}_{\text{ext}} - \mu\mathcal{L}(\mathbf{J}_t) - \nabla\phi + \mathbf{u} \times \mathbf{B})$$

or equivalently,

$$\mu\mathcal{L}(\mathbf{J}_t) + \sigma^{-1}\mathbf{J} + \nabla\phi - \mathbf{u} \times \mathbf{B} = \mathbf{E}_{\text{ext}}\,. \tag{6}$$

This is the desired evolution equation for the current density $\mathbf{J}$ in the fluid domain. Analogous to the pressure p in the Navier-Stokes equations, the scalar potential ϕ plays the role of a Lagrange multiplier associated with the divergence constraint

$$\nabla \cdot \mathbf{J} = 0\,. \tag{7}$$

Obviously the system of equations (1)–(2) and (6)–(7), with $\mathbf{B}$ and $\mathbf{E}_{\text{ext}}$ given by (4) and (5), is *closed* only if the external current distribution $\mathbf{J}_{\text{ext}}$ is assumed to be known. If this is not the case, Equations (6)–(7) must be solved in a larger region of space, including the fluid and adjacent external conductors (with $\mathbf{u} = 0$ outside the fluid, of course). It should be noted, however, that $\mathbf{J}_{\text{ext}}$ enters the equations only via the induced magnetic field, $\mu\nabla \times \mathcal{L}(\mathbf{J}_{\text{ext}})$. In many applications the effect of this field on the fluid motion will be negligible. In fact, the applied magnetic field $\mathbf{B}_{\text{ext}}$ is typically much stronger than *any* induced field, so that it may well be reasonable to neglect induction effects altogether. Formally, this amounts to setting $\mu = 0$ in (4)–(6), in which case Equation (6) becomes quasi-stationary.

The system of equations (1)–(2) and (6)–(7) must be supplemented with initial conditions for $\mathbf{u}$ and $\mathbf{J}$ and suitable boundary conditions for $(\mathbf{u}, p)$ and $(\mathbf{J}, \phi)$. Let Ω denote the fluid domain, Γ its surface, and $\mathbf{n}$ the outward unit normal vector field on Γ. The simplest physically reasonable and mathematically feasible boundary conditions are $\mathbf{u} = 0$ and $\mathbf{J} \cdot \mathbf{n} = 0$ on Γ. Here we allow for both mass and current flux across Γ, which leads to inhomogeneous Dirichlet or Neumann type boundary conditions. Specifically, we prescribe the *velocity* $\mathbf{u}$ on an open subset Γ_1 of Γ and the *stress* $\eta\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) \cdot \mathbf{n} - p\,\mathbf{n}$ on its complement $\Gamma_2 = \Gamma \setminus \overline{\Gamma_1}$; we prescribe the *current flux* $\mathbf{J} \cdot \mathbf{n}$ on an open subset Γ_3 of Γ and the *electric potential* ϕ on its complement $\Gamma_4 = \Gamma \setminus \overline{\Gamma_3}$:

$$\mathbf{u} = \mathbf{g}_1 \quad \text{on } \Gamma_1, \qquad \eta\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) \cdot \mathbf{n} - p\,\mathbf{n} = \mathbf{g}_2 \quad \text{on } \Gamma_2,$$

$$\mathbf{J} \cdot \mathbf{n} = g_3 \quad \text{on } \Gamma_3, \qquad \phi = g_4 \quad \text{on } \Gamma_4.$$

In certain cases, the boundary data $\mathbf{g}_1$, $\mathbf{g}_2$, g_3, g_4 must satisfy compatibility conditions. For example, if $\Gamma_2 = \emptyset$, then $\mathbf{g}_1 \cdot \mathbf{n}$ must have mean zero on Γ (since $\nabla \cdot \mathbf{u} = 0$ in Ω); if $\Gamma_4 = \emptyset$, then g_3 must have mean zero on Γ (since $\nabla \cdot \mathbf{J} = 0$ in Ω).

