SIAM J. NUMER. ANAL.
Vol. 36, No. 4, pp. 1304–1332


# ANALYSIS AND NUMERICAL APPROXIMATION OF A STATIONARY MHD FLOW PROBLEM WITH NONIDEAL BOUNDARY*

A. J. MEIR† AND PAUL G. SCHMIDT†

**Abstract.** We are concerned with the steady flow of a conducting fluid, confined to a bounded region of space and driven by a combination of body forces, externally generated magnetic fields, and currents entering and leaving the fluid through electrodes attached to the surface. The flow is governed by the Navier–Stokes equations (in the fluid region) and Maxwell's equations (in all of space), coupled via Ohm's law and the Lorentz force. By means of the Biot–Savart law, we reduce the problem to a system of integro-differential equations in the fluid region, derive a mixed variational formulation, and prove its well-posedness under a small-data assumption. We then study the finite-element approximation of solutions (in the case of unique solvability) and establish optimal-order error estimates. Finally, an implementation of the method is described and illustrated with the results of some numerical experiments.

**Key words.** magnetohydrodynamics, Navier–Stokes equations, Maxwell's equations, mixed variational methods, finite elements

**AMS subject classifications.** Primary, 65N30, 76W05; Secondary, 65N12, 65N15, 35A15, 35Q30, 35Q60

**PII.** S003614299732615X

**Introduction.** Magnetohydrodynamics (MHD) is the theory of the macroscopic interaction of electrically conducting fluids and electromagnetic fields. Applications arise in astronomy and geophysics as well as in connection with numerous engineering problems, such as liquid-metal cooling of nuclear reactors, electromagnetic casting of metals, MHD power generation, and MHD ion propulsion. We refer to [8] or [20] for general information and to [12] for more specific references.

Assuming the fluid to be incompressible, viscous, and finitely conducting, MHD flow is governed by the Navier–Stokes and pre-Maxwell equations, coupled via Ohm's law and the Lorentz force (see, for example, [20, Chapter 2]). While the fluid may be confined to a bounded region of space, it typically interacts with the outside world (in particular, with current-carrying external conductors) through the universal electromagnetic field. This interaction entails formidable difficulties in the mathematical analysis and numerical solution of realistic MHD flow problems. In particular, while the Navier–Stokes equations are posed in the body of conducting fluid, Maxwell's equations need to be solved in all of space, and interior and exterior fields must be suitably matched at the interfaces separating media with different electromagnetic properties.

Only in special circumstances, most notably if the fluid is confined by perfectly conducting walls, can attention be restricted to the fluid region itself. A fair amount of mathematical work has been devoted to this case, that is, the case of MHD flow with "ideal" boundaries (see, for example, [4, 9, 10, 19, 24] for general results regarding

*Received by the editors August 15, 1997; accepted for publication (in revised form) May 6, 1998; published electronically June 29, 1999. This material is based upon work supported by the National Science Foundation under grants DMS-9404440 and DMS-9625096.
http://www.siam.org/journals/sinum/36-4/32615.html

†Department of Mathematics, Parker Hall, Auburn University, Auburn, AL 36849-5310 (ajm@math. auburn.edu, pgs@math.auburn.edu).

the existence, uniqueness, regularity, and asymptotic behavior of solutions and [7] for a finite-element analysis). Only a few authors have dealt with more general scenarios, involving "nonideal" boundaries (see [9, 10, 16, 17, 21]). Our earlier paper [12] contains a detailed discussion of the relevant mathematical literature and many additional references.

Of course, the physics and engineering literature abounds with experimental studies, asymptotic analyses, and computational simulations of a wide spectrum of MHD-dominated processes. Computer codes have been developed and applied to the solution of industrial-strength MHD flow problems, but the rigorous numerical analysis of such problems, which usually involve several fluid and solid conductors, complicated geometries, and, frequently, free surfaces, is still largely terra incognita (see [1, 13, 14] for a case study in a typical situation).

Our own approach to MHD flow with nonideal boundaries is based on the observation that the unknown magnetic field can frequently be eliminated from the equations by means of the Biot–Savart law, thereby reducing the problem to a system of integro-differential equations (for the velocity, pressure, current density, and electric potential) *in the fluid region*. As discussed in [11] and [12, Section 5], this "velocity-current formulation" avoids some of the difficulties inherent in the traditional formulation of the MHD equations.

Here, we exploit the velocity-current formulation for the analysis and numerical approximation of a simple, yet typical model problem: the steady flow of a conducting fluid, confined to a bounded region of space and driven by a combination of body forces, externally generated magnetic fields, and currents entering and leaving the fluid through electrodes attached to the surface. A precise statement of the problem is given in section 1. In section 2 we derive a mixed variational formulation of the problem and prove its well-posedness for small data; this extends earlier results in [11]. Section 3, the central part of the paper, is devoted to the finite-element approximation of solutions in the case of unique solvability. Optimal-order error estimates are established under quite general assumptions on the discretization. Finally, in section 4, we describe an implementation of the method and report on some numerical experiments.

**1. The problem.** We are concerned with the stationary flow of a viscous, incompressible, electrically conducting fluid, confined to a region $\Omega$ (a bounded Lipschitz domain in $\mathbb{R}^3$), in the presence of various body forces, electric and magnetic fields, and electric currents. Assuming all external field sources (if any) to be known, the flow can be completely described in terms of the following unknown quantities: the fluid velocity $\mathbf{u}$ and pressure $p$, the current density $\mathbf{J}$ in the fluid, the electric potential $\phi$, and the magnetic field $\mathbf{B}$. The governing equations are the Navier–Stokes equations and Ohm's law,

$$-\eta\Delta\mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mathbf{J}\times\mathbf{B} = \mathbf{F} \tag{1.1}$$

and

$$\sigma^{-1}\mathbf{J} + \nabla\phi - \mathbf{u}\times\mathbf{B} = \mathbf{E}, \tag{1.2}$$

along with the continuity equations

$$\nabla\cdot\mathbf{u} = 0 \quad \text{and} \quad \nabla\cdot\mathbf{J} = 0, \tag{1.3}$$

reflecting the conservation of mass and charge. The viscosity $\eta$, density $\rho$, and conductivity $\sigma$ of the fluid are positive parameters; $\mathbf{F}$ is a given body force, and $\mathbf{E}$ represents a given, externally generated electric field. (Physically, $\mathbf{E}$ should be assumed to be irrotational and could then be absorbed into the potential gradient, but we allow an arbitrary field $\mathbf{E}$, for reasons of symmetry in the equations.)

The magnetic field $\mathbf{B}$ can be written as

$$\mathbf{B} = \mathbf{B}_0 + \mathcal{B}(\mathbf{J}), \tag{1.4}$$

where $\mathbf{B}_0$ comprises field components generated by known external sources (permanent magnets or electric currents flowing in circuits outside the fluid), while $\mathcal{B}(\mathbf{J})$ is induced by the unknown current $\mathbf{J}$ in the fluid. Under mild assumptions on $\mathbf{J}$, the Biot–Savart law implies that

$$\mathcal{B}(\mathbf{J})(x) = -\frac{\mu}{4\pi} \int_\Omega \frac{x - y}{|x - y|^3} \times \mathbf{J}(y)\, dy, \tag{1.5}$$

for $x \in \mathbb{R}^3$, where $\mu$ is the magnetic permeability. (For simplicity we assume the fluid, as well as any materials outside, to be nonmagnetic, so that $\mu$ is constant throughout space.)

Equations (1.1)–(1.3) need to be supplemented by suitable boundary conditions for $\mathbf{u}$ and $\mathbf{J}$ on the boundary $\Gamma$ of the region $\Omega$ occupied by the fluid; in the simplest case, $\mathbf{u} = 0$ and $\mathbf{J} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ denotes the outward-pointing unit normal vector field on $\Gamma$. Here we allow the fluid to be mechanically driven through boundary forcing; this leads to a nonhomogeneous Dirichlet boundary condition,

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma,$$

where $\mathbf{g}$ must satisfy $\int_\Gamma \mathbf{g} \cdot \mathbf{n} = 0$ (since $\nabla \cdot \mathbf{u} = 0$ in $\Omega$). We also allow electric current to enter and leave $\Omega$ through the boundary. Obviously the current loop must then be closed in the exterior of $\Omega$; that is, we must have an external current distribution $\mathbf{J}_{\text{ext}}$ in $\mathbb{R}^3 \setminus \overline{\Omega}$ such that

$$\mathbf{J} \cdot \mathbf{n} = \mathbf{J}_{\text{ext}} \cdot \mathbf{n} \quad \text{on } \Gamma.$$

Of course, $\mathbf{J}_{\text{ext}}$ should satisfy $\nabla \cdot \mathbf{J}_{\text{ext}} = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$ and $\int_\Gamma \mathbf{J}_{\text{ext}} \cdot \mathbf{n} = 0$ (since $\nabla \cdot \mathbf{J} = 0$ in $\Omega$). Given $\mathbf{J}_{\text{ext}}$, the magnetic field $\mathbf{B}_0$ (generated by sources outside the fluid) can be written as

$$\mathbf{B}_0(x) = \mathbf{B}_{\text{ext}}(x) - \frac{\mu}{4\pi} \int_{\mathbb{R}^3 \setminus \overline{\Omega}} \frac{x - y}{|x - y|^3} \times \mathbf{J}_{\text{ext}}(y)\, dy, \tag{1.6}$$

for $x \in \mathbb{R}^3$, where $\mathbf{B}_{\text{ext}}$ comprises field components generated by external sources other than $\mathbf{J}_{\text{ext}}$ (if any). The field $\mathbf{B}_{\text{ext}}$ is assumed to be given with $\nabla \cdot \mathbf{B}_{\text{ext}} = 0$ in $\mathbb{R}^3$ and $\nabla \times \mathbf{B}_{\text{ext}} = 0$ in $\Omega$.

The current $\mathbf{J}_{\text{ext}}$ should be thought of as flowing in an external conductor, connected to $\Omega$ through two or more electrodes. It could be generated by a voltage source, somewhere in the external circuit, for the purpose of driving the fluid in $\Omega$ (this is the principle of an MHD propulsion device). To *prescribe* $\mathbf{J}_{\text{ext}}$ amounts to the assumption that the voltage source is *adjustable* (or that there is an adjustable resistor in the external circuit), so that a specified external current can be maintained no matter what the fluid's response. For practical purposes, it would be more feasible to prescribe only the potential difference generated at the voltage source (and the resistance of the external circuit) and to treat $\mathbf{J}_{\text{ext}}$ as an additional unknown. This obviously more complicated situation will be the subject of future investigation. For now, we will concern ourselves with the following problem.

*Problem $P_0$.* Given parameters $\eta, \rho, \sigma, \mu > 0$ and data

$$\begin{aligned}
&\mathbf{F} \in \mathbf{H}^{-1}(\Omega),\ \mathbf{E} \in \mathbf{L}^2(\Omega),\\
&\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma) \text{ with } \textstyle\int_\Gamma \mathbf{g}\cdot\mathbf{n} = 0,\\
&\mathbf{J}_{\text{ext}} \in \mathbf{L}^2(\mathbb{R}^3 \setminus \overline{\Omega}) \text{ with } \nabla\cdot\mathbf{J}_{\text{ext}} = 0 \text{ in } \mathbb{R}^3\setminus\overline{\Omega} \text{ and } \textstyle\int_\Gamma \mathbf{J}_{\text{ext}}\cdot\mathbf{n} = 0,\\
&\mathbf{B}_{\text{ext}} \in \mathbf{W}^1(\mathbb{R}^3) \text{ with } \nabla\cdot\mathbf{B}_{\text{ext}} = 0 \text{ in } \mathbb{R}^3 \text{ and } \nabla\times\mathbf{B}_{\text{ext}} = 0 \text{ in } \Omega,
\end{aligned}$$

find functions

$$\begin{aligned}
&\mathbf{u} \in \mathbf{H}^1(\Omega) \text{ with } \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega \text{ and } \mathbf{u} = \mathbf{g} \text{ on } \Gamma,\\
&\mathbf{J} \in \mathbf{L}^2(\Omega) \text{ with } \nabla\cdot\mathbf{J} = 0 \text{ in } \Omega \text{ and } \mathbf{J}\cdot\mathbf{n} = \mathbf{J}_{\text{ext}}\cdot\mathbf{n} \text{ on } \Gamma,\\
&p \in L^2(\Omega)/\mathbb{R},\ \phi \in H^1(\Omega)/\mathbb{R},
\end{aligned}$$

such that (1.1) and (1.2) are satisfied, with $\mathbf{B}$ given by (1.4), (1.5), and (1.6).

Here and in what follows, $L^2$ and $H^1$ denote the usual Lebesgue and Sobolev spaces of square-integrable functions on the respective domains (that is, on $\Omega$, $\mathbb{R}^3 \setminus \overline{\Omega}$, or $\mathbb{R}^3$); $W^1(\mathbb{R}^3)$ is the completion of $H^1(\mathbb{R}^3)$ with respect to the norm $f \mapsto \|\nabla f\|_{\mathbf{L}^2(\mathbb{R}^3)}$. We think of $H^{-1}(\Omega)$ as the norm dual of $H^1_0(\Omega)$, which is the subspace of $H^1(\Omega)$ comprised of the functions that vanish on $\Gamma$ (in the sense of traces). Finally, $H^{1/2}(\Gamma)$ denotes the trace space of $H^1(\Omega)$, endowed with the usual infimum norm, and $H^{-1/2}(\Gamma)$ is the norm dual of $H^{1/2}(\Gamma)$. Throughout, boldface type indicates a space of $\mathbb{R}^3$-valued functions, so that, for example, $\mathbf{L}^2(\Omega) = (L^2(\Omega))^3$.

The spaces in Problem $P_0$ have been chosen so that all the equations and boundary conditions are meaningful and the singular integrals in the decomposition of $\mathbf{B}$ are well defined. Note that if we define $\tilde{\mathbf{J}} \in \mathbf{L}^2(\mathbb{R}^3)$ to coincide with $\mathbf{J}$ in $\Omega$ and with $\mathbf{J}_{\text{ext}}$ in $\mathbb{R}^3 \setminus \overline{\Omega}$, then $\nabla\cdot\tilde{\mathbf{J}} = 0$ (in the sense of distributions on $\mathbb{R}^3$), and we have $\mathbf{B} = \mathbf{B}_{\text{ext}} + \tilde{\mathbf{B}}$ with $\tilde{\mathbf{B}}$ given by

$$\tilde{\mathbf{B}}(x) = -\frac{\mu}{4\pi}\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \tilde{\mathbf{J}}(y)\,dy,$$

for $x \in \mathbb{R}^3$. The latter is the unique solution, in $\mathbf{W}^1(\mathbb{R}^3)$, of Maxwell's equations,

$$\nabla\cdot\tilde{\mathbf{B}} = 0 \quad \text{and} \quad \nabla\times\tilde{\mathbf{B}} = \mu\tilde{\mathbf{J}} \tag{1.7}$$

(see [12, Section 2]). For simplicity, we require that $\mathbf{B}_{\text{ext}}$ belongs to $\mathbf{W}^1(\mathbb{R}^3)$ as well (although the only technical condition on $\mathbf{B}_{\text{ext}}$ needed later is that its restriction to $\Omega$ belongs to $\mathbf{L}^3(\Omega)$).

**2. Weak formulation and well-posedness.** Deriving a weak formulation of Problem $P_0$ and proving its well-posedness (for small data) is fairly straightforward, following the reasoning in [11] with appropriate modifications. Unfortunately, we have to introduce a considerable amount of notation in order to state our results and to set the stage for the subsequent numerical analysis of the problem.