Summarizing, our problem is the following: Given the fluid domain Ω (a bounded region of space with sufficiently regular boundary $\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2} = \overline{\Gamma_3} \cup \overline{\Gamma_4}$), given the positive parameters ρ, η, μ, and σ, given the external fields $\mathbf{F}_{\text{ext}}$, $\mathbf{J}_{\text{ext}}$, $\mathbf{B}_{\text{ext}}$, and $\mathbf{E}_{\text{ext}} = -\nabla \times \mathcal{L}(\mathbf{B}_{\text{ext},t}) - \mu\mathcal{L}(\mathbf{J}_{\text{ext},t})$, given compatible boundary data $\mathbf{g}_1$, $\mathbf{g}_2$, g_3, and g_4, and given initial values $\mathbf{u}_0$ and $\mathbf{J}_0$, find vector fields $\mathbf{u} = \mathbf{u}(t,x)$, $\mathbf{J} = \mathbf{J}(t,x)$ and scalar fields $p = p(t,x)$, $\phi = \phi(t,x)$ such that the following equations are satisfied with $\mathbf{B} = \mathbf{B}_{\text{ext}} + \mu\nabla \times \mathcal{L}(\mathbf{J}_{\text{ext}}) + \mu\nabla \times \mathcal{L}(\mathbf{J})$:

$$\rho\mathbf{u}_t - \eta\Delta\mathbf{u} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \mathbf{J} \times \mathbf{B} = \mathbf{F}_{\text{ext}} \quad (t > 0,\ x \in \Omega),$$

$$\nabla \cdot \mathbf{u} = 0 \quad (t > 0,\ x \in \Omega),$$

$$\mu\mathcal{L}(\mathbf{J}_t) + \sigma^{-1}\mathbf{J} + \nabla\phi - \mathbf{u} \times \mathbf{B} = \mathbf{E}_{\text{ext}} \quad (t > 0,\ x \in \Omega),$$

$$\nabla \cdot \mathbf{J} = 0 \quad (t > 0,\ x \in \Omega),$$

$$\mathbf{u} = \mathbf{g}_1 \quad (t > 0,\ x \in \Gamma_1),$$

$$\eta\Big(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\Big) \cdot \mathbf{n} - p\,\mathbf{n} = \mathbf{g}_2 \quad (t > 0,\ x \in \Gamma_2),$$

$$\mathbf{J} \cdot \mathbf{n} = g_3 \quad (t > 0,\ x \in \Gamma_3), \qquad \phi = g_4 \quad (t > 0,\ x \in \Gamma_4),$$

$$\mathbf{u} = \mathbf{u}_0 \quad (t = 0,\ x \in \Omega), \qquad \mathbf{J} = \mathbf{J}_0 \quad (t = 0,\ x \in \Omega).$$

Under mild regularity assumptions on the data, this problem has a weak solution $(\mathbf{u}, \mathbf{J}, p, \phi)$, defined for all time $t > 0$. If the boundary data are sufficiently small (or if the viscosity η and resistivity σ^{-1} of the fluid are sufficiently large), the solution remains bounded as $t \to \infty$. For further details and a rigorous proof (if only in the case $\Gamma_2 = \Gamma_4 = \emptyset$), the reader is referred to [12].

III. A VARIATIONAL FORMULATION FOR THE STATIONARY PROBLEM

As a first step towards the numerical analysis and finite-element approximation of the full, time-dependent problem described in Section II, we consider the steady-state version where data and unknowns are independent of time. In this case Equations (1)–(2) and (6)–(7) reduce to

$$-\eta\Delta\mathbf{u} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \mathbf{J} \times \mathbf{B} = \mathbf{F}_{\text{ext}}\,, \tag{8}$$

$$\nabla \cdot \mathbf{u} = 0\,, \tag{9}$$

$$\sigma^{-1}\mathbf{J} + \nabla\phi - \mathbf{u} \times \mathbf{B} = \mathbf{E}_{\text{ext}}\,, \tag{10}$$

$$\nabla \cdot \mathbf{J} = 0\,, \tag{11}$$

all posed in the fluid domain Ω and supplemented with boundary conditions on the surface $\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2} = \overline{\Gamma_3} \cup \overline{\Gamma_4}$:

$$\mathbf{u} = \mathbf{g}_1 \quad \text{on } \Gamma_1, \qquad \eta\Big(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\Big) \cdot \mathbf{n} - p\,\mathbf{n} = \mathbf{g}_2 \quad \text{on } \Gamma_2, \tag{12}$$

$$\mathbf{J} \cdot \mathbf{n} = g_3 \quad \text{on } \Gamma_3, \qquad \phi = g_4 \quad \text{on } \Gamma_4. \tag{13}$$

As before, the magnetic field is given by

$$\mathbf{B} = \mathbf{B}_{\text{ext}} + \mu\nabla \times \mathcal{L}(\mathbf{J}_{\text{ext}}) + \mu\nabla \times \mathcal{L}(\mathbf{J})\,. \tag{14}$$

On physical grounds, $\mathbf{E}_{\text{ext}}$ should be zero in the stationary case, but for reasons of symmetry in the equations we allow for an arbitrary field $\mathbf{E}_{\text{ext}}$. We assume that Ω is a bounded Lipschitz domain and that the subsets Γ_i of the surface Γ are non-empty, open Lipschitz surfaces with $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\overline{\Gamma_1} \cup \overline{\Gamma_2} = \Gamma$ and $\Gamma_3 \cap \Gamma_4 = \emptyset$, $\overline{\Gamma_3} \cup \overline{\Gamma_4} = \Gamma$. (The subsequent analysis would remain valid, with only minor modifications, if one of the sets Γ_1, Γ_2 and/or one of the sets Γ_3, Γ_4 was empty.)

We will seek weak solutions $(\mathbf{u}, \mathbf{J}, p, \phi)$ of Equations (8)–(14) with

$$\mathbf{u} \in \mathbf{X}_1 := \mathbf{H}^1(\Omega), \quad \mathbf{J} \in \mathbf{X}_2 := \mathbf{L}^2(\Omega),$$

$$p \in M_1 := L^2(\Omega), \quad \phi \in M_2 := H^1(\Omega).$$

In addition to the above, we will need the subspaces

$$\tilde{\mathbf{X}}_1 := \{\mathbf{v} \in \mathbf{X}_1 \mid \mathbf{v} = 0 \text{ on } \Gamma_1\}$$

and

$$\tilde{M}_2 := \{\psi \in M_2 \mid \psi = 0 \text{ on } \Gamma_4\}.$$

Here and in the sequel, $L^2(\Omega)$ denotes the space of square-integrable scalar functions on Ω, and $H^1(\Omega)$ is the subspace of $L^2(\Omega)$ comprised of functions with square-integrable first-order derivatives. Both $L^2(\Omega)$ and $H^1(\Omega)$ are Hilbert spaces with norms given by

$$\|f\|_{L^2(\Omega)} := \Big(\int_\Omega |f|^2\Big)^{1/2}$$

and

$$\|f\|_{H^1(\Omega)} := \Big(\|f\|^2_{L^2(\Omega)} + \|\nabla f\|^2_{\mathbf{L}^2(\Omega)}\Big)^{1/2}.$$

Bold-face type is used for the corresponding spaces of vector functions.

The following assumptions on the data guarantee that all the equations are meaningful (in the weak sense):

$$\mathbf{F}_{\text{ext}} \in \mathbf{L}^2(\Omega), \quad \mathbf{E}_{\text{ext}} \in \mathbf{L}^2(\Omega),$$

$$\mathbf{J}_{\text{ext}} \in \mathbf{L}^2(\mathbf{R}^3 \setminus \overline{\Omega}), \quad \mathbf{B}_{\text{ext}} \in \mathbf{H}^1(\Omega),$$

$$\mathbf{g}_1 \in \mathbf{H}^{1/2}(\Gamma_1), \quad \mathbf{g}_2 \in \mathbf{H}^{-1/2}(\Gamma_2),$$

$$g_3 \in H^{-1/2}(\Gamma_3), \quad g_4 \in H^{1/2}(\Gamma_4).$$

The space $H^{1/2}(\Gamma_i)$, for $1 \le i \le 4$, consists of the traces (or generalized boundary values) on Γ_i of functions in $H^1(\Omega)$, and $H^{-1/2}(\Gamma_i)$ is the dual of $H^{1/2}(\Gamma_i)$. These are Hilbert spaces with norms derived from that of $H^1(\Omega)$. Again, bold-face type is used for the corresponding spaces of vector functions.

To derive a weak or variational form of the problem at hand, we multiply Equations (8) and (10) by test functions $\mathbf{v} \in \tilde{\mathbf{X}}_1$ and $\mathbf{K} \in \mathbf{X}_2$, respectively, and Equations (9) and (11) by test functions $q \in M_1$ and $\psi \in \tilde{M}_2$, respectively. We then integrate over Ω, perform several integrations by parts, regroup terms, and add the equations obtained from (8) and (10) and those obtained from (9) and (11). This procedure results in two equations of the form