To begin, let us define

$$\begin{aligned}
&\mathbf{Y}_1 := \mathbf{H}^1(\Omega), && \mathbf{Y}_2 := \mathbf{L}^2(\Omega), && \mathbf{Y} := \mathbf{Y}_1 \times \mathbf{Y}_2,\\
&\mathbf{X}_1 := \mathbf{H}^1_0(\Omega), && \mathbf{X}_2 := \mathbf{L}^2(\Omega), && \mathbf{X} := \mathbf{X}_1 \times \mathbf{X}_2
\end{aligned}$$

and

$$M_1 := L^2(\Omega)/\mathbb{R}, \qquad M_2 := H^1(\Omega)/\mathbb{R}, \qquad M := M_1 \times M_2.$$

All these spaces are understood to be endowed with their natural Hilbert-space structures, inherited from $L^2(\Omega)$ and $H^1(\Omega)$.

For all of the following, we fix a set of parameters $\eta$, $\rho$, $\sigma$, $\mu$ and a set of data $\mathbf{F}$, $\mathbf{E}$, $\mathbf{g}$, $\mathbf{J}_{\mathrm{ext}}$, $\mathbf{B}_{\mathrm{ext}}$ as in Problem $P_0$. For notational convenience, we define

$$j := \mathbf{J}_{\mathrm{ext}} \cdot \mathbf{n}.$$

Furthermore, we let $\mathcal{B} : \mathbf{L}^2(\Omega) \to \mathbf{W}^1(\mathbb{R}^3)$ denote the bounded linear operator (see [12, Section 2]) given by

$$\mathcal{B}(\mathbf{f})(x) := -\frac{\mu}{4\pi} \int_\Omega \frac{x-y}{|x-y|^3} \times \mathbf{f}(y)\, dy,$$

for $x \in \mathbb{R}^3$ and $\mathbf{f} \in \mathbf{L}^2(\Omega)$, and define $\mathbf{B}_0$ as in (1.6). Finally, let $\mathcal{P}$ denote the orthogonal projection in $L^2(\Omega)$ given by

$$\mathcal{P}(f) := f - \frac{1}{|\Omega|} \int_\Omega f.$$

Multiplying the equations in (1.1)–(1.3) by test functions $\mathbf{v} \in \mathbf{X}_1$, $\mathbf{K} \in \mathbf{X}_2$, $q \in M_1$, and $\psi \in M_2$, respectively, then integrating over $\Omega$ and regrouping terms, we obtain two variational equations of the form

$$\begin{aligned} a_0\big((\mathbf{u},\mathbf{J}),(\mathbf{v},\mathbf{K})\big) + a_1\big((\mathbf{u},\mathbf{J}),(\mathbf{u},\mathbf{J}),(\mathbf{v},\mathbf{K})\big) \\ + b\big((\mathbf{v},\mathbf{K}),(p,\phi)\big) = \int_\Omega \mathbf{F}\cdot\mathbf{v} + \int_\Omega \mathbf{E}\cdot\mathbf{K} \end{aligned} \tag{2.1}$$

and

$$b\big((\mathbf{u},\mathbf{J}),(q,\psi)\big) = \int_\Gamma j\,\psi, \tag{2.2}$$

where $a_0 : \mathbf{Y}\times\mathbf{Y} \to \mathbb{R}$ (a bilinear form), $a_1 : \mathbf{Y}\times\mathbf{Y}\times\mathbf{Y} \to \mathbb{R}$ (a trilinear form), and $b : \mathbf{Y} \times M \to \mathbb{R}$ (a bilinear form) are given by

$$\begin{aligned} a_0\big((\mathbf{v}_1,\mathbf{K}_1),(\mathbf{v}_2,\mathbf{K}_2)\big) &:= \eta \int_\Omega (\nabla\mathbf{v}_1) : (\nabla\mathbf{v}_2) + \sigma^{-1} \int_\Omega \mathbf{K}_1\cdot\mathbf{K}_2 \\ &\quad + \int_\Omega \Big(\big(\mathbf{K}_2\times\mathbf{B}_0\big)\cdot\mathbf{v}_1 - \big(\mathbf{K}_1\times\mathbf{B}_0\big)\cdot\mathbf{v}_2\Big), \\ a_1\big((\mathbf{v}_1,\mathbf{K}_1),(\mathbf{v}_2,\mathbf{K}_2),(\mathbf{v}_3,\mathbf{K}_3)\big) &:= \frac{\rho}{2} \int_\Omega \Big(\big((\mathbf{v}_1\cdot\nabla)\mathbf{v}_2\big)\cdot\mathbf{v}_3 - \big((\mathbf{v}_1\cdot\nabla)\mathbf{v}_3\big)\cdot\mathbf{v}_2\Big) \\ &\quad + \int_\Omega \Big(\big(\mathbf{K}_3\times\mathcal{B}(\mathbf{K}_1)\big)\cdot\mathbf{v}_2 - \big(\mathbf{K}_2\times\mathcal{B}(\mathbf{K}_1)\big)\cdot\mathbf{v}_3\Big), \end{aligned}$$

and

$$b\big((\mathbf{v},\mathbf{K}),(q,\psi)\big) := -\int_\Omega (\nabla\cdot\mathbf{v})\mathcal{P}(q) + \int_\Omega \mathbf{K}\cdot(\nabla\psi).$$

*Remark* 2.1. (a) The first integral on the right-hand side of (2.1) and the integral on the right-hand side of (2.2) are understood in the sense of duality pairings, between $\mathbf{H}^{-1}(\Omega)$ and $\mathbf{H}_0^1(\Omega)$ and between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, respectively.

(b) For reasons that will become clear in section 3, the term in $a_1$ that stems from the inertial force in (1.1) has been "skew-symmetrized." Note that

$$\frac{1}{2}\int_\Omega \Big( \big((\mathbf{v}_1\cdot\nabla)\mathbf{v}_2\big)\cdot\mathbf{v}_3 - \big((\mathbf{v}_1\cdot\nabla)\mathbf{v}_3\big)\cdot\mathbf{v}_2 \Big) = \int_\Omega \big((\mathbf{v}_1\cdot\nabla)\mathbf{v}_2\big)\cdot\mathbf{v}_3$$

whenever $\nabla\cdot\mathbf{v}_1 = 0$ and $\mathbf{v}_3|_\Gamma = 0$. The advantage of defining $a_1$ in this way is that for every $(\mathbf{v}_0, \mathbf{K}_0) \in \mathbf{Y}$, the bilinear form $a_1\big((\mathbf{v}_0,\mathbf{K}_0),(\cdot,\cdot),(\cdot,\cdot)\big)$ is skew-symmetric on $\mathbf{Y}\times\mathbf{Y}$.

(c) The projection $\mathcal{P}$ has been inserted in the definition of the form $b$ so that $b$ is well defined on $\mathbf{Y}\times M$, independent of the choice of representatives for the equivalence classes in $M_1 = L^2(\Omega)/\mathbb{R}$ and $M_2 = H^1(\Omega)/\mathbb{R}$. In (2.1) and (2.2), this projection has no effect at all since

$$\int_\Omega (\nabla\cdot\mathbf{v})\mathcal{P}(q) = \int_\Omega (\nabla\cdot\mathbf{v})q - \frac{1}{|\Omega|}\Big(\int_\Gamma \mathbf{v}\cdot\mathbf{n}\Big)\Big(\int_\Omega q\Big)$$

and $\int_\Gamma \mathbf{v}\cdot\mathbf{n} = 0$ if $\nabla\cdot\mathbf{v} = 0$ or $\mathbf{v}|_\Gamma = 0$. In dealing with the discretized equations of section 3, the presence of $\mathcal{P}$ will allow us to work with approximate boundary values for the fluid velocity that do not have to satisfy a compatibility condition (see section 4 for details).

Routine arguments show that the original problem $P_0$ is equivalent to the following variational version.

*Problem* $P_1$. Find $(\mathbf{u},\mathbf{J}) \in \mathbf{Y}$ with $\mathbf{u}|_\Gamma = \mathbf{g}$ and $(p,\phi) \in M$ such that (2.1) and (2.2) are satisfied for all $(\mathbf{v},\mathbf{K}) \in \mathbf{X}$ and $(q,\psi) \in M$, respectively.

It should be noted that while we have to enforce the Dirichlet boundary condition on $\mathbf{u}$, the boundary condition on $\mathbf{J}$ is a natural one. In fact, $\mathbf{J} \in \mathbf{L}^2(\Omega)$ satisfies

$$\int_\Omega \mathbf{J}\cdot(\nabla\psi) = \int_\Gamma j\,\psi \quad \forall\,\psi \in M_2$$

if and only if

$$\nabla\cdot\mathbf{J} = 0 \text{ in } \Omega \quad\text{and}\quad \mathbf{J}\cdot\mathbf{n} = j \text{ on } \Gamma.$$

In the following lemma, we gather the properties of the forms $a_0$, $a_1$, and $b$ that will be needed in the subsequent analysis of Problem $P_1$. Here and in what follows, $c$ denotes a fixed constant depending only on the domain $\Omega$.

LEMMA 2.2. (a) *The forms $a_0$, $a_1$, and $b$ are bounded on $\mathbf{Y}\times\mathbf{Y}$, $\mathbf{Y}\times\mathbf{Y}\times\mathbf{Y}$, and $\mathbf{Y}\times M$, respectively, with norms*

$$\|a_0\| \le c\max\{1,\eta,\sigma^{-1},\mu\}(1 + \|\mathbf{J}_{\text{ext}}\|_{\mathbf{L}^2(\mathbb{R}^3\setminus\overline{\Omega})} + \|\mathbf{B}_{\text{ext}}|_\Omega\|_{\mathbf{L}^3(\Omega)}),$$
$$\|a_1\| \le c\max\{\rho,\mu\}, \quad \textit{and} \quad \|b\| \le \sqrt{3}.$$

(b) *The form $a_0$ is positive definite on $\mathbf{X}\times\mathbf{X}$; more precisely, there exists a number $\alpha \ge c^{-1}\min\{\eta,\sigma^{-1}\}$ such that*

$$a_0((\mathbf{v},\mathbf{K}),(\mathbf{v},\mathbf{K})) \ge \alpha\|(\mathbf{v},\mathbf{K})\|_{\mathbf{Y}}^2 \qquad \forall\,(\mathbf{v},\mathbf{K}) \in \mathbf{X}. \tag{2.3}$$

(c) *The form b satisfies the Ladyzhenskaya–Babuska–Brezzi condition* (*LBB-condition*) *on* $\mathbf{X} \times M$; *that is*, *there exists a number* $\beta > 0$ (*depending only on* $\Omega$) *such that*

$$\inf_{(q,\psi)\in M} \sup_{(\mathbf{v},\mathbf{K})\in\mathbf{X}} \frac{b\big((\mathbf{v},\mathbf{K}),(q,\psi)\big)}{\|(\mathbf{v},\mathbf{K})\|_{\mathbf{Y}}\|(q,\psi)\|_M} \geq \beta.$$

*Proof.* Part (a) follows from elementary estimates, using the boundedness of the operator $\mathcal{B} : \mathbf{L}^2(\Omega) \to \mathbf{W}^1(\mathbb{R}^3)$ and the continuity of the embeddings of $\mathbf{W}^1(\mathbb{R}^3)$ into $\mathbf{H}^1_{\text{loc}}(\mathbb{R}^3)$ and of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^6(\Omega)$. Part (b) is an immediate consequence of Poincaré's inequality. The LBB-condition in (c) is equivalent to the invertibility of the gradient operator as a mapping from $M_1 = L^2(\Omega)/\mathbb{R}$ into $\mathbf{X}_1^* = \mathbf{H}^{-1}(\Omega)$ (see [5, Corollary I.2.4]) and from $M_2 = H^1(\Omega)/\mathbb{R}$ into $\mathbf{X}_2^* = \mathbf{L}^2(\Omega)$ (see [5, Theorem I.1.9]). ◻

To further reduce Problem $P_1$, we write

$$\mathbf{u} = \mathbf{u}_0 + \hat{\mathbf{u}} \quad \text{and} \quad \mathbf{J} = \mathbf{J}_0 + \hat{\mathbf{J}},$$

where $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$ and $\mathbf{J}_0 \in \mathbf{L}^2(\Omega)$ are chosen so that

$$(2.4) \quad \nabla\cdot\mathbf{u}_0 = 0 \text{ in } \Omega, \quad \mathbf{u}_0 = \mathbf{g} \text{ on } \Gamma \quad \text{and} \quad \nabla\cdot\mathbf{J}_0 = 0 \text{ in } \Omega, \quad \mathbf{J}_0\cdot\mathbf{n} = j \text{ on } \Gamma.$$

The existence of $\mathbf{u}_0$ and $\mathbf{J}_0$ follows from the LBB-condition, Lemma 2.2(c). Since this fact (and its proof) will be very important in section 3, we briefly recall the argument. Note that thanks to the LBB-condition, the operator $B : \mathbf{X} \to M^*$, defined by

$$B(\mathbf{v},\mathbf{K}) := b\big((\mathbf{v},\mathbf{K}),(\cdot,\cdot)\big),$$

for $(\mathbf{v},\mathbf{K}) \in \mathbf{X}$, is onto and restricts to an isomorphism between $\mathbf{V}^\perp$ and $M^*$, where $\mathbf{V}^\perp$ denotes the orthogonal complement (in $\mathbf{X}$) of the space

$$(2.5) \qquad \mathbf{V} := \{(\mathbf{v},\mathbf{K}) \in \mathbf{X};\ b((\mathbf{v},\mathbf{K}),(q,\psi)) = 0 \ \ \forall\,(q,\psi) \in M\}.$$

Moreover,

$$\|B(\mathbf{v},\mathbf{K})\|_{M^*} \geq \beta\,\|(\mathbf{v},\mathbf{K})\|_{\mathbf{Y}} \quad \forall\,(\mathbf{v},\mathbf{K}) \in \mathbf{V}^\perp,$$

where $\beta$ is the constant in the LBB-condition (see [5, Lemma I.4.1]). This means that the Moore–Penrose pseudoinverse of $B$, that is, the operator $B^+ : M^* \to \mathbf{X}$ that assigns to every $\varphi \in M^*$ the unique element $(\mathbf{v},\mathbf{K}) \in \mathbf{V}^\perp$ with $B(\mathbf{v},\mathbf{K}) = \varphi$, satisfies $\|B^+\| \leq \beta^{-1}$.

Now let $\Lambda : \mathbf{H}^{1/2}(\Gamma) \to \mathbf{H}^1(\Omega)$ denote a bounded linear lifting operator, say, the Moore–Penrose pseudoinverse of the trace operator $\mathbf{v} \mapsto \mathbf{v}|_\Gamma$ (in which case $\|\Lambda\| = 1$). Given boundary data $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$ and $j = \mathbf{J}_{\text{ext}}\cdot\mathbf{n} \in H^{-1/2}(\Gamma)$, define $\varphi(\mathbf{g},j) \in M^*$ by

$$\varphi(\mathbf{g},j)(q,\psi) := \int_\Gamma j\,\psi - b\big((\Lambda\mathbf{g},0),(q,\psi)\big),$$

for $(q,\psi) \in M$, and let

$$(\mathbf{u}_0,\mathbf{J}_0) := (\Lambda\mathbf{g},0) + B^+\big(\varphi(\mathbf{g},j)\big).$$

By construction, $(\mathbf{u}_0, \mathbf{J}_0) \in \mathbf{Y}$, $\mathbf{u}_0|_\Gamma = \mathbf{g}$, and $b\big((\mathbf{u}_0, \mathbf{J}_0), (q, \psi)\big) = \int_\Gamma j\, \psi$ for all $(q, \psi) \in M$. That is, $(\mathbf{u}_0, \mathbf{J}_0)$ satisfies (2.4), as required. Moreover,

$$\|(\mathbf{u}_0, \mathbf{J}_0)\|_{\mathbf{Y}} \leq \lambda \, \|(\mathbf{g}, j)\|_{\mathbf{H}^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)}, \tag{2.6}$$

with a constant $\lambda$ depending only on $\Omega$ (for example, $\lambda = 1 + \sqrt{3}\, \beta^{-1}$ if $\|\Lambda\| = 1$).