$$\begin{aligned} a_0\Big((\mathbf{u},\mathbf{J}),(\mathbf{v},\mathbf{K})\Big) + a_1\Big((\mathbf{u},\mathbf{J}),(\mathbf{u},\mathbf{J}),(\mathbf{v},\mathbf{K})\Big)& \\ + b\Big((\mathbf{v},\mathbf{K}),(p,\phi)\Big) = \ell_0(\mathbf{v},\mathbf{K})& \end{aligned} \tag{15}$$

and

$$b\Big((\mathbf{u},\mathbf{J}),(q,\psi)\Big) = \ell_1(q,\psi)\,, \tag{16}$$

where a_0 (a bilinear form), a_1 (a trilinear form), b (a bilinear form), ℓ_0 and ℓ_1 (linear forms) are given by

$$\begin{aligned} &a_0\Big((\mathbf{v}_1,\mathbf{K}_1),(\mathbf{v}_2,\mathbf{K}_2)\Big) \\ &\quad := \frac{\eta}{2}\int_\Omega \Big(\nabla\mathbf{v}_1 + (\nabla\mathbf{v}_2)^T\Big) : \Big(\nabla\mathbf{v}_2 + (\nabla\mathbf{v}_2)^T\Big) + \sigma^{-1}\int_\Omega \mathbf{K}_1 \cdot \mathbf{K}_2 \\ &\qquad + \int_\Omega \Big(\Big(\mathbf{K}_2 \times \mathbf{B}_0\Big) \cdot \mathbf{v}_1 - \Big(\mathbf{K}_1 \times \mathbf{B}_0\Big) \cdot \mathbf{v}_2\Big)\,, \end{aligned}$$

where $\mathbf{B}_0 := \mathbf{B}_{\text{ext}} + \mu\nabla \times \mathcal{L}(\mathbf{J}_{\text{ext}})$, for $(\mathbf{v}_1,\mathbf{K}_1),(\mathbf{v}_2,\mathbf{K}_2) \in \mathbf{X}_1 \times \mathbf{X}_2$,

$$\begin{aligned} &a_1\Big((\mathbf{v}_1,\mathbf{K}_1),(\mathbf{v}_2,\mathbf{K}_2),(\mathbf{v}_3,\mathbf{K}_3)\Big) \\ &\quad := \frac{\rho}{2}\int_\Omega \Big(\Big((\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2\Big) \cdot \mathbf{v}_3 - \Big((\mathbf{v}_1 \cdot \nabla)\mathbf{v}_3\Big) \cdot \mathbf{v}_2\Big) \\ &\qquad + \mu\int_\Omega \Big(\Big(\mathbf{K}_3 \times (\nabla \times \mathcal{L}(\mathbf{K}_1))\Big) \cdot \mathbf{v}_2 - \Big(\mathbf{K}_2 \times (\nabla \times \mathcal{L}(\mathbf{K}_1))\Big) \cdot \mathbf{v}_3\Big)\,, \end{aligned}$$

for $(\mathbf{v}_1,\mathbf{K}_1),(\mathbf{v}_2,\mathbf{K}_2),(\mathbf{v}_3,\mathbf{K}_3) \in \mathbf{X}_1 \times \mathbf{X}_2$,

$$b\Big((\mathbf{v},\mathbf{K}),(q,\psi)\Big) := -\int_\Omega (\nabla \cdot \mathbf{v})q + \int_\Omega \mathbf{K} \cdot (\nabla\psi)\,,$$

for $(\mathbf{v},\mathbf{K}) \in \mathbf{X}_1 \times \mathbf{X}_2$, $(q,\psi) \in M_1 \times M_2$,

$$\ell_0(\mathbf{v},\mathbf{K}) := \int_\Omega \mathbf{F}_{\text{ext}} \cdot \mathbf{v} + \int_\Omega \mathbf{E}_{\text{ext}} \cdot \mathbf{K} + \int_{\Gamma_2} \mathbf{g}_2 \cdot \mathbf{v}\,,$$

for $(\mathbf{v}, \mathbf{K}) \in \mathbf{X}_1 \times \mathbf{X}_2$, and

$$\ell_1(q, \psi) := \int_{\Gamma_3} g_3 \psi \,,$$

for $(q, \psi) \in M_1 \times M_2$.

Routine arguments show that finding a weak solution $(\mathbf{u}, \mathbf{J}, p, \phi)$ of Equations (8)–(14) is equivalent to solving the following variational problem.

Problem (P)**.** Find $\mathbf{u} \in \mathbf{X}_1$ with $\mathbf{u} = \mathbf{g}_1$ on Γ_1, $\mathbf{J} \in \mathbf{X}_2$, $p \in M_1$, and $\phi \in M_2$ with $\phi = g_4$ on Γ_4 such that Equations (15) and (16) are satisfied for all $(\mathbf{v}, \mathbf{K}) \in \tilde{\mathbf{X}}_1 \times \mathbf{X}_2$ and $(q, \psi) \in M_1 \times \tilde{M}_2$, respectively.

After homogenization of the essential boundary conditions for $\mathbf{u}$ and ϕ, Problem (P) reduces to a mixed variational problem in the sense of the Ladyzhenskaya-Babuska-Brezzi theory (see, for example, [2, Chapter IV.1]). This allows us to prove the well-posedness of Problem (P), at least under a small-data assumption.

Theorem 1. If the data $\mathbf{F}_{\text{ext}}$, $\mathbf{E}_{\text{ext}}$, $\mathbf{J}_{\text{ext}}$, $\mathbf{B}_{\text{ext}}$ and $\mathbf{g}_1$, $\mathbf{g}_2$, g_3, g_4 are sufficiently small (or if the viscosity η and resistivity σ^{-1} are sufficiently large), then Problem (P) has a unique solution $(\mathbf{u}, \mathbf{J}, p, \phi)$, which depends continuously on the data and parameters of the problem.

Roughly speaking, Theorem 1 guarantees the existence, uniqueness, and stability of a steady solution to the MHD equations in the case of low Reynolds and magnetic Reynolds numbers. For a much more precise statement of the theorem, including specific bounds on the allowable size of the data (relative to the parameters of the problem), we refer to [8].

IV. FINITE-ELEMENT DISCRETIZATION AND ERROR ESTIMATES

In order to discretize Problem (P), we choose finite-dimensional approximations $\mathbf{X}_1^h$, $\mathbf{X}_2^h$, M_1^h, and M_2^h of the spaces $\mathbf{X}_1 := \mathbf{H}^1(\Omega)$, $\mathbf{X}_2 := \mathbf{L}^2(\Omega)$, $M_1 := L^2(\Omega)$, and $M_2 := H^1(\Omega)$. Furthermore, we set $\tilde{\mathbf{X}}_1^h := \{\mathbf{v}^h \in \mathbf{X}_1^h \mid \mathbf{v}^h = 0 \text{ on } \Gamma_1\}$ and $\tilde{M}_2^h := \{\psi^h \in M_2^h \mid \psi^h = 0 \text{ on } \Gamma_4\}$ and choose approximate essential boundary data $\mathbf{g}_1^h \in \{\mathbf{v}^h|_{\Gamma_1} \mid \mathbf{v}^h \in \mathbf{X}_1^h\}$ and $g_4^h \in \{\psi^h|_{\Gamma_4} \mid \psi^h \in M_2^h\}$. Here h is a discretization parameter, for example, the meshsize of a triangulation of the domain Ω. We assume that the spaces $\mathbf{X}_i^h$ and M_i^h approximate $\mathbf{X}_i$ and M_i in the sense that the error of best approximation of a function in $\mathbf{X}_i$ or M_i by elements of $\mathbf{X}_i^h$ or M_i^h tends to 0 as $h \to 0$; of course, we also assume that $\mathbf{g}_1^h \to \mathbf{g}_1$ and $g_4^h \to g_4$ (in the respective trace spaces). We then consider the following finite-dimensional approximation of Problem (P).

Problem (P^h)**.** Find $\mathbf{u}^h \in \mathbf{X}_1^h$ with $\mathbf{u}^h = \mathbf{g}_1^h$ on Γ_1, $\mathbf{J}^h \in \mathbf{X}_2^h$, $p^h \in M_1^h$, and $\phi^h \in M_2^h$ with $\phi^h = g_4^h$ on Γ_4 such that the equations

$$\begin{aligned} a_0\Big((\mathbf{u}^h, \mathbf{J}^h), (\mathbf{v}^h, \mathbf{K}^h)\Big) + a_1\Big((\mathbf{u}^h, \mathbf{J}^h), (\mathbf{u}^h, \mathbf{J}^h), (\mathbf{v}^h, \mathbf{K}^h)\Big) \\ + b\Big((\mathbf{v}^h, \mathbf{K}^h), (p^h, \phi^h)\Big) = \ell_0(\mathbf{v}^h, \mathbf{K}^h) \end{aligned} \tag{17}$$

and

$$b\Big((\mathbf{u}^h, \mathbf{J}^h), (q^h, \psi^h)\Big) = \ell_1(q^h, \psi^h) \tag{18}$$

are satisfied for all $(\mathbf{v}^h, \mathbf{K}^h) \in \tilde{\mathbf{X}}_1^h \times \mathbf{X}_2^h$ and $(q^h, \psi^h) \in M_1^h \times \tilde{M}_2^h$, respectively.