Substituting $\mathbf{u} = \mathbf{u}_0 + \hat{\mathbf{u}}$ and $\mathbf{J} = \mathbf{J}_0 + \hat{\mathbf{J}}$ in (2.1) and (2.2), we obtain an equivalent pair of equations of the form

$$a\big((\hat{\mathbf{u}}, \hat{\mathbf{J}}), (\hat{\mathbf{u}}, \hat{\mathbf{J}}), (\mathbf{v}, \mathbf{K})\big) + b\big((\mathbf{v}, \mathbf{K}), (p, \phi)\big) = \ell(\mathbf{v}, \mathbf{K})$$

and

$$b\big((\hat{\mathbf{u}}, \hat{\mathbf{J}}), (q, \psi)\big) = 0,$$

where $a : \mathbf{X} \times \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ and $\ell \in \mathbf{X}^*$ are defined by

$$\begin{aligned} &a\big((\mathbf{v}_1, \mathbf{K}_1), (\mathbf{v}_2, \mathbf{K}_2), (\mathbf{v}_3, \mathbf{K}_3)\big) \\ &\quad := a_0\big((\mathbf{v}_2, \mathbf{K}_2), (\mathbf{v}_3, \mathbf{K}_3)\big) + a_1\big((\mathbf{v}_1, \mathbf{K}_1), (\mathbf{v}_2, \mathbf{K}_2), (\mathbf{v}_3, \mathbf{K}_3)\big) \\ &\quad\quad + a_1\big((\mathbf{v}_2, \mathbf{K}_2), (\mathbf{u}_0, \mathbf{J}_0), (\mathbf{v}_3, \mathbf{K}_3)\big) + a_1\big((\mathbf{u}_0, \mathbf{J}_0), (\mathbf{v}_2, \mathbf{K}_2), (\mathbf{v}_3, \mathbf{K}_3)\big) \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \ell(\mathbf{v}, \mathbf{K}) := &\int_\Omega \mathbf{F} \cdot \mathbf{v} + \int_\Omega \mathbf{E} \cdot \mathbf{K} \\ &- a_0\big((\mathbf{u}_0, \mathbf{J}_0), (\mathbf{v}, \mathbf{K})\big) - a_1\big((\mathbf{u}_0, \mathbf{J}_0), (\mathbf{u}_0, \mathbf{J}_0), (\mathbf{v}, \mathbf{K})\big). \end{aligned} \tag{2.8}$$

After this reduction, the problem at hand fits into a nonlinear version of the classical Ladyzhenskaya–Babuska–Brezzi theory (see, for example, [5, Chapter IV.1]). Moreover, Problem $P_1$ is equivalent to the following variational problem in the space $\mathbf{V}$, defined in (2.5).

*Problem* $P_2$. Find $(\hat{\mathbf{u}}, \hat{\mathbf{J}}) \in \mathbf{V}$ such that $a\big((\hat{\mathbf{u}}, \hat{\mathbf{J}}), (\hat{\mathbf{u}}, \hat{\mathbf{J}}), (\mathbf{v}, \mathbf{K})\big) = \ell(\mathbf{v}, \mathbf{K})$ for all $(\mathbf{v}, \mathbf{K}) \in \mathbf{V}$.

In fact, if $(\mathbf{u}, \mathbf{J}, p, \phi)$ is a solution of $P_1$, then $(\mathbf{u} - \mathbf{u}_0, \mathbf{J} - \mathbf{J}_0)$ solves $P_2$, and if $(\hat{\mathbf{u}}, \hat{\mathbf{J}})$ is a solution of $P_2$, then there exists a unique pair $(p, \phi) \in M$ such that $(\mathbf{u}_0 + \hat{\mathbf{u}}, \mathbf{J}_0 + \hat{\mathbf{J}}, p, \phi)$ solves $P_1$ (see [5, Chapter IV.1, Theorem 1.4]).

To infer the well-posedness (for small data) of Problem $P_2$, we need only verify certain continuity and coercivity properties of the form $a$.

LEMMA 2.3. (a) *The mapping* $(\mathbf{v}, \mathbf{K}) \mapsto a\big((\mathbf{v}, \mathbf{K}), (\mathbf{v}, \mathbf{K}), (\mathbf{v}_0, \mathbf{K}_0)\big)$, *for any* $(\mathbf{v}_0, \mathbf{K}_0) \in \mathbf{V}$, *is weakly sequentially continuous on* $\mathbf{V}$.

(b) *For every* $(\mathbf{v}_0, \mathbf{K}_0) \in \mathbf{V}$ *and all* $(\mathbf{v}, \mathbf{K}) \in \mathbf{V}$, *we have*

$$a\big((\mathbf{v}_0, \mathbf{K}_0), (\mathbf{v}, \mathbf{K}), (\mathbf{v}, \mathbf{K})\big) \geq \Big(\alpha - \lambda \|a_1\| \|(\mathbf{g}, j)\|\Big) \|(\mathbf{v}, \mathbf{K})\|_{\mathbf{Y}}^2,$$

*where* $\alpha$ *and* $\lambda$ *are the constants in* (2.3) *and* (2.6), *respectively,* $\|a_1\|$ *is the norm of the trilinear form* $a_1$, *and* $\|(\mathbf{g}, j)\|$ *denotes the norm of* $(\mathbf{g}, j)$ *in* $\mathbf{H}^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.

(c) *The mapping* $(\mathbf{v}_0, \mathbf{K}_0) \mapsto a\big((\mathbf{v}_0, \mathbf{K}_0), (\cdot, \cdot), (\cdot, \cdot)\big)$ *is uniformly Lipschitz continuous, with Lipschitz constant* $\|a_1\|$, *from* $\mathbf{V}$ *into the space* $\mathcal{L}(\mathbf{V}, \mathbf{V}^*)$ *of bounded linear operators from* $\mathbf{V}$ *into* $\mathbf{V}^*$.

*Proof.* Part (a) is readily checked, recalling the boundedness of $\mathcal{B}$ as a mapping from $\mathbf{L}^2(\Omega)$ into $\mathbf{W}^1(\mathbb{R}^3)$, the continuous embedding of $\mathbf{W}^1(\mathbb{R}^3)$ into $\mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$, and the compact embedding of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$. Part (b) follows from (2.3) and (2.6), observing the skew-symmetry of the form $a_1$ with respect to its second and third arguments (see Remark 2.1(b)), while (c) is an immediate consequence of the definition of $a$ and the boundedness of $a_1$. ∎

The above and [5, Chapter IV, Theorems 1.2 and 1.3] yield the following existence and uniqueness result.

THEOREM 2.4. *Let* $N = N(\mathbf{F}, \mathbf{E}, \mathbf{g}, \mathbf{J}_{\rm ext}, \mathbf{B}_{\rm ext})$ *denote the norm of the functional* $\ell|_{\mathbf{V}}$. *Let* $\|(\mathbf{g}, j)\|$ *denote the norm of* $(\mathbf{g}, j)$ *in* $\mathbf{H}^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, *where* $j = \mathbf{J}_{\rm ext} \cdot \mathbf{n}$, *and choose constants* $\alpha$ *and* $\lambda$ *as in* (2.3) *and* (2.6).

(a) *If* $\|(\mathbf{g}, j)\| < \frac{\alpha}{\lambda\|a_1\|}$, *then there exists at least one solution* $(\hat{\mathbf{u}}, \hat{\mathbf{J}})$ *of Problem* $P_2$ *that satisfies*

$$\|(\hat{\mathbf{u}}, \hat{\mathbf{J}})\|_{\mathbf{Y}} \le \frac{N}{\alpha - \lambda\|a_1\|\|(\mathbf{g}, j)\|}.$$

(b) *If* $\|(\mathbf{g}, j)\| < \frac{\alpha}{\lambda\|a_1\|}$ *and* $N < \frac{1}{\|a_1\|}\big(\alpha - \lambda\|a_1\|\|(\mathbf{g}, j)\|\big)^2$, *then the solution of Problem* $P_2$ *is unique.*

*Remark* 2.5. (a) The quantity $N = N(\mathbf{F}, \mathbf{E}, \mathbf{g}, \mathbf{J}_{\rm ext}, \mathbf{B}_{\rm ext})$ measures the size of the data $\mathbf{F}$, $\mathbf{E}$, $\mathbf{g}$, $\mathbf{J}_{\rm ext}$, $\mathbf{B}_{\rm ext}$; in fact, we have

$$N \le \|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)} + \|\mathbf{E}\|_{\mathbf{L}^2(\Omega)} + c\max\{1, \eta, \rho, \sigma^{-1}, \mu\}\Big(\|\mathbf{g}\|_{\mathbf{H}^{1/2}(\Gamma)} + \|\mathbf{J}_{\rm ext}\|_{\mathbf{L}^2(\mathbb{R}^3\setminus\overline{\Omega})}\Big) \Big(1 + \|\mathbf{g}\|_{\mathbf{H}^{1/2}(\Gamma)} + \|\mathbf{J}_{\rm ext}\|_{\mathbf{L}^2(\mathbb{R}^3\setminus\overline{\Omega})} + \|\mathbf{B}_{\rm ext}|_\Omega\|_{\mathbf{L}^3(\Omega)}\Big).$$

Theorem 2.4 thus asserts the *existence* of a solution of Problem $P_2$ and, consequently, of Problems $P_1$ and $P_0$ if the *boundary data* $\mathbf{g}$ and $j = \mathbf{J}_{\rm ext} \cdot \mathbf{n}$ are sufficiently small; *uniqueness* is guaranteed if *all the data*, $\mathbf{F}$, $\mathbf{E}$, $\mathbf{g}$, $\mathbf{J}_{\rm ext}$, and $\mathbf{B}_{\rm ext}$, are sufficiently small. (Note that the constants $\alpha$, $\lambda$, and $\|a_1\|$ are independent of the data.)

While it seems natural that *uniqueness* holds only for small data, it is somewhat disturbing that the *existence* of a solution should require small boundary data. No such assumption is needed in the case of the Navier–Stokes equations, but there the proof of existence relies on the construction (due to E. Hopf) of a special lifting of the boundary values of the fluid velocity (see, for example, [5, Chapter IV.2, Lemma 2.3]). Hopf's device does not seem to work for the MHD equations, due to the presence of additional nonlinear terms. However, for the purposes of the present paper, this issue is of minor importance since the subsequent finite-element analysis is anyway restricted to the case of unique solvability.

(b) The smallness assumptions of Theorem 2.4 must be interpreted relative to the parameters of the problem. For example, given *any* data set $(\mathbf{F}, \mathbf{E}, \mathbf{g}, \mathbf{J}_{\rm ext}, \mathbf{B}_{\rm ext})$, the assumptions of Theorem 2.4(b) are satisfied and the problem has a unique solution provided that the viscosity $\eta$ and resistivity $\sigma^{-1}$ of the fluid are sufficiently large. (Note that according to Lemma 2.2(a), $\alpha \ge c^{-1}\min\{\eta, \sigma^{-1}\}$ and $\|a_1\| \le c\max\{\rho, \mu\}$, while $\lambda$ depends only on $\Omega$.)

(c) Under the conditions of Theorem 2.4(b), the unique solution $(\hat{\mathbf{u}}, \hat{\mathbf{J}})$ of Problem $P_2$ satisfies

$$\|(\hat{\mathbf{u}}, \hat{\mathbf{J}})\|_{\mathbf{Y}} \le \frac{N}{\alpha - \lambda\|a_1\|\|(\mathbf{g}, j)\|} < \frac{\alpha - \lambda\|a_1\|\|(\mathbf{g}, j)\|}{\|a_1\|} = \frac{\alpha}{\|a_1\|} - \lambda\|(\mathbf{g}, j)\|.$$

For the corresponding unique solution $(\mathbf{u}, \mathbf{J}, p, \phi)$ of Problem $P_1$ this implies

$$\|(\mathbf{u}, \mathbf{J})\|_{\mathbf{Y}} \le \|(\mathbf{u}_0, \mathbf{J}_0)\|_{\mathbf{Y}} + \|(\hat{\mathbf{u}}, \hat{\mathbf{J}})\|_{\mathbf{Y}} < \frac{\alpha}{\|a_1\|}.$$

Conversely, if Problem $P_1$ has a solution $(\mathbf{u}, \mathbf{J}, p, \phi)$ with $\|(\mathbf{u}, \mathbf{J})\|_{\mathbf{Y}} < \alpha/\|a_1\|$, then the solution is necessarily unique. To prove this, suppose that both $(\mathbf{u}, \mathbf{J}, p, \phi)$ and $(\tilde{\mathbf{u}}, \tilde{\mathbf{J}}, \tilde{p}, \tilde{\phi})$ are solutions of $P_1$ and that $\|(\mathbf{u}, \mathbf{J})\|_{\mathbf{Y}} < \alpha/\|a_1\|$. Then we have

$$a_0\big((\mathbf{u}-\tilde{\mathbf{u}}, \mathbf{J}-\tilde{\mathbf{J}}), (\mathbf{v}, \mathbf{K})\big) + a_1\big((\mathbf{u}-\tilde{\mathbf{u}}, \mathbf{J}-\tilde{\mathbf{J}}), (\mathbf{u}, \mathbf{J}), (\mathbf{v}, \mathbf{K})\big) \\ + a_1\big((\tilde{\mathbf{u}}, \tilde{\mathbf{J}}), (\mathbf{u}-\tilde{\mathbf{u}}, \mathbf{J}-\tilde{\mathbf{J}}), (\mathbf{v}, \mathbf{K})\big) + b\big((\mathbf{v}, \mathbf{K}), (p-\tilde{p}, \phi-\tilde{\phi})\big) = 0$$

for all $(\mathbf{v}, \mathbf{K}) \in \mathbf{X}$ and

$$b\big((\mathbf{u}-\tilde{\mathbf{u}}, \mathbf{J}-\tilde{\mathbf{J}}), (q, \psi)\big) = 0$$

for all $(q, \psi) \in M$. In particular, $(\mathbf{u}-\tilde{\mathbf{u}}, \mathbf{J}-\tilde{\mathbf{J}}) \in \mathbf{V}$ and therefore

$$a_0\big((\mathbf{u}-\tilde{\mathbf{u}}, \mathbf{J}-\tilde{\mathbf{J}}), (\mathbf{u}-\tilde{\mathbf{u}}, \mathbf{J}-\tilde{\mathbf{J}})\big) + a_1\big((\mathbf{u}-\tilde{\mathbf{u}}, \mathbf{J}-\tilde{\mathbf{J}}), (\mathbf{u}, \mathbf{J}), (\mathbf{u}-\tilde{\mathbf{u}}, \mathbf{J}-\tilde{\mathbf{J}})\big) = 0.$$

Thanks to (2.3), this implies

$$0 \geq \big(\alpha - \|a_1\|\|(\mathbf{u}, \mathbf{J})\|_{\mathbf{Y}}\big)\|(\mathbf{u}-\tilde{\mathbf{u}}, \mathbf{J}-\tilde{\mathbf{J}})\|_{\mathbf{Y}}^2,$$

and since $\alpha - \|a_1\|\|(\mathbf{u}, \mathbf{J})\|_{\mathbf{Y}} > 0$ (by assumption), we conclude that $(\mathbf{u}, \mathbf{J}) = (\tilde{\mathbf{u}}, \tilde{\mathbf{J}})$.

(d) A generalization of the above estimate shows that in the regime of unique solvability, the solution of Problem $P_1$ depends Lipschitz continuously on the data $\mathbf{F}$, $\mathbf{E}$, $\mathbf{g}$, $\mathbf{J}_{\text{ext}}$, and $\mathbf{B}_{\text{ext}}$.

**3. Finite-dimensional approximation.** Let $B$ denote a Banach space and $(B^h)_{h\in I}$ a family of finite-dimensional subspaces of $B$, where $I$ is a subset of the interval $(0, 1)$ having 0 as its only limit point. We say that $(B^h)_{h\in I}$ is a finite-dimensional approximation of $B$ (or that $B^h$ approximates $B$, for short) if for every $f \in B$, we have $\inf_{f^h\in B^h} \|f - f^h\|_B \to 0$ as $h \to 0$.