Under certain technical conditions on the finite-dimensional spaces $\mathbf{X}_i^h$ and M_i^h, an analog of Theorem 1 holds for Problem (P^h), and we obtain an optimal-order estimate for the discretization error (see [8] for details).

Theorem 2. If the data $\mathbf{F}_{\text{ext}}$, $\mathbf{E}_{\text{ext}}$, $\mathbf{J}_{\text{ext}}$, $\mathbf{B}_{\text{ext}}$ and $\mathbf{g}_1$, $\mathbf{g}_2$, g_3, g_4 are sufficiently small (or if the viscosity η and resistivity σ^{-1} of the fluid are sufficiently large) and if h is sufficiently small, then both Problem (P) and Problem (P^h) have unique solutions $(\mathbf{u}, \mathbf{J}, p, \phi)$ and $(\mathbf{u}^h, \mathbf{J}^h, p^h, \phi^h)$, respectively. Moreover, the discretization error (that is, the distance between $(\mathbf{u}, \mathbf{J}, p, \phi)$ and $(\mathbf{u}^h, \mathbf{J}^h, p^h, \phi^h)$ in the norm of the product space $\mathbf{X}_1 \times \mathbf{X}_2 \times M_1 \times M_2$) is of the same order as the sum of the error of best approximation of $(\mathbf{u}, \mathbf{J}, p, \phi)$ by elements of $\mathbf{X}_1 \times \mathbf{X}_2 \times M_1 \times M_2$ plus the error in the approximate boundary data, $\|\mathbf{g}_1 - \mathbf{g}_1^h\|_{\mathbf{H}^{1/2}(\Gamma_1)} + \|g_4 - g_4^h\|_{H^{1/2}(\Gamma_4)}$. In particular, $(\mathbf{u}^h, \mathbf{J}^h, p^h, \phi^h) \to (\mathbf{u}, \mathbf{J}, p, \phi)$ as $h \to 0$.

Theorem 2 and general results of finite-element theory suggest that Problem (P^h) will be a k-th order approximation of Problem (P) (for some positive integer k) if we use appropriate piecewise polynomial approximations of degree k for the velocity and electric potential and of degree $k-1$ for the pressure and current density. Assuming, for simplicity, that the domain Ω is a polyhedron and that we are given a regular decomposition of $\overline{\Omega}$ into simplicial or rectangular elements, we may approximate $H^1(\Omega)$ and $L^2(\Omega)$ by the spaces $\mathcal{P}_2^h$ and $\mathcal{P}_1^h$ of continuous piecewise quadratics (or triquadratics) and continuous piecewise linears (or trilinears) on tetrahedra (or rectangular parallelepipeds), respectively, and then set $\mathbf{X}_1^h := \mathcal{P}_2^h \times \mathcal{P}_2^h \times \mathcal{P}_2^h$ and $M_1^h := \mathcal{P}_1^h$. These so-called Taylor-Hood type velocity-pressure pairs are widely used in computational fluid dynamics and well understood (see, for example, [1, Chapter VI.6] or [3, Chapter 3]); in particular, they satisfy all the technical conditions needed to prove Theorem 2, the most important of which is the so-called LBB-condition.

In view of the above choices of velocity-pressure pairs, it is natural to set $M_2^h := \mathcal{P}_2^h$. In order to satisfy the LBB-condition, the space $\mathbf{X}_2^h$ should then contain the gradients of all continuous piecewise quadratics (on tetrahedra) or triquadratics (on rectangular parallelepipeds). Thus, in the case of a *simplicial* triangulation, we choose for $\mathbf{X}_2^h$ the subspace of $\mathbf{L}^2(\Omega)$ comprised of all vector functions on Ω whose components are (generally discontinuous) piecewise linears. When using *rectangular* elements, we let $\mathbf{X}_2^h := \mathbf{X}_{2,1}^h \times \mathbf{X}_{2,2}^h \times \mathbf{X}_{2,3}^h$ and choose for $\mathbf{X}_{2,i}^h$ the tensor product of the space of (generally discontinuous) piecewise linears in the i-th variable and the space of continuous piecewise biquadratics in the remaining two variables. Note that in any case, $\mathbf{X}_2^h$ contains $\mathcal{P}_1^h \times \mathcal{P}_1^h \times \mathcal{P}_1^h$. Pairs of spaces like $\mathbf{X}_2^h$ and M_2^h are commonly used in connection with so-called primal mixed methods (see, for example, [11, Section 12]).