In all of the following, we assume that $(\mathbf{Y}_1^h)_{h\in I}$, $(\mathbf{Y}_2^h)_{h\in I}$, $(M_1^h)_{h\in I}$, and $(M_2^h)_{h\in I}$ are finite-dimensional approximations of $\mathbf{Y}_1 := \mathbf{H}^1(\Omega)$, $\mathbf{Y}_2 := \mathbf{L}^2(\Omega)$, $M_1 := L^2(\Omega)/\mathbb{R}$, and $M_2 := H^1(\Omega)/\mathbb{R}$, respectively. This implies, of course, that the product spaces $\mathbf{Y}^h := \mathbf{Y}_1^h \times \mathbf{Y}_2^h$ and $M^h := M_1^h \times M_2^h$ approximate $\mathbf{Y} := \mathbf{Y}_1 \times \mathbf{Y}_2$ and $M := M_1 \times M_2$, respectively. Recalling that $\mathbf{X}_1 := \mathbf{H}_0^1(\Omega)$, $\mathbf{X}_2 := \mathbf{Y}_2$, and $\mathbf{X} := \mathbf{X}_1 \times \mathbf{X}_2$, we also set $\mathbf{X}_1^h := \mathbf{Y}_1^h \cap \mathbf{X}_1$, $\mathbf{X}_2^h := \mathbf{Y}_2^h$, and $\mathbf{X}^h := \mathbf{X}_1^h \times \mathbf{X}_2^h$. Finally, we let $\mathbf{Y}_{1,\Gamma}^h$ denote the trace space of $\mathbf{Y}_1^h$, that is, the subspace $\{\mathbf{v}^h|_\Gamma;\, \mathbf{v}^h \in \mathbf{Y}_1^h\}$ of $\mathbf{Y}_{1,\Gamma} := \mathbf{H}^{1/2}(\Gamma)$. Note that automatically, $(\mathbf{Y}_{1,\Gamma}^h)_{h\in I}$ is a finite-dimensional approximation of $\mathbf{Y}_{1,\Gamma}$. However, an extra condition (see below) will be needed to guarantee that $(\mathbf{X}_1^h)_{h\in I}$ approximates $\mathbf{X}_1$.

Again, we assume a set of parameters $\eta$, $\rho$, $\sigma$, $\mu$ and a set of data $\mathbf{F}$, $\mathbf{E}$, $\mathbf{g}$, $\mathbf{J}_{\text{ext}}$, $\mathbf{B}_{\text{ext}}$ to be given as in Problem $P_0$ and let $j := \mathbf{J}_{\text{ext}} \cdot \mathbf{n}$. Moreover, we choose a family $(\mathbf{g}^h)_{h\in I}$ of approximate boundary values $\mathbf{g}^h \in \mathbf{Y}_{1,\Gamma}^h$ such that $\mathbf{g}^h \to \mathbf{g}$ in $\mathbf{Y}_{1,\Gamma}$ as $h \to 0$. We then consider a family $P_1^h$ $(h \in I)$ of finite-dimensional approximations to Problem $P_1$, as follows.

*Problem* $P_1^h$. Find $(\mathbf{u}^h, \mathbf{J}^h) \in \mathbf{Y}^h$ with $\mathbf{u}^h|_\Gamma = \mathbf{g}^h$ and $(p^h, \phi^h) \in M^h$ such that

$$a_0\big((\mathbf{u}^h, \mathbf{J}^h), (\mathbf{v}^h, \mathbf{K}^h)\big) + a_1\big((\mathbf{u}^h, \mathbf{J}^h), (\mathbf{u}^h, \mathbf{J}^h), (\mathbf{v}^h, \mathbf{K}^h)\big) \\ + b\big((\mathbf{v}^h, \mathbf{K}^h), (p^h, \phi^h)\big) = \int_\Omega \mathbf{F} \cdot \mathbf{v}^h + \int_\Omega \mathbf{E} \cdot \mathbf{K}^h \qquad \forall\, (\mathbf{v}^h, \mathbf{K}^h) \in \mathbf{X}^h \tag{3.1}$$

and

$$b\big((\mathbf{u}^h, \mathbf{J}^h), (q^h, \psi^h)\big) = \int_\Gamma j\,\psi^h \qquad \forall\,(q^h, \psi^h) \in M^h. \tag{3.2}$$

To prove the well-posedness (for small data) of Problem $P_1^h$ and to establish optimal-order error estimates in the spirit of the Ladyzhenskaya–Babuska–Brezzi theory (see, for example, [2, 5, 15]), we need to impose two conditions on the finite-dimensional spaces involved. Our first assumption is that the form $b$ satisfies the LBB-condition on $\mathbf{X}^h \times M^h$ uniformly with respect to $h \in I$.

*Assumption $A_1$*. There exists a number $\beta > 0$ such that

$$\inf_{(q^h,\psi^h)\in M^h} \sup_{(\mathbf{v}^h,\mathbf{K}^h)\in \mathbf{X}^h} \frac{b\big((\mathbf{v}^h, \mathbf{K}^h), (q^h, \psi^h)\big)}{\|(\mathbf{v}^h, \mathbf{K}^h)\|_{\mathbf{Y}} \|(q^h, \psi^h)\|_M} \geq \beta \quad \forall\, h \in I.$$

Our second assumption is needed to deal with the nonhomogeneous essential boundary condition for the velocity field.

*Assumption $A_2$*. There exist a number $\gamma > 0$ and a family $(\Pi^h)_{h\in I}$ of bounded linear projections from $\mathbf{Y}_1$ onto $\mathbf{Y}_1^h$ such that $\Pi^h(\mathbf{X}_1) \subset \mathbf{X}_1$ and $\|\Pi^h\| \leq \gamma$ for all $h \in I$.

*Remark* 3.1. (a) Note that the uniform boundedness of the projections $\Pi^h$ in Assumption $A_2$ implies their strong convergence in $\mathbf{Y}_1$. Indeed, we have

$$\|\mathbf{v} - \Pi^h \mathbf{v}\|_{\mathbf{Y}_1} \leq (\gamma + 1) \inf_{\mathbf{v}^h \in \mathbf{Y}_1^h} \|\mathbf{v} - \mathbf{v}^h\|_{\mathbf{Y}_1}$$

for every $\mathbf{v} \in \mathbf{Y}_1$ and $h \in I$.

The crucial property that distinguishes the projections $\Pi^h$ from, say, the *orthogonal* projections of $\mathbf{Y}_1$ onto $\mathbf{Y}_1^h$, is that the former *preserve homogeneous Dirichlet boundary values.* One immediate consequence of this property is that the spaces $\mathbf{X}_1^h = \mathbf{Y}_1^h \cap \mathbf{X}_1$ approximate $\mathbf{X}_1$.

Obviously, the spaces $\mathbf{X}^h \times M^h$ will then approximate $\mathbf{X} \times M$, and as a consequence, Assumption $A_1$ (the uniform LBB-condition on $\mathbf{X}^h \times M^h$) implies that Lemma 2.2(c) (the LBB-condition on $\mathbf{X} \times M$) holds *with the same constant $\beta$.*

(b) Another consequence of Assumption $A_2$ is the existence of a uniformly bounded family $(\Pi_\Gamma^h)_{h\in I}$ of linear projections from $\mathbf{Y}_{1,\Gamma}$ onto $\mathbf{Y}_{1,\Gamma}^h$ (the trace spaces of $\mathbf{Y}_1$ and $\mathbf{Y}_1^h$, respectively). Define $\Pi_\Gamma^h : \mathbf{Y}_{1,\Gamma} \to \mathbf{Y}_{1,\Gamma}^h$ by

$$\Pi_\Gamma^h \mathbf{k} := (\Pi^h \Lambda \mathbf{k})|_\Gamma,$$

for $\mathbf{k} \in \mathbf{Y}_{1,\Gamma}$ and $h \in I$, where $\Lambda$ denotes, as before, a bounded linear lifting operator from $\mathbf{Y}_{1,\Gamma}$ into $\mathbf{Y}_1$. Since $\Pi^h(\mathbf{X}_1) \subset \mathbf{X}_1$, the above definition is actually *independent of the choice of the lifting operator*, and this implies that $\Pi_\Gamma^h \mathbf{k}^h = \mathbf{k}^h$ for all $\mathbf{k}^h \in \mathbf{Y}_{1,\Gamma}^h$; that is, $\Pi_\Gamma^h$ is indeed a projection. Furthermore, since we can choose $\Lambda$ so that $\|\Lambda\| = 1$, we have $\|\Pi_\Gamma^h\| \leq \gamma$ (with the constant $\gamma$ of Assumption $A_2$) and thus

$$\|\mathbf{k} - \Pi_\Gamma^h \mathbf{k}\|_{\mathbf{Y}_{1,\Gamma}} \leq (\gamma + 1) \inf_{\mathbf{k}^h \in \mathbf{Y}_{1,\Gamma}^h} \|\mathbf{k} - \mathbf{k}^h\|_{\mathbf{Y}_{1,\Gamma}}$$

for every $\mathbf{k} \in \mathbf{Y}_{1,\Gamma}$ and $h \in I$. In particular, if $\mathbf{k} \in \mathbf{Y}_{1,\Gamma}$ and if $\mathbf{v} \in \mathbf{Y}_1$ is any lifting of $\mathbf{k}$, then

$$\|\mathbf{k} - \Pi_\Gamma^h \mathbf{k}\|_{\mathbf{Y}_{1,\Gamma}} \leq (\gamma + 1) \inf_{\mathbf{v}^h \in \mathbf{Y}_1^h} \|\mathbf{v} - \mathbf{v}^h\|_{\mathbf{Y}_1}$$

for all $h \in I$.

(c) With reasoning similar to (b), we infer the existence of a uniformly bounded family $(\Lambda^h)_{h\in I}$ of linear lifting operators from $\mathbf{Y}^h_{1,\Gamma}$ into $\mathbf{Y}^h_1$. Indeed, if we define $\Lambda^h : \mathbf{Y}^h_{1,\Gamma} \to \mathbf{Y}^h_1$ by

$$\Lambda^h \mathbf{k}^h := \Pi^h \Lambda \mathbf{k}^h,$$

for $\mathbf{k}^h \in \mathbf{Y}^h_{1,\Gamma}$ and $h \in I$, then $(\Lambda^h \mathbf{k}^h)|_\Gamma = \Pi^h_\Gamma \mathbf{k}^h = \mathbf{k}^h$ for all $\mathbf{k}^h \in \mathbf{Y}^h_{1,\Gamma}$, that is, $\Lambda^h$ is a lifting operator, and $\|\Lambda^h\| \le \gamma$ for all $h \in I$, provided we choose $\Lambda$ so that $\|\Lambda\| = 1$. Moreover, $\mathbf{k}^h \to \mathbf{k}$ in $\mathbf{Y}_{1,\Gamma}$ implies that $\Lambda^h\mathbf{k}^h \to \Lambda\mathbf{k}$ in $\mathbf{Y}_1$.

(d) All three of the above remarks will play a role in the subsequent analysis. We note that finite-dimensional spaces satisfying Assumptions $A_1$ and $A_2$ have been devised and analyzed in the finite-element literature. Specific examples, relevant in connection with Problem $P_1$, will be discussed in section 4.

In the same way that we used the operator $\Lambda$ and the LBB-condition on $\mathbf{X} \times M$ to “homogenize” Problem $P_1$ in section 2, we now employ the operators $\Lambda^h$ and Assumption $A_1$ to “homogenize” Problem $P^h_1$. Define

$$(\mathbf{u}^h_0, \mathbf{J}^h_0) := (\Lambda^h \mathbf{g}^h, 0) + (B^h)^+\big(\varphi^h(\mathbf{g}^h, j)\big),$$

where $(B^h)^+$ is the Moore–Penrose pseudoinverse of the operator $B^h : \mathbf{X}^h \to (M^h)^*$ defined by

$$B^h(\mathbf{v}^h, \mathbf{K}^h) := b\big((\mathbf{v}^h, \mathbf{K}^h), (\cdot,\cdot)\big)\big|_{M^h},$$

for $(\mathbf{v}^h, \mathbf{K}^h) \in \mathbf{X}^h$, and where $\varphi^h(\mathbf{g}^h, j) \in (M^h)^*$ is given by

$$\varphi^h(\mathbf{g}^h, j)(q^h, \psi^h) := \int_\Gamma j\,\psi^h - b\big((\Lambda^h\mathbf{g}^h, 0), (q^h, \psi^h)\big),$$

for $(q^h, \psi^h) \in M^h$. By construction, we have $(\mathbf{u}^h_0, \mathbf{J}^h_0) \in \mathbf{Y}^h$, $\mathbf{u}^h_0|_\Gamma = \mathbf{g}^h$, and $b\big((\mathbf{u}^h_0, \mathbf{J}^h_0), (q^h, \psi^h)\big) = \int_\Gamma j\,\psi^h$ for all $(q^h, \psi^h) \in M^h$. Moreover,

$$\|(\mathbf{u}^h_0, \mathbf{J}^h_0)\|_{\mathbf{Y}} \le \lambda \|(\mathbf{g}^h, j)\|_{\mathbf{H}^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} \quad \forall\, h \in I, \tag{3.3}$$

with a constant $\lambda$ depending only on $\Omega$ and the choice of spaces $(\mathbf{Y}^h)_{h\in I}$ and $(M^h)_{h\in I}$ (for example, $\lambda = (1+\sqrt{3}\,\beta^{-1})\gamma$, where $\beta$ and $\gamma$ are the constants in Assumptions $A_1$ and $A_2$). Inequality (2.6), the corresponding estimate for $(\mathbf{u}_0, \mathbf{J}_0)$, as constructed in section 2, will automatically hold with the same constant $\lambda$. It can also be verified that

$$\mathbf{g}^h \to \mathbf{g} \text{ in } \mathbf{Y}_{1,\Gamma} \implies (\mathbf{u}^h_0, \mathbf{J}^h_0) \to (\mathbf{u}_0, \mathbf{J}_0) \text{ in } \mathbf{Y}. \tag{3.4}$$

In essence, the proof of (3.4) amounts to showing that the Moore–Penrose pseudoinverses $(B^h)^+$ of the operators $B^h$ converge strongly to the Moore–Penrose pseudoinverse $B^+$ of the operator $B$. More precisely, $(B^h)^+(\varphi|_{M^h}) \to B^+\varphi$ for every $\varphi \in M^*$. This is a (not quite trivial) consequence of Assumption $A_1$ and can be established in the abstract setting of [5, Chapter II.1.1].

Substituting $\mathbf{u}^h = \mathbf{u}^h_0 + \hat{\mathbf{u}}^h$ and $\mathbf{J}^h = \mathbf{J}^h_0 + \hat{\mathbf{J}}^h$ in (3.1) and (3.2), we arrive at the finite-dimensional analogue of Problem $P_2$.

*Problem* $P^h_2$. Find $(\hat{\mathbf{u}}^h, \hat{\mathbf{J}}^h) \in \mathbf{V}^h$ such that

$$a^h\big((\hat{\mathbf{u}}^h, \hat{\mathbf{J}}^h), (\hat{\mathbf{u}}^h, \hat{\mathbf{J}}^h), (\mathbf{v}^h, \mathbf{K}^h)\big) = \ell^h(\mathbf{v}^h, \mathbf{K}^h) \qquad \forall\,(\mathbf{v}^h, \mathbf{K}^h) \in \mathbf{V}^h.$$

Here, the forms $a^h$ and $\ell^h$ are defined just like $a$ and $\ell$ in (2.7) and (2.8), except that $(\mathbf{u}_0^h, \mathbf{J}_0^h)$ replaces $(\mathbf{u}_0, \mathbf{J}_0)$. The space $\mathbf{V}^h$ is the finite-dimensional analogue of $\mathbf{V}$, as defined in (2.5); that is,

$$\mathbf{V}^h := \{(\mathbf{v}^h, \mathbf{K}^h) \in \mathbf{X}^h;\ b\big((\mathbf{v}^h, \mathbf{K}^h), (q^h, \psi^h)\big) = 0\ \ \forall (q^h, \psi^h) \in M^h\}.$$

Due to Assumption $A_1$, Problem $P_2^h$ is equivalent to Problem $P_1^h$ in the same way that Problem $P_2$ is equivalent to Problem $P_1$.

It is readily checked that, although $\mathbf{V}^h$ is in general not a subspace of $\mathbf{V}$, the continuity and coercivity properties of the form $a$ (on $\mathbf{V}$), as stated in Lemma 2.3, carry over to the form $a^h$ (on $\mathbf{V}^h$). In particular, for every $(\mathbf{v}_0^h, \mathbf{K}_0^h) \in \mathbf{V}^h$ and all $(\mathbf{v}^h, \mathbf{K}^h) \in \mathbf{V}^h$, we have

$$a^h\big((\mathbf{v}_0^h, \mathbf{K}_0^h), (\mathbf{v}^h, \mathbf{K}^h), (\mathbf{v}^h, \mathbf{K}^h)\big) \geq \Big(\alpha - \lambda \|a_1\| \|(\mathbf{g}^h, j)\|\Big) \|(\mathbf{v}^h, \mathbf{K}^h)\|_{\mathbf{Y}}^2,$$

with constants $\alpha$ and $\lambda$ as in (2.3) and (3.3). This yields an existence and uniqueness result analogous to Theorem 2.4 for Problem $P_2^h$.

THEOREM 3.2. *Let $N^h = N^h(\mathbf{F}, \mathbf{E}, \mathbf{g}^h, \mathbf{J}_{\text{ext}}, \mathbf{B}_{\text{ext}})$ denote the norm of the functional $\ell^h|_{\mathbf{V}^h}$. Let $\|(\mathbf{g}^h, j)\|$ denote the norm of $(\mathbf{g}^h, j)$ in $\mathbf{H}^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, where $j = \mathbf{J}_{\text{ext}} \cdot \mathbf{n}$, and choose constants $\alpha$ and $\lambda$ as in (2.3) and (3.3).*

(a) *If $\|(\mathbf{g}^h, j)\| < \frac{\alpha}{\lambda \|a_1\|}$, then there exists at least one solution $(\hat{\mathbf{u}}^h, \hat{\mathbf{J}}^h)$ of Problem $P_2^h$ that satisfies*

$$\|(\hat{\mathbf{u}}^h, \hat{\mathbf{J}}^h)\|_{\mathbf{Y}} \leq \frac{N^h}{\alpha - \lambda \|a_1\| \|(\mathbf{g}^h, j)\|}.$$

(b) *If $\|(\mathbf{g}^h, j)\| < \frac{\alpha}{\lambda \|a_1\|}$ and $N^h < \frac{1}{\|a_1\|}\big(\alpha - \lambda \|a_1\| \|(\mathbf{g}^h, j)\|\big)^2$, then the solution of Problem $P_2^h$ is unique.*

Of course, the remarks following Theorem 2.4 apply accordingly. Furthermore, (3.4) implies that

$$N^h(\mathbf{F}, \mathbf{E}, \mathbf{g}^h, \mathbf{J}_{\text{ext}}, \mathbf{B}_{\text{ext}}) \to N(\mathbf{F}, \mathbf{E}, \mathbf{g}, \mathbf{J}_{\text{ext}}, \mathbf{B}_{\text{ext}});$$

that is, $\|\ell^h|_{\mathbf{V}^h}\| \to \|\ell|_{\mathbf{V}}\|$, as $h \to 0$. Based on these observations, we obtain the following corollary to Theorems 2.4 and 3.2.

COROLLARY 3.3. *Let $N = N(\mathbf{F}, \mathbf{E}, \mathbf{g}, \mathbf{J}_{\text{ext}}, \mathbf{B}_{\text{ext}})$ denote the norm of the functional $\ell|_{\mathbf{V}}$. Let $\|(\mathbf{g}, j)\|$ denote the norm of $(\mathbf{g}, j)$ in $\mathbf{H}^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, where $j = \mathbf{J}_{\text{ext}} \cdot \mathbf{n}$, and choose constants $\alpha$ and $\lambda$ as in (2.3) and (3.3).*

*If $\|(\mathbf{g}, j)\| < \frac{\alpha}{\lambda \|a_1\|}$ and $N < \frac{1}{\|a_1\|}\big(\alpha - \lambda \|a_1\| \|(\mathbf{g}, j)\|\big)^2$, and if $h$ is sufficiently small, then Problems $P_1$ and $P_1^h$ both have unique solutions $(\mathbf{u}, \mathbf{J}, p, \phi)$ and $(\mathbf{u}^h, \mathbf{J}^h, p^h, \phi^h)$, respectively, and these solutions satisfy $\|(\mathbf{u}, \mathbf{J})\|_{\mathbf{Y}} < \frac{\alpha}{\|a_1\|}$ and $\|(\mathbf{u}^h, \mathbf{J}^h)\|_{\mathbf{Y}} < \frac{\alpha}{\|a_1\|}$.*

The central result of this section is an estimate for the discretization error that arises when, in the case of unique solvability, the solution of Problem $P_1$ is approximated by that of $P_1^h$.

THEOREM 3.4. *Let the constants $\alpha$, $\beta$, and $\gamma$ be chosen as in Lemma 2.2(b), Assumption $A_1$, and Assumption $A_2$. Suppose $(\mathbf{u}, \mathbf{J}, p, \phi)$ and $(\mathbf{u}^h, \mathbf{J}^h, p^h, \phi^h)$ are solutions of Problems $P_1$ and $P_1^h$, respectively. Define $\nu := \|(\mathbf{u}, \mathbf{J})\|_{\mathbf{Y}}$ and $\nu^h := \|(\mathbf{u}^h, \mathbf{J}^h)\|_{\mathbf{Y}}$, and assume that $\nu < \frac{\alpha}{\|a_1\|}$. Finally, let*

$$\theta^h := \sup_{\substack{(\mathbf{v}^h, \mathbf{K}^h) \in \mathbf{V}^h \\ \|(\mathbf{v}^h, \mathbf{K}^h)\|_{\mathbf{Y}} = 1}} \ \inf_{(\mathbf{v}, \mathbf{K}) \in \mathbf{V}} \|(\mathbf{v}^h, \mathbf{K}^h) - (\mathbf{v}, \mathbf{K})\|_{\mathbf{Y}}.$$

*Then we have*

$$\|(\mathbf{u},\mathbf{J})-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}} \le \Big(1+\frac{\|a_0\|+(\nu+\nu^h)\|a_1\|}{\alpha-\nu\|a_1\|}\Big)\Big(1+\frac{\|b\|}{\beta}\Big) \Big((1+\gamma)\inf_{(\mathbf{v}^h,\mathbf{K}^h)\in\mathbf{Y}^h}\|(\mathbf{u},\mathbf{J})-(\mathbf{v}^h,\mathbf{K}^h)\|_{\mathbf{Y}}+\gamma\|\mathbf{g}-\mathbf{g}^h\|_{\mathbf{Y}_{1,\Gamma}}\Big) + \frac{\theta^h\|b\|}{\alpha-\nu\|a_1\|}\inf_{(q^h,\psi^h)\in M^h}\|(p,\phi)-(q^h,\psi^h)\|_M$$

*and*

$$\|(p,\phi)-(p^h,\phi^h)\|_M \le \frac{\|a_0\|+(\nu+\nu^h)\|a_1\|}{\beta}\|(\mathbf{u},\mathbf{J})-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}} + \Big(1+\frac{\|b\|}{\beta}\Big)\inf_{(q^h,\psi^h)\in M^h}\|(p,\phi)-(q^h,\psi^h)\|_M.$$

*Remark* 3.5. The quantity $\theta^h$ measures the "angle" between the spaces $\mathbf{V}^h$ and $\mathbf{V}$. In fact, $\theta^h=0$ if and only if $\mathbf{V}^h\subset\mathbf{V}$. In any case, we have $0\le\theta^h\le 1$ and

$$\inf_{(\mathbf{v},\mathbf{K})\in\mathbf{V}}\|(\mathbf{v}^h,\mathbf{K}^h)-(\mathbf{v},\mathbf{K})\|_{\mathbf{Y}}\le\theta^h\|(\mathbf{v}^h,\mathbf{K}^h)\|_{\mathbf{Y}}\qquad\forall\,(\mathbf{v}^h,\mathbf{K}^h)\in\mathbf{V}^h.$$

Before turning to the proof of Theorem 3.4, we state a simple corollary.

COROLLARY 3.6. *In the situation of Corollary* 3.3, *let the unique solutions of Problems* $P_1$ *and* $P_1^h$ *be denoted by* $(\mathbf{u},\mathbf{J},p,\phi)$ *and* $(\mathbf{u}^h,\mathbf{J}^h,p^h,\phi^h)$, *respectively. Then there exists a number* $\delta>0$ *such that the following estimate holds for all sufficiently small* $h\in I$*:*

$$\|(\mathbf{u},\mathbf{J})-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}}+\|(p,\phi)-(p^h,\phi^h)\|_M \le\delta\Big(\inf_{(\mathbf{v}^h,\mathbf{K}^h)\in\mathbf{Y}^h}\|(\mathbf{u},\mathbf{J})-(\mathbf{v}^h,\mathbf{K}^h)\|_{\mathbf{Y}} + \inf_{(q^h,\psi^h)\in M^h}\|(p,\phi)-(q^h,\psi^h)\|_M+\|\mathbf{g}-\mathbf{g}^h\|_{\mathbf{Y}_{1,\Gamma}}\Big). \tag{3.5}$$

*In particular,* $(\mathbf{u}^h,\mathbf{J}^h)\to(\mathbf{u},\mathbf{J})$ *in* $\mathbf{Y}$ *and* $(p^h,\phi^h)\to(p,\phi)$ *in* $M$, *as* $h\to 0$.

*Proof.* Under the stated assumptions, we have

$$\nu=\|(\mathbf{u},\mathbf{J})\|_{\mathbf{Y}}<\frac{\alpha}{\|a_1\|}\quad\text{and}\quad\nu^h=\|(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}}<\frac{\alpha}{\|a_1\|}.$$

Also, by Remark 3.5, $\theta^h\le 1$. Using these bounds and the two estimates of Theorem 3.4, we obtain (3.5) with, for example,

$$\delta:=(1+\gamma)\Big(1+\frac{2\alpha+\|a_0\|}{\beta}\Big)\Big(1+\frac{2\alpha+\|a_0\|}{\alpha-\nu\|a_1\|}\Big)\Big(1+\frac{\|b\|}{\beta}+\frac{\|b\|}{\alpha-\nu\|a_1\|}\Big).$$

Convergence follows immediately, thanks to the approximation properties of $(\mathbf{Y}^h)_{h\in I}$ and $(M^h)_{h\in I}$ and the assumption that $\|\mathbf{g}-\mathbf{g}^h\|_{\mathbf{Y}_{1,\Gamma}}\to 0$ as $h\to 0$. □

The estimate (3.5) is *of optimal order* in the sense that it shows the total discretization error $\|(\mathbf{u},\mathbf{J})-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}}+\|(p,\phi)-(p^h,\phi^h)\|_M$ to be proportional to the sum of the errors of best approximation of $(\mathbf{u},\mathbf{J})$ and $(p,\phi)$ by elements of $\mathbf{Y}^h$ and $M^h$, respectively, and of the error in the approximate boundary values $\mathbf{g}^h$. Thus, we can obtain specific rates of convergence by employing finite-dimensional spaces

$\mathbf{Y}^h$ and $M^h$ with suitable approximation properties and by choosing sufficiently good approximate boundary values. Note that it is always possible to choose $\mathbf{g}^h$ so that $\|\mathbf{g}-\mathbf{g}^h\|_{\mathbf{Y}_{1,\Gamma}}$ is of the same order as $\inf_{\mathbf{v}^h\in\mathbf{Y}_1^h}\|\mathbf{u}-\mathbf{v}^h\|_{\mathbf{Y}_1}$. As will be seen in section 4, the projections $\Pi_\Gamma^h$ of Remark 3.1(b) provide a numerically feasible way of doing this.

We now turn to the proof of Theorem 3.4. Throughout, we assume that all hypotheses of the theorem are satisfied.

First, we observe that $(\mathbf{u},\mathbf{J},p,\phi)$ and $(\mathbf{u}^h,\mathbf{J}^h,p^h,\phi^h)$ satisfy (2.1) and (3.1), respectively. Using the same test functions $(\mathbf{v}^h,\mathbf{K}^h)\in\mathbf{X}^h$ in both equations, subtracting the latter from the former, and regrouping the trilinear terms, we obtain

$$\begin{aligned}0 &= a_0\big((\mathbf{u}-\mathbf{u}^h,\mathbf{J}-\mathbf{J}^h),(\mathbf{v}^h,\mathbf{K}^h)\big)+a_1\big((\mathbf{u}-\mathbf{u}^h,\mathbf{J}-\mathbf{J}^h),(\mathbf{u},\mathbf{J}),(\mathbf{v}^h,\mathbf{K}^h)\big)\\ &\quad+a_1\big((\mathbf{u}^h,\mathbf{J}^h),(\mathbf{u}-\mathbf{u}^h,\mathbf{J}-\mathbf{J}^h),(\mathbf{v}^h,\mathbf{K}^h)\big)+b\big((\mathbf{v}^h,\mathbf{K}^h),(p-p^h,\phi-\phi^h)\big).\end{aligned}\tag{3.6}$$

Now let $(\mathbf{w}^h,\mathbf{L}^h)\in\mathbf{Y}^h$, $(r^h,\chi^h)\in M^h$, and $(\mathbf{v}^h,\mathbf{K}^h)\in\mathbf{X}^h$. Then we have

$$\begin{aligned}&a_0\big((\mathbf{w}^h-\mathbf{u}^h,\mathbf{L}^h-\mathbf{J}^h),(\mathbf{v}^h,\mathbf{K}^h)\big)+a_1\big((\mathbf{w}^h-\mathbf{u}^h,\mathbf{L}^h-\mathbf{J}^h),(\mathbf{u},\mathbf{J}),(\mathbf{v}^h,\mathbf{K}^h)\big)\\ &\quad+a_1\big((\mathbf{u}^h,\mathbf{J}^h),(\mathbf{w}^h-\mathbf{u}^h,\mathbf{L}^h-\mathbf{J}^h),(\mathbf{v}^h,\mathbf{K}^h)\big)\\ &\quad+b\big((\mathbf{v}^h,\mathbf{K}^h),(r^h-p^h,\chi^h-\phi^h)\big)\\ &=a_0\big((\mathbf{w}^h-\mathbf{u},\mathbf{L}^h-\mathbf{J}),(\mathbf{v}^h,\mathbf{K}^h)\big)+a_1\big((\mathbf{w}^h-\mathbf{u},\mathbf{L}^h-\mathbf{J}),(\mathbf{u},\mathbf{J}),(\mathbf{v}^h,\mathbf{K}^h)\big)\\ &\quad+a_1\big((\mathbf{u}^h,\mathbf{J}^h),(\mathbf{w}^h-\mathbf{u},\mathbf{L}^h-\mathbf{J}),(\mathbf{v}^h,\mathbf{K}^h)\big)+b\big((\mathbf{v}^h,\mathbf{K}^h),(r^h-p,\chi^h-\phi)\big).\end{aligned}\tag{3.7}$$

(Note that the difference of the left- and right-hand sides of (3.7) equals the right-hand side of (3.6).) Equations (3.6) and (3.7) will be used repeatedly in the following estimates. We proceed in several steps.