With the above choices of finite-element spaces, the error of best approximation of the exact solution of Problem (P) will be of order h^2 provided that the exact solution is sufficiently regular (that is, if $\mathbf{u} \in \mathbf{H}^2(\Omega)$, $\mathbf{J} \in \mathbf{H}^1(\Omega)$, $p \in H^1(\Omega)$, $\phi \in H^2(\Omega)$). Approximate essential boundary data can be chosen in such a way that the error in those is of the same order. For example, if $\mathbf{g}_1$ and g_4 are sufficiently smooth, one can take for $\mathbf{g}_1^h$ and g_4^h the Lagrange interpolants of $\mathbf{g}_1$ and g_4 in the respective trace spaces of $\mathbf{X}_1^h$ and M_2^h. In general, independent of the smoothness of $\mathbf{g}_1$ and g_4, one can utilize generalized interpolants of Scott-Zhang type (see [13, Section 5]. In any case, Theorem 2 then guarantees that the solution of Problem (P^h) will approximate the exact solution of Problem (P) with an error of order h^2.

Several methods suggest themselves naturally for solving the discrete problem (P^h). Most straightforward is a simple linearization-iteration scheme where one lags the first argument $(\mathbf{u}^h, \mathbf{J}^h)$ of the trilinear form a_1. In the situation of Theorem 2, this scheme converges globally, that is, for every initial guess $(\mathbf{u}_0^h, \mathbf{J}_0^h)$. Despite the presence of the nonlocal operator $\mathcal{L}$, the resulting linear systems are sparse and can be solved either directly or iteratively. Intermediate computations of the induced magnetic field $\mu \nabla \times \mathcal{L}(\mathbf{J})$ are expensive, but can be handled efficiently, for example, with fast multi-pole methods.

Further speed-up may be achieved through the use of multi-level methods. In [5], for example, we describe a simple two-level algorithm, which yields optimal-order approximations by first solving the nonlinear problem (P^h) on a rather coarse grid (with $h \sim H$, say) and then solving a linearization of (P^h) on a much finer grid (with $h \sim H^2$). Finally, parts of the method are inherently parallelizable — a feature that will have to be exploited in order to deal with industrial-strength applications.

V. NUMERICAL EXPERIMENT

We implemented the method, as described, to simulate MHD flow around a circular cylinder in a channel with square cross section (see Figure 1). The flow domain was discretized by first mapping it to a rectangular channel with a rectangular cavity and then decomposing the latter into cubes of equal size (see Figures 2 and 3). In view of the remarks about suitable finite-element spaces in Section IV, we used standard triquadratic Lagrange elements for the velocity and electric potential, standard trilinear Lagrange elements for the pressure. For the i-th component of the current density, we chose Hermite elements with nine nodes, namely, those nodes of the principal lattice of degree two (on the reference cube) that are not on faces perpendicular to the i-th coordinate axis; two degrees of freedom were associated with each such node a, namely, $f \mapsto f(a)$ and $f \mapsto \partial_i f(a)$. This choice is convenient in constructing a basis for the somewhat nonstandard space $\mathbf{X}^h_{2,i}$. We used Lagrange interpolation to approximate the essential boundary data and employed the simple iteration scheme described in Section IV to solve Problem (P^h).

We prescribed a parabolic inflow velocity profile at the left end of the channel, zero velocity on the channel walls and on the cylinder surface, and zero stress on the outflow boundary (the right end of the channel). A permanent magnet, generating a dipole field $\mathbf{B}_{\text{ext}}$, was positioned along the cylinder axis (north pole facing the front), and a pair of electrodes was located on the down-stream part of the cylinder surface, one near the top, the other near the bottom. On the electrodes we specified the electric potential (negative on the upper, positive on the lower one); on all other boundaries we required zero current flux. No external body forces, external currents, or external electric fields were accounted for.