*Step* 1.

$$\begin{aligned}&\|(\mathbf{u},\mathbf{J})-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}}\\ &\quad\le\left(1+\frac{\|a_0\|+(\nu+\nu^h)\|a_1\|}{\alpha-\nu\|a_1\|}\right)\inf_{\substack{(\mathbf{w}^h,\mathbf{L}^h)\in\mathbf{Y}^h\\(\mathbf{w}^h,\mathbf{L}^h)-(\mathbf{u}^h,\mathbf{J}^h)\in\mathbf{V}^h}}\|(\mathbf{u},\mathbf{J})-(\mathbf{w}^h,\mathbf{L}^h)\|_{\mathbf{Y}}\\ &\qquad+\frac{\theta^h\|b\|}{\alpha-\nu\|a_1\|}\inf_{(r^h,\chi^h)\in M^h}\|(p,\phi)-(r^h,\chi^h)\|_M.\end{aligned}\tag{3.8}$$

*Proof.* Let $(\mathbf{w}^h,\mathbf{L}^h)\in\mathbf{Y}^h$ be such that $(\mathbf{w}^h,\mathbf{L}^h)-(\mathbf{u}^h,\mathbf{J}^h)\in\mathbf{V}^h$ (which means that $\mathbf{w}^h|_\Gamma=\mathbf{u}^h|_\Gamma=\mathbf{g}^h$ and $b\big((\mathbf{w}^h,\mathbf{L}^h),(q^h,\psi^h)\big)=b\big((\mathbf{u}^h,\mathbf{J}^h),(q^h,\psi^h)\big)=\int_\Gamma j\,\psi^h$ for all $(q^h,\psi^h)\in M^h$). Also, let $(r^h,\chi^h)\in M^h$ and $(\mathbf{w},\mathbf{L})\in\mathbf{V}$. Using $(\mathbf{v}^h,\mathbf{K}^h):=(\mathbf{w}^h-\mathbf{u}^h,\mathbf{L}^h-\mathbf{J}^h)$ as a test function in (3.7), we obtain

$$\begin{aligned}&a_0\big((\mathbf{w}^h-\mathbf{u}^h,\mathbf{L}^h-\mathbf{J}^h),(\mathbf{w}^h-\mathbf{u}^h,\mathbf{L}^h-\mathbf{J}^h)\big)\\ &\qquad+a_1\big((\mathbf{w}^h-\mathbf{u}^h,\mathbf{L}^h-\mathbf{J}^h),(\mathbf{u},\mathbf{J}),(\mathbf{w}^h-\mathbf{u}^h,\mathbf{L}^h-\mathbf{J}^h)\big)\\ &\quad=a_0\big((\mathbf{w}^h-\mathbf{u},\mathbf{L}^h-\mathbf{J}),(\mathbf{w}^h-\mathbf{u}^h,\mathbf{L}^h-\mathbf{J}^h)\big)\\ &\qquad+a_1\big((\mathbf{w}^h-\mathbf{u},\mathbf{L}^h-\mathbf{J}),(\mathbf{u},\mathbf{J}),(\mathbf{w}^h-\mathbf{u}^h,\mathbf{L}^h-\mathbf{J}^h)\big)\\ &\qquad+a_1\big((\mathbf{u}^h,\mathbf{J}^h),(\mathbf{w}^h-\mathbf{u},\mathbf{L}^h-\mathbf{J}),(\mathbf{w}^h-\mathbf{u}^h,\mathbf{L}^h-\mathbf{J}^h)\big)\\ &\qquad+b\big((\mathbf{w}^h-\mathbf{u}^h-\mathbf{w},\mathbf{L}^h-\mathbf{J}^h-\mathbf{L}),(r^h-p,\chi^h-\phi)\big).\end{aligned}\tag{3.9}$$

(Note that the third and fourth terms on the left-hand side of (3.7) vanish by the skew-symmetry of $a_1\big((\mathbf{u}^h,\mathbf{J}^h),(\cdot,\cdot),(\cdot,\cdot)\big)$ and by definition of $\mathbf{V}^h$, respectively, and that $b\big((\mathbf{w},\mathbf{L}),(r^h-p,\chi^h-\phi)\big)=0$ by definition of $\mathbf{V}$.) Thanks to (2.3), the left-hand side of (3.9) is bounded from below by

$$\big(\alpha-\nu\|a_1\|\big)\|(\mathbf{w}^h,\mathbf{L}^h)-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}}^2$$

(where $\nu=\|(\mathbf{u},\mathbf{J})\|_{\mathbf{Y}}$). The right-hand side of (3.9) can be estimated from above by

$$\begin{aligned}&\big(\|a_0\|+(\nu+\nu^h)\|a_1\|\big)\|(\mathbf{w}^h,\mathbf{L}^h)-(\mathbf{u},\mathbf{J})\|_{\mathbf{Y}}\,\|(\mathbf{w}^h,\mathbf{L}^h)-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}}\\&\qquad+\|b\|\|(\mathbf{w}^h,\mathbf{L}^h)-(\mathbf{u}^h,\mathbf{J}^h)-(\mathbf{w},\mathbf{L})\|_{\mathbf{Y}}\,\|(r^h,\chi^h)-(p,\phi)\|_M\end{aligned}$$

(where $\nu^h=\|(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}}$) and, taking the infimum over $(\mathbf{w},\mathbf{L})\in\mathbf{V}$, by

$$\Big(\big(\|a_0\|+(\nu+\nu^h)\|a_1\|\big)\|(\mathbf{w}^h,\mathbf{L}^h)-(\mathbf{u},\mathbf{J})\|_{\mathbf{Y}}+\theta^h\|b\|\|(r^h,\chi^h)-(p,\phi)\|_M\Big)\\ \|(\mathbf{w}^h,\mathbf{L}^h)-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}}$$

(see Remark 3.5). Hence,

$$\begin{aligned}&\big(\alpha-\nu\|a_1\|\big)\|(\mathbf{w}^h,\mathbf{L}^h)-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}}\\&\qquad\le\big(\|a_0\|+(\nu+\nu^h)\|a_1\|\big)\|(\mathbf{w}^h,\mathbf{L}^h)-(\mathbf{u},\mathbf{J})\|_{\mathbf{Y}}+\theta^h\|b\|\|(r^h,\chi^h)-(p,\phi)\|_M.\end{aligned}$$

Using the triangle inequality and recalling that by assumption $\nu<\alpha/\|a_1\|$, we conclude that

$$\begin{aligned}\|(\mathbf{u},\mathbf{J})-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}}&\le\|(\mathbf{u},\mathbf{J})-(\mathbf{w}^h,\mathbf{L}^h)\|_{\mathbf{Y}}+\|(\mathbf{w}^h,\mathbf{L}^h)-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}}\\&\le\left(1+\frac{\|a_0\|+(\nu+\nu^h)\|a_1\|}{\alpha-\nu\|a_1\|}\right)\|(\mathbf{u},\mathbf{J})-(\mathbf{w}^h,\mathbf{L}^h)\|_{\mathbf{Y}}\\&\qquad+\frac{\theta^h\|b\|}{\alpha-\nu\|a_1\|}\,\|(p,\phi)-(r^h,\chi^h)\|_M,\end{aligned}$$

which implies (3.8). □

*Step* 2.

$$\inf_{\substack{(\mathbf{w}^h,\mathbf{L}^h)\in\mathbf{Y}^h\\(\mathbf{w}^h,\mathbf{L}^h)-(\mathbf{u}^h,\mathbf{J}^h)\in\mathbf{V}^h}}\|(\mathbf{u},\mathbf{J})-(\mathbf{w}^h,\mathbf{L}^h)\|_{\mathbf{Y}}\le\left(1+\frac{\|b\|}{\beta}\right)\inf_{\substack{(\mathbf{v}^h,\mathbf{K}^h)\in\mathbf{Y}^h\\(\mathbf{v}^h,\mathbf{K}^h)-(\mathbf{u}^h,\mathbf{J}^h)\in\mathbf{X}^h}}\|(\mathbf{u},\mathbf{J})-(\mathbf{v}^h,\mathbf{K}^h)\|_{\mathbf{Y}}. \tag{3.10}$$

*Proof.* Let $(\mathbf{v}^h,\mathbf{K}^h)\in\mathbf{Y}^h$ be such that $(\mathbf{v}^h,\mathbf{K}^h)-(\mathbf{u}^h,\mathbf{J}^h)\in\mathbf{X}^h$ (that is, $\mathbf{v}^h|_\Gamma=\mathbf{u}^h|_\Gamma=\mathbf{g}^h$). Define $(\mathbf{w}^h,\mathbf{L}^h):=(\mathbf{v}^h,\mathbf{K}^h)+(B^h)^+\big(b\big((\mathbf{u}-\mathbf{v}^h,\mathbf{J}-\mathbf{K}^h),(\cdot,\cdot)\big)|_{M^h}\big)$ (see the discussion preceding Problem $P_2^h$ for the definition of $B^h$ and $(B^h)^+$). Then we have $(\mathbf{w}^h,\mathbf{L}^h)\in\mathbf{Y}^h$, $\mathbf{w}^h|_\Gamma=\mathbf{g}^h$, and

$$\begin{aligned}b\big((\mathbf{w}^h,\mathbf{L}^h),(q^h,\psi^h)\big)&=b\big((\mathbf{v}^h,\mathbf{K}^h),(q^h,\psi^h)\big)+b\big((\mathbf{u}-\mathbf{v}^h,\mathbf{J}-\mathbf{K}^h),(q^h,\psi^h)\big)\\&=b\big((\mathbf{u},\mathbf{J}),(q^h,\psi^h)\big)=\int_\Gamma j\,\psi^h=b\big((\mathbf{u}^h,\mathbf{J}^h),(q^h,\psi^h)\big)\end{aligned}$$

for all $(q^h,\psi^h)\in M^h$; that is, $(\mathbf{w}^h,\mathbf{L}^h)-(\mathbf{u}^h,\mathbf{J}^h)\in\mathbf{V}^h$. Moreover,

$$\|(\mathbf{w}^h,\mathbf{L}^h)-(\mathbf{v}^h,\mathbf{K}^h)\|_{\mathbf{Y}}\le\frac{\|b\|}{\beta}\|(\mathbf{u},\mathbf{J})-(\mathbf{v}^h,\mathbf{K}^h)\|_{\mathbf{Y}}$$

and thus

$$\begin{aligned}\|(\mathbf{u},\mathbf{J})-(\mathbf{w}^h,\mathbf{L}^h)\|_{\mathbf{Y}} &\le \|(\mathbf{u},\mathbf{J})-(\mathbf{v}^h,\mathbf{K}^h)\|_{\mathbf{Y}}+\|(\mathbf{v}^h,\mathbf{K}^h)-(\mathbf{w}^h,\mathbf{L}^h)\|_{\mathbf{Y}}\\ &\le \left(1+\frac{\|b\|}{\beta}\right)\|(\mathbf{u},\mathbf{J})-(\mathbf{v}^h,\mathbf{K}^h)\|_{\mathbf{Y}}.\end{aligned}$$

The estimate (3.10) follows by taking infima. ◻

*Step* 3.

$$\begin{aligned}&\inf_{\substack{(\mathbf{v}^h,\mathbf{K}^h)\in\mathbf{Y}^h\\(\mathbf{v}^h,\mathbf{K}^h)-(\mathbf{u}^h,\mathbf{J}^h)\in\mathbf{X}^h}}\|(\mathbf{u},\mathbf{J})-(\mathbf{v}^h,\mathbf{K}^h)\|_{\mathbf{Y}}\\ &\qquad\le(1+\gamma)\inf_{(\mathbf{w}^h,\mathbf{L}^h)\in\mathbf{Y}^h}\|(\mathbf{u},\mathbf{J})-(\mathbf{w}^h,\mathbf{L}^h)\|_{\mathbf{Y}}+\gamma\|\mathbf{g}-\mathbf{g}^h\|_{\mathbf{Y}_{1,\Gamma}}.\end{aligned}\tag{3.11}$$

*Proof.* Let $(\mathbf{w}^h,\mathbf{L}^h)\in\mathbf{Y}^h$. Then define $\mathbf{v}^h:=\mathbf{w}^h+\Lambda^h(\mathbf{g}^h-\mathbf{w}^h|_\Gamma)$, where $\Lambda^h:\mathbf{Y}^h_{1,\Gamma}\to\mathbf{Y}^h_1$ denotes the lifting operator of Remark 3.1(c), and let $\mathbf{K}^h:=\mathbf{L}^h$. Clearly, $\mathbf{v}^h|_\Gamma=\mathbf{g}^h$ and thus, $(\mathbf{v}^h,\mathbf{K}^h)-(\mathbf{u}^h,\mathbf{J}^h)\in\mathbf{X}^h$. Moreover,

$$\begin{aligned}\|(\mathbf{u},\mathbf{J})-(\mathbf{v}^h,\mathbf{K}^h)\|_{\mathbf{Y}} &\le \|(\mathbf{u},\mathbf{J})-(\mathbf{w}^h,\mathbf{L}^h)\|_{\mathbf{Y}}+\|(\Lambda^h(\mathbf{g}^h-\mathbf{w}^h|_\Gamma),0)\|_{\mathbf{Y}}\\ &\le \|(\mathbf{u},\mathbf{J})-(\mathbf{w}^h,\mathbf{L}^h)\|_{\mathbf{Y}}+\gamma\|\mathbf{g}^h-\mathbf{w}^h|_\Gamma\|_{\mathbf{Y}_{1,\Gamma}}\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{g}^h-\mathbf{w}^h|_\Gamma\|_{\mathbf{Y}_{1,\Gamma}} &\le \|\mathbf{g}^h-\mathbf{g}\|_{\mathbf{Y}_{1,\Gamma}}+\|\mathbf{g}-\mathbf{w}^h|_\Gamma\|_{\mathbf{Y}_{1,\Gamma}}\\ &\le \|\mathbf{g}^h-\mathbf{g}\|_{\mathbf{Y}_{1,\Gamma}}+\|\mathbf{u}-\mathbf{w}^h\|_{\mathbf{Y}_1}\end{aligned}$$

(since $\mathbf{g}=\mathbf{u}|_\Gamma$). This implies

$$\|(\mathbf{u},\mathbf{J})-(\mathbf{v}^h,\mathbf{K}^h)\|_{\mathbf{Y}}\le(1+\gamma)\|(\mathbf{u},\mathbf{J})-(\mathbf{w}^h,\mathbf{L}^h)\|_{\mathbf{Y}}+\gamma\|\mathbf{g}-\mathbf{g}^h\|_{\mathbf{Y}_{1,\Gamma}},$$

from which (3.11) follows by taking infima. ◻

*Step* 4.

$$\begin{aligned}&\|(\mathbf{u},\mathbf{J})-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}}\\ &\quad\le\left(1+\frac{\|a_0\|+(\nu+\nu^h)\|a_1\|}{\alpha-\nu\|a_1\|}\right)\left(1+\frac{\|b\|}{\beta}\right)\\ &\qquad\left((1+\gamma)\inf_{(\mathbf{w}^h,\mathbf{L}^h)\in\mathbf{Y}^h}\|(\mathbf{u},\mathbf{J})-(\mathbf{w}^h,\mathbf{L}^h)\|_{\mathbf{Y}}+\gamma\|\mathbf{g}-\mathbf{g}^h\|_{\mathbf{Y}_{1,\Gamma}}\right)\\ &\qquad+\frac{\theta^h\|b\|}{\alpha-\nu\|a_1\|}\inf_{(r^h,\chi^h)\in M^h}\|(p,\phi)-(r^h,\chi^h)\|_M.\end{aligned}\tag{3.12}$$

*Proof.* Combine the estimates in Steps 1–3. ◻

*Step* 5.

$$\begin{aligned}\|(p,\phi)-(p^h,\phi^h)\|_M &\le \frac{\|a_0\|+(\nu+\nu^h)\|a_1\|}{\beta}\|(\mathbf{u},\mathbf{J})-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}}\\ &\quad+\left(1+\frac{\|b\|}{\beta}\right)\inf_{(r^h,\chi^h)\in M^h}\|(p,\phi)-(r^h,\chi^h)\|_M.\end{aligned}\tag{3.13}$$

*Proof.* Let $(r^h,\chi^h)\in M^h$. Recalling (3.6), we see that for all $(\mathbf{v}^h,\mathbf{K}^h)\in\mathbf{X}^h$, we have

$$
\begin{aligned}
&b\big((\mathbf{v}^h,\mathbf{K}^h),(r^h-p^h,\chi^h-\phi^h)\big)\\
&\quad= b\big((\mathbf{v}^h,\mathbf{K}^h),(r^h-p,\chi^h-\phi)\big)+b\big((\mathbf{v}^h,\mathbf{K}^h),(p-p^h,\phi-\phi^h)\big)\\
&\quad= b\big((\mathbf{v}^h,\mathbf{K}^h),(r^h-p,\chi^h-\phi)\big)-a_0\big((\mathbf{u}-\mathbf{u}^h,\mathbf{J}-\mathbf{J}^h),(\mathbf{v}^h,\mathbf{K}^h)\big)\\
&\qquad\qquad -a_1\big((\mathbf{u}-\mathbf{u}^h,\mathbf{J}-\mathbf{J}^h),(\mathbf{u},\mathbf{J}),(\mathbf{v}^h,\mathbf{K}^h)\big)\\
&\qquad\qquad -a_1\big((\mathbf{u}^h,\mathbf{J}^h),(\mathbf{u}-\mathbf{u}^h,\mathbf{J}-\mathbf{J}^h),(\mathbf{v}^h,\mathbf{K}^h)\big)\\
&\quad\le \Big(\|b\|\,\|(r^h,\chi^h)-(p,\phi)\|_M\\
&\qquad\qquad +\big(\|a_0\|+(\nu+\nu^h)\|a_1\|\big)\|(\mathbf{u},\mathbf{J})-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}}\Big)\|(\mathbf{v}^h,\mathbf{K}^h)\|_{\mathbf{Y}}
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\sup_{(\mathbf{v}^h,\mathbf{K}^h)\in\mathbf{X}^h}\frac{b\big((\mathbf{v}^h,\mathbf{K}^h),(r^h-p^h,\chi^h-\phi^h)\big)}{\|(\mathbf{v}^h,\mathbf{K}^h)\|_{\mathbf{Y}}}\\
&\qquad\le \|b\|\,\|(r^h,\chi^h)-(p,\phi)\|_M+\big(\|a_0\|+(\nu+\nu^h)\|a_1\|\big)\|(\mathbf{u},\mathbf{J})-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}}.
\end{aligned}
$$

By Assumption $A_1$, the supremum on the left-hand side is bounded from below by $\beta\,\|(r^h,\chi^h)-(p^h,\phi^h)\|_M$, and it follows that

$$
\begin{aligned}
&\|(r^h,\chi^h)-(p^h,\phi^h)\|_M\\
&\qquad\le \frac{\|b\|}{\beta}\,\|(r^h,\chi^h)-(p,\phi)\|_M+\frac{\|a_0\|+(\nu+\nu^h)\|a_1\|}{\beta}\,\|(\mathbf{u},\mathbf{J})-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
\|(p,\phi)-(p^h,\phi^h)\|_M \;&\le\; \|(p,\phi)-(r^h,\chi^h)\|_M+\|(r^h,\chi^h)-(p^h,\phi^h)\|_M\\
&\le \frac{\|a_0\|+(\nu+\nu^h)\|a_1\|}{\beta}\,\|(\mathbf{u},\mathbf{J})-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}}+\left(1+\frac{\|b\|}{\beta}\right)\|(p,\phi)-(r^h,\chi^h)\|_M
\end{aligned}
$$

and hence the estimate (3.13). ◻

With (3.12) and (3.13), the assertions of Theorem 3.4 are proved.

**4. Implementation and numerical experiments.** Here we describe a simple iteration scheme for the solution of Problem $P_1^h$ and possible choices for finite-element spaces and approximate boundary values. We also discuss expected convergence rates and illustrate the performance of our method with numerical experiments.

**4.1. Iteration scheme.** Several iterative methods suggest themselves naturally for the numerical solution of Problem $P_1^h$. The following scheme is simple and efficient.

Given $(\mathbf{u}_0^h,\mathbf{J}_0^h)\in\mathbf{Y}^h$ with $\mathbf{u}_0^h|_\Gamma=\mathbf{g}^h$, for $n\in\mathbb{N}$ find $(\mathbf{u}_n^h,\mathbf{J}_n^h)\in\mathbf{Y}^h$ with $\mathbf{u}_n^h|_\Gamma=\mathbf{g}^h$ and $(p_n^h,\phi_n^h)\in M^h$ such that

$$
\begin{aligned}
&a_0\big((\mathbf{u}_n^h,\mathbf{J}_n^h),(\mathbf{v}^h,\mathbf{K}^h)\big)+a_1\big((\mathbf{u}_{n-1}^h,\mathbf{J}_{n-1}^h),(\mathbf{u}_n^h,\mathbf{J}_n^h),(\mathbf{v}^h,\mathbf{K}^h)\big)\\
&\qquad +b\big((\mathbf{v}^h,\mathbf{K}^h),(p_n^h,\phi_n^h)\big)=\int_\Omega \mathbf{F}\cdot\mathbf{v}^h+\int_\Omega \mathbf{E}\cdot\mathbf{K}^h\qquad \forall\,(\mathbf{v}^h,\mathbf{K}^h)\in\mathbf{X}^h
\end{aligned}
\tag{4.1}
$$

and

$$
b\big((\mathbf{u}_n^h,\mathbf{J}_n^h),(q^h,\psi^h)\big)=\int_\Gamma j\,\psi^h\qquad \forall\,(q^h,\psi^h)\in M^h.
\tag{4.2}
$$

It is a routine exercise to prove that this iteration scheme is well posed for every initial guess $(\mathbf{u}_0^h, \mathbf{J}_0^h)$. Moreover, if $(\mathbf{u}^h, \mathbf{J}^h, p^h, \phi^h)$ is a solution of Problem $P_1^h$ and if $(\mathbf{u}_n^h, \mathbf{J}_n^h, p_n^h, \phi_n^h)$ is a sequence of iterates satisfying (4.1) and (4.2), then

$$\|(\mathbf{u}^h, \mathbf{J}^h) - (\mathbf{u}_n^h, \mathbf{J}_n^h)\|_{\mathbf{Y}} \le \frac{\nu^h \|a_1\|}{\alpha} \|(\mathbf{u}^h, \mathbf{J}^h) - (\mathbf{u}_{n-1}^h, \mathbf{J}_{n-1}^h)\|_{\mathbf{Y}}$$

and

$$\|(p^h, \phi^h) - (p_n^h, \phi_n^h)\|_M \le \frac{\alpha + \|a_0\| + \nu_{n-1}^h \|a_1\|}{\beta} \, \frac{\nu^h \|a_1\|}{\alpha} \, \|(\mathbf{u}^h, \mathbf{J}^h) - (\mathbf{u}_{n-1}^h, \mathbf{J}_{n-1}^h)\|_{\mathbf{Y}}$$

for all $n \in \mathbb{N}$, where $\nu^h := \|(\mathbf{u}^h, \mathbf{J}^h)\|_{\mathbf{Y}}$, $\nu_{n-1}^h := \|(\mathbf{u}_{n-1}^h, \mathbf{J}_{n-1}^h)\|_{\mathbf{Y}}$, and $\alpha$ and $\beta$ are the constants of Lemma 2.2(b) and Assumption $A_1$. In the situation of Theorem 3.2(b), that is, if Problem $P_1^h$ has a (necessarily unique) solution $(\mathbf{u}^h, \mathbf{J}^h, p^h, \phi^h)$ with $\nu^h = \|(\mathbf{u}^h, \mathbf{J}^h)\|_{\mathbf{Y}} < \alpha / \|a_1\|$, we infer the convergence of the iteration scheme, along with the usual a priori and a posteriori error estimates. In fact, we have

$$\|(\mathbf{u}^h, \mathbf{J}^h) - (\mathbf{u}_n^h, \mathbf{J}_n^h)\|_{\mathbf{Y}} \le \frac{\lambda^h}{1 - \lambda^h} \|(\mathbf{u}_n^h, \mathbf{J}_n^h) - (\mathbf{u}_{n-1}^h, \mathbf{J}_{n-1}^h)\|_{\mathbf{Y}}$$

and

$$\|(\mathbf{u}^h, \mathbf{J}^h) - (\mathbf{u}_n^h, \mathbf{J}_n^h)\|_{\mathbf{Y}} \le \frac{(\lambda^h)^n}{1 - \lambda^h} \|(\mathbf{u}_1^h, \mathbf{J}_1^h) - (\mathbf{u}_0^h, \mathbf{J}_0^h)\|_{\mathbf{Y}}$$

for all $n \in \mathbb{N}$, where

$$\lambda^h := \frac{\nu^h \|a_1\|}{\alpha} < 1.$$

A question we need to address at this point is whether the nonlocal operators $\mathcal{B}$ and $\mathcal{P}$ intervening in the definition of the forms $a_1$ and $b$ (see the beginning of section 2) will adversely affect the sparsity of the matrix associated with the linear equations (4.1) and (4.2)—assuming, of course, that basis functions with small support are used to span the spaces $\mathbf{Y}^h$ and $M^h$. Fortunately, there is no problem: the operator $\mathcal{B}$ does not affect the sparsity at all since we are lagging the first argument of the trilinear form $a_1$, and the projection $\mathcal{P}$ can be eliminated by rewriting (4.2) in the form

$$b_0\big((\mathbf{u}_n^h, \mathbf{J}_n^h), (q^h, \psi^h)\big) = \int_\Gamma j\, \psi^h - \frac{1}{|\Omega|} \Big(\int_\Gamma \mathbf{g}^h \cdot \mathbf{n}\Big)\Big(\int_\Omega q^h\Big) \qquad \forall\, (q^h, \psi^h) \in M^h,$$

where $b_0$ is defined just like $b$ except for the omission of the projection $\mathcal{P}$. (Recall Remark 2.1(c), and note that $\mathcal{P}$ intervenes only in (4.2) and only if the approximate boundary values $\mathbf{g}^h$ do not satisfy the condition $\int_\Gamma \mathbf{g}^h \cdot \mathbf{n} = 0$.)

**4.2. Finite-element spaces.** The error estimate (3.5) and classical approximation theory of finite-element spaces suggest that Problem $P_1^h$ will be a $k$th order approximation of Problem $P_1$ (for some $k \in \mathbb{N}$) if we use appropriate piecewise polynomial approximations of degree $k$ for the velocity and electric potential, and of degree $k-1$ for the pressure and current density. Since the convergence and error analysis in section 3 is partially based on Assumption $A_1$ (the uniform LBB-condition), the velocity-pressure pairs $(\mathbf{X}_1^h, M_1^h)$ and the current-potential pairs $(\mathbf{X}_2^h, M_2^h)$ should be chosen so that the inf-sup conditions

$$\inf_{q^h \in M_1^h} \sup_{\mathbf{v}^h \in \mathbf{X}_1^h} \frac{\int_\Omega (\nabla \cdot \mathbf{v}^h) q^h}{\|\mathbf{v}^h\|_{\mathbf{X}_1} \|q^h\|_{M_1}} \ge \beta_1 \quad \forall\, h \in I \tag{4.3}$$

and

$$\inf_{\psi^h \in M_2^h} \sup_{\mathbf{K}^h \in \mathbf{X}_2^h} \frac{\int_\Omega \mathbf{K}^h \cdot (\nabla \psi^h)}{\|\mathbf{K}^h\|_{\mathbf{X}_2} \|\psi^h\|_{M_2}} \geq \beta_2 \quad \forall\, h \in I \tag{4.4}$$

hold with positive constants $\beta_1$ and $\beta_2$, respectively; Assumption $A_1$ would then be satisfied with $\beta = \min\{\beta_1, \beta_2\}$.

The condition (4.3) is familiar from the theory of the Stokes and Navier–Stokes equations; it severely limits the choices for $(\mathbf{X}_1^h, M_1^h)$, but by now many suitable pairs of spaces are known. Assuming, for simplicity, that the domain $\Omega$ is a polyhedron and that we are given a regular family of finite-element decompositions of $\overline{\Omega}$ into simplicial or rectangular elements, we may approximate $H^1(\Omega)$ and $L^2(\Omega)$ by the spaces $\mathcal{Q}^h$ and $\mathcal{L}^h$ of continuous piecewise quadratics (or triquadratics) and continuous piecewise linears (or trilinears) on tetrahedra (or rectangular parallelepipeds), respectively, and then set $\mathbf{Y}_1^h := (\mathcal{Q}^h)^3$, $\mathbf{X}_1^h := \mathbf{Y}_1^h \cap \mathbf{X}_1 = (\mathcal{Q}^h \cap H_0^1(\Omega))^3$, and $M_1^h := \mathcal{L}^h/\mathbb{R}$. These so-called Taylor–Hood-type velocity-pressure pairs are widely used in computational fluid dynamics and well understood (see, for example, [2, Chapter VI.6], [6, Chapter 3]), or [22, 23]); in particular, they satisfy the inf-sup condition (4.3) under mild assumptions on the geometry of the underlying triangulations.

Fortunately, the spaces $\mathbf{Y}_1^h$, as defined above, do satisfy the second crucial hypothesis of section 3, namely, Assumption $A_2$. This is a by-product of the work of Scott and Zhang [18], who constructed generalized interpolants in spaces of continuous piecewise polynomials (associated with regular triangulations of polyhedral domains) that exhibit optimal approximation properties while preserving homogeneous Dirichlet boundary values. In particular, these interpolants are uniformly bounded projections from $H^1(\Omega)$ onto $\mathcal{Q}^h$ that leave $H_0^1(\Omega)$ invariant—as required in Assumption $A_2$. (The construction in [18] is based on simplicial triangulations but carries over, mutatis mutandis, to the case of rectangular elements.)

The inf-sup condition (4.4) for the current-potential pairs $(\mathbf{X}_2^h, M_2^h)$ is trivially satisfied whenever $\mathbf{X}_2^h$ contains the gradients of the functions in $M_2^h$. In view of our choices for the velocity-pressure pairs, it is natural to set $M_2^h := \mathcal{Q}^h/\mathbb{R}$. The space $\mathbf{X}_2^h$ should then contain the gradients of all continuous piecewise quadratics (on tetrahedra) or triquadratics (on rectangular parallelepipeds). Thus, in the case of a *simplicial* triangulation, we choose for $\mathbf{X}_2^h$ the subspace of $\mathbf{L}^2(\Omega)$ comprised of all vector functions on $\Omega$ whose components are (generally discontinuous) piecewise linears. When using *rectangular* elements, we let $\mathbf{X}_2^h := \mathbf{X}_{2,1}^h \times \mathbf{X}_{2,2}^h \times \mathbf{X}_{2,3}^h$ and choose for $\mathbf{X}_{2,i}^h$ the tensor product of the space of (generally discontinuous) piecewise linears in the $i$th variable and the space of continuous piecewise biquadratics in the remaining two variables. Note that in any case, $\mathbf{X}_2^h \supset (\mathcal{L}^h)^3$. Although the inf-sup condition (4.4) arises naturally in connection with so-called primal mixed methods (see, for example, [15, Section 12]), it seems that not very many suitable pairs of spaces $(\mathbf{X}_2^h, M_2^h)$ have been devised.

We mention that the quotient spaces $M_1^h = \mathcal{L}^h/\mathbb{R}$ and $M_2^h = \mathcal{Q}^h/\mathbb{R}$ are most easily realized by dropping one basis function each from $\mathcal{L}^h$ and $\mathcal{Q}^h$ (which amounts to setting the pressure and electric potential equal to zero at one node each of the triangulation). For purposes of error analysis, however, the computed pressures and electric potentials should be normalized, in a postprocessing step, to have mean zero on $\Omega$.

All the preceding observations can be generalized to allow for regular decompositions of $\overline{\Omega}$ into convex hexahedra (instead of rectangular parallelepipeds) or even more general isoparametric elements (if $\Omega$ is not a polyhedron). However, the fact that the local basis functions are then isoparametric images of polynomials on a reference element (rather than genuine polynomials) necessitates some care in the choice of $\mathbf{X}_2^h$ (see [15, Remark 2.4]).