Since the experiment was anyway of an academic nature, we set all parameters equal to one. Moreover, all non-zero data (inflow velocity, applied magnetic field, and boundary values of the electric potential) were roughly of order one. We first solved the problem without magnetism and electricity; Figure 4 shows the resulting (purely hydrodynamic) velocity field. We then repeated the computation with magnetism and electricity switched on. The resulting velocity field, depicted in Figure 5, reveals a significant change in the flow pattern in the wake of the cylinder. In both cases, we also computed the total force acting on the cylinder, that is, the integral of the stress over the cylinder surface. In both cases, this force is parallel to the channel axis, but its direction is reversed when magnetism and electricity are switched on. The numerical values obtained were +230 versus -148. Most of the change in the total force is due to a reversal of the pressure gradient near the cylinder. Computing only the skin friction component, we found a drag reduction from 56 to 17.

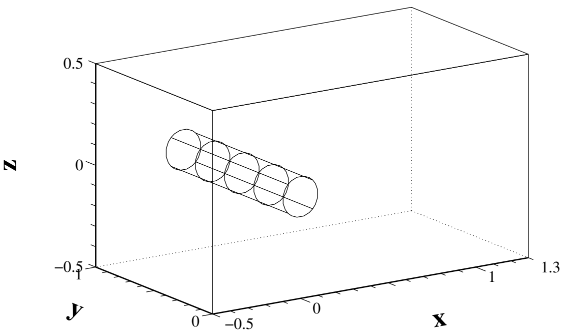

Fig. 1. The channel and cylinder.

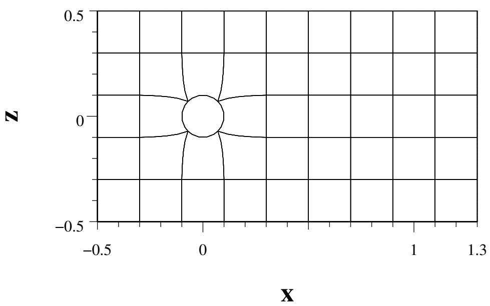

Fig. 2. Physical grid.

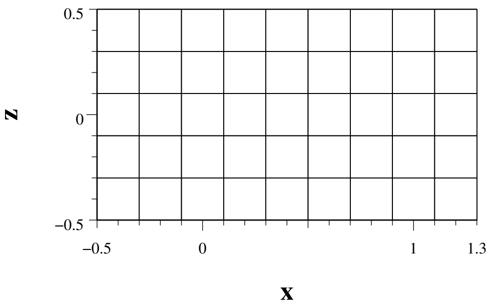

Fig. 3. Logical grid.

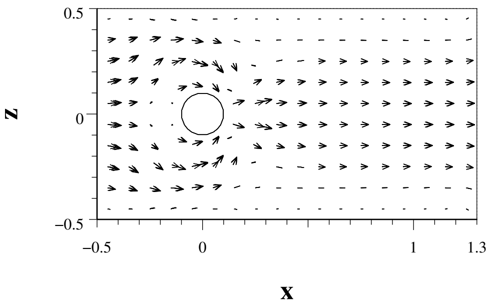

Fig. 4. Velocity field.

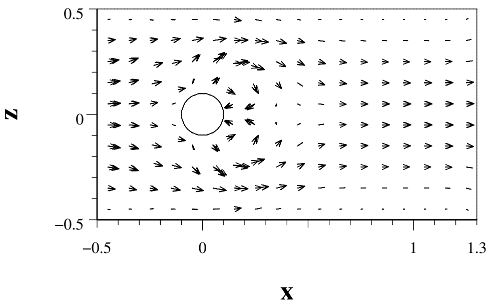

Fig. 5. Velocity field with MHD.

VI. CONCLUDING REMARKS

A novel formulation of the equations of viscous incompressible MHD was presented that allows for realistic boundary and interface conditions and accounts for the electromagnetic interaction of the fluid with the outside world while restricting computations to the region occupied by the fluid (and possibly, adjacent solid conductors). A mixed variational method was developed for the corresponding steady-state problem, which lends itself naturally to a finite-element discretization. The method was successfully implemented and tested by simulating a simple drag reduction experiment. The method can be used to solve a variety of MHD flow control and design problems, where the controls are applied magnetic fields, electric currents, and electric potentials. In its present implementation, the method is limited to the simulation of steady, laminar flows in the case of low Reynolds and magnetic Reynolds numbers, but the approach is potentially applicable to the simulation of unsteady and turbulent flows as well.

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