**4.3. Approximate boundary data.** In order for Problem $P_1^h$ to be a viable approximation of Problem $P_1$, we need to choose approximations $\mathbf{g}^h \in \mathbf{Y}_{1,\Gamma}^h$ for the boundary values $\mathbf{g}$ of the fluid velocity such that $\|\mathbf{g}-\mathbf{g}^h\|_{\mathbf{Y}_{1,\Gamma}} \to 0$ as $h \to 0$. In view of the error estimate (3.5), the choice of $\mathbf{g}^h$ would be *optimal* if

$$\|\mathbf{g}-\mathbf{g}^h\|_{\mathbf{Y}_{1,\Gamma}} = O\Big(\inf_{\mathbf{v}^h\in\mathbf{Y}_1^h} \|\mathbf{u}-\mathbf{v}^h\|_{\mathbf{Y}_1}\Big) \quad \text{as } h \to 0, \tag{4.5}$$

where $\mathbf{u}$ is the velocity component of the solution of Problem $P_1$.

Recall that it is *not* necessary to enforce the compatibility condition $\int_\Gamma \mathbf{g}^h \cdot \mathbf{n} = 0$. If $\mathbf{g}$ is continuous, we can simply work with the Lagrange interpolant of $\mathbf{g}$ in $\mathbf{Y}_{1,\Gamma}^h$, although in general this choice will fail to be optimal in the sense of (4.5) unless $\mathbf{g}$ is sufficiently smooth. An optimal choice, feasible for *arbitrary* $\mathbf{g} \in \mathbf{Y}_{1,\Gamma}$, is to let $\mathbf{g}^h := \Pi_\Gamma^h \mathbf{g}$, where $\Pi_\Gamma^h$ is the projection operator of Remark 3.1(b). With our choice of finite-element spaces, where the operators $\Pi^h$ of Assumption $A_2$ are Scott–Zhang-type interpolants, the operators $\Pi_\Gamma^h$ are locally defined boundary interpolants and thus, easy to compute (see [18, Section 5]).

**4.4. Predicted convergence rates.** Our error estimate (3.5) and standard results of approximation theory for finite-element spaces (which can be found, for example, in [3]), imply that with the above choices of finite elements and approximate boundary values, Problem $P_1^h$ is a second-order approximation of Problem $P_1$. Specifically, if the assumptions of Corollary 3.3 are satisfied, if $(\mathbf{u},\mathbf{J},p,\phi)$ and $(\mathbf{u}^h,\mathbf{J}^h,p^h,\phi^h)$ denote the (unique) solutions of Problems $P_1$ and $P_1^h$, respectively, and if $(\mathbf{u},\mathbf{J},p,\phi)$ belongs to the space $\mathbf{H}^{s+1}(\Omega)\times\mathbf{H}^s(\Omega)\times H^s(\Omega)\times H^{s+1}(\Omega)$ for some $s\in[0,2]$, then

$$\|(\mathbf{u},\mathbf{J})-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}} + \|(p,\phi)-(p^h,\phi^h)\|_M = O(h^s) \quad \text{as } h \to 0.$$

**4.5. Numerical experiments.** We implemented the method, as described, on the unit cube, $\Omega=(0,1)^3$. Following the preceding general remarks about suitable finite-element spaces, we decomposed $\overline{\Omega}$ into cubes of equal size and used standard triquadratic Lagrange elements for the velocity and electric potential, standard trilinear Lagrange elements for the pressure. For the $i$th component of the current density, we chose Hermite elements with 9 nodes, namely, those nodes of the principal lattice of degree two that are not on faces perpendicular to the $i$th coordinate axis; two degrees of freedom were associated with each such node $a$, namely, $f \mapsto f(a)$ and $f \mapsto \partial_i f(a)$. (Recall that the space $\mathbf{X}_{2,i}^h$ should be the tensor product of the space of generally discontinuous piecewise linears in the $i$th variable and the space of continuous piecewise biquadratics in the remaining two variables. Instead of the above 9-node Hermite elements, we could, of course, use 18-node Lagrange elements to construct a basis for this space, but we would then be unable to utilize the same nodes as for the velocity, pressure, and potential.) For simplicity, Lagrange interpolation was used to approximate the boundary values of the fluid velocity.

We solved Problem $P_1^h$ with the iterative method described at the beginning of this section. The procedure was stopped once the distance (in $\mathbf{Y}\times M$) between consecutive iterates dropped below a given tolerance ($10^{-8}$ in the experiments below). Stiffness matrices and load vectors were computed with a 27-point Gaussian quadrature rule

TABLE 4.1

| $h$ | $\mathbf{H}^1$-error in $\mathbf{u}$ / $\mathbf{H}^1$-rate in $\mathbf{u}$ | $\mathbf{L}^2$-error in $\mathbf{J}$ / $\mathbf{L}^2$-rate in $\mathbf{J}$ | $L^2$-error in $p$ / $L^2$-rate in $p$ | $H^1$-error in $\phi$ / $H^1$-rate in $\phi$ |
|---|---|---|---|---|
| $\frac{1}{3}$ | 0.1065E-00 | 0.4510E-01 | 0.5954E-02 | 0.5712E-01 |
| | 2.00 | 1.98 | 2.62 | 1.98 |
| $\frac{1}{4}$ | 0.5996E-01 | 0.2549E-01 | 0.2806E-02 | 0.3230E-01 |
| | 2.00 | 1.99 | 2.42 | 1.99 |
| $\frac{1}{5}$ | 0.3839E-01 | 0.1635E-01 | 0.1637E-02 | 0.2072E-01 |
| | 2.00 | 1.99 | 2.29 | 2.00 |
| $\frac{1}{6}$ | 0.2667E-01 | 0.1137E-01 | 0.1078E-02 | 0.1440E-01 |

on the reference element. In each iteration and at each node, the Biot–Savart formula (1.5), with $\mathbf{J}$ replaced by the approximate current density from the previous iteration, was numerically evaluated with 8-point Gaussian quadrature rules, avoiding the weak singularity of the integral. The sparse linear systems resulting from (4.1) and (4.2) were solved directly, using a standard linear-algebra package.

To test the predicted quadratic convergence of our method, we contrived a simple (albeit unphysical) example, where an exact solution of Problem $P_0$ is given by

$$\begin{aligned}\mathbf{u}(x,y,z) &:= -\pi\cos(\pi x)\exp(-s)\mathbf{t},\\ \mathbf{J}(x,y,z) &:= 2\sin(\pi x)(1-s)\exp(-s)\mathbf{i} + \pi\cos(\pi x)\exp(-s)\mathbf{r},\\ p(x,y,z) &:= -\frac{1}{2}\sin^2(\pi x)\, s\exp(-2s),\\ \phi(x,y,z) &:= \frac{2}{\pi}\cos(\pi x),\end{aligned}$$

with $\mathbf{i} := (1,0,0)$, $\mathbf{r} := (0, y-\frac{1}{2}, z-\frac{1}{2})$, $\mathbf{t} := (0, \frac{1}{2}-z, y-\frac{1}{2})$, and $s := |\mathbf{r}|^2$. The associated magnetic field is

$$\mathbf{B}(x,y,z) := \mathbf{i} + \sin(\pi x)\exp(-s)\mathbf{t}.$$

Since $\mathbf{u}$ and $\mathbf{B}$ are divergence-free and $\nabla\times\mathbf{B} = \mathbf{J}$, the above is indeed a smooth solution of Problem $P_0$ provided that the parameters $\eta$, $\rho$, $\sigma$, and $\mu$ are all unity and the data $\mathbf{F}$, $\mathbf{E}$, $\mathbf{g}$, $\mathbf{J}_{\text{ext}}$, and $\mathbf{B}_{\text{ext}}$ are chosen in the obvious way, namely,

$$\begin{aligned}\mathbf{F} &:= -\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mathbf{J}\times\mathbf{B},\\ \mathbf{E} &:= \mathbf{J} + \nabla\phi - \mathbf{u}\times\mathbf{B},\end{aligned}$$

and

$$\mathbf{g} := \mathbf{u}|_{\Gamma}, \quad \mathbf{J}_{\text{ext}} := \mathbf{J}|_{\mathbb{R}^3\setminus\overline{\Omega}}, \quad \mathbf{B}_{\text{ext}} := \mathbf{i}.$$

The magnetic field induced by $\mathbf{J}_{\text{ext}}$ can be obtained as $\mathbf{B} - \mathbf{B}_{\text{ext}} - \mathcal{B}(\mathbf{J}|_{\Omega})$ (instead of integrating the Biot–Savart formula over $\mathbb{R}^3\setminus\overline{\Omega}$).

We computed four approximations $(\mathbf{u}^{h_i}, \mathbf{J}^{h_i}, p^{h_i}, \phi^{h_i})$ of $(\mathbf{u}, \mathbf{J}, p, \phi)$, using rather coarse grids with 27 ($h_0 = \frac{1}{3}$), 64 ($h_1 = \frac{1}{4}$), 125 ($h_2 = \frac{1}{5}$), and 216 ($h_3 = \frac{1}{6}$) elements, respectively. (This corresponds to 2318, 4985, 9170, and 15215 degrees of freedom,

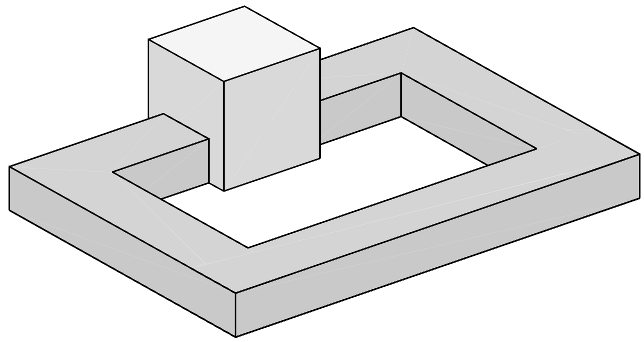

FIG. 4.1. *Fluid region and external conductor.*

respectively, not counting the magnetic field.) Table 4.1 shows the discretization errors $e_i$ and the convergence rates $s_i := \ln(\frac{e_{i-1}}{e_i})/\ln(\frac{h_{i-1}}{h_i})$. Despite the coarseness of the grids, the rates $s_i$ are found to be in good agreement with the predicted asymptotic rate, $s = 2$ (except for the initially faster-than-predicted convergence in the pressure).

As a somewhat less contrived model problem, consider a fluid in $\Omega$, driven by a uniform, externally generated current $\mathbf{J}_{\text{ext}}$ that enters and leaves the fluid through electrodes attached to two opposite faces of the cube; the current loop is closed via an external conductor of uniform, rectangular cross section (see Figure 4.1). In the absence of any other driving mechanisms, we have $\mathbf{F} = 0$, $\mathbf{E} = 0$, $\mathbf{g} = 0$, and $\mathbf{B}_{\text{ext}} = 0$. Let $|\mathbf{J}_{\text{ext}}| = 100$, and assume, for simplicity, that the parameters $\eta$, $\rho$, $\sigma$, and $\mu$ are all unity. Figures 4.2–4.5 depict the approximate current density, electric potential, velocity, and pressure fields, as computed on a grid of 125 cubic elements. The magnetic field induced by $\mathbf{J}_{\text{ext}}$ was obtained by numerically integrating the Biot–Savart formula over the external conductor.

As expected, all the fields are symmetric about the plane $x = \frac{1}{2}$. The current and potential fields are also fairly symmetric about the plane $y = \frac{1}{2}$, but not so the velocity and pressure fields. Their pronounced asymmetry is due to the magnetic field generated by the current in the external conductor. To reveal this effect more clearly, we repeated the computation, but this time suppressing the magnetic field contribution from the external current. Figures 4.6 and 4.7 show the resulting velocity and pressure fields, now perfectly symmetric about the plane $y = \frac{1}{2}$. A comparison of Figures 4.4 and 4.6, which use the same scale, and of Figures 4.5 and 4.7 reveals yet another effect: the presence of the externally generated magnetic field greatly magnifies the fluid motion and pressure gradients (by way of increasing the Lorentz forces). To substantiate this observation, we performed a third computation, adding a uniform magnetic field $\mathbf{B}_{\text{ext}}$, downward-pointing, of magnitude $|\mathbf{B}_{\text{ext}}| = 10$, to the fields induced by $\mathbf{J}$ and $\mathbf{J}_{\text{ext}}$. In the fluid region, this additional magnetic field reinforces the one induced by $\mathbf{J}_{\text{ext}}$. The ensuing velocity and pressure fields are depicted in Figures 4.8 and 4.9; the scale in Figure 4.8 is the same as that in Figures 4.4 and 4.6. This experiment, although of limited physical relevance, underscores one of the main points of our analysis: that even in the simplest MHD flow problems, it is critically important to account for the fluid's electromagnetic interaction with the outside world.

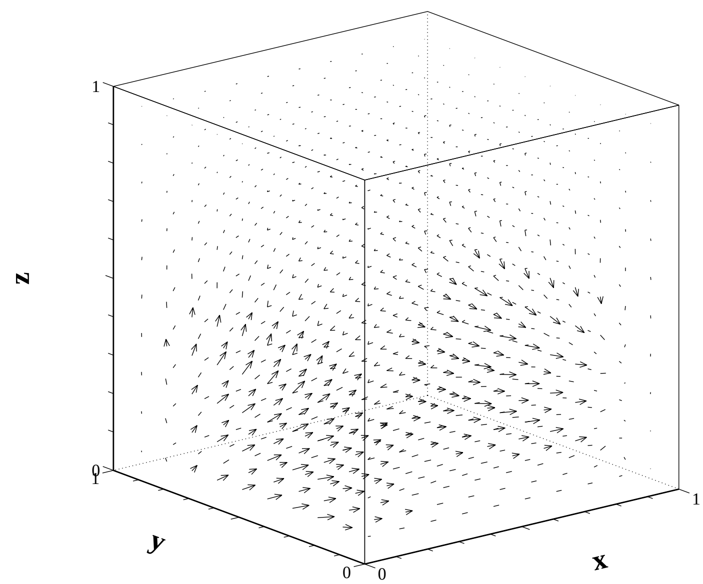


FIG. 4.2. *Current density.*

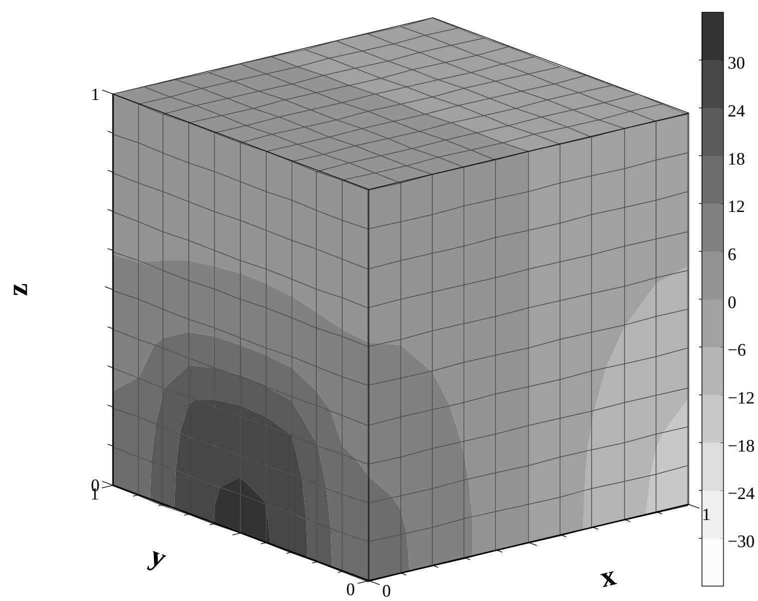


FIG. 4.3. *Electric potential.*

Despite their limited scope and somewhat academic nature, the above experiments allow the conclusion that our method and implementation work adequately and efficiently. The simple iteration scheme that was employed required between five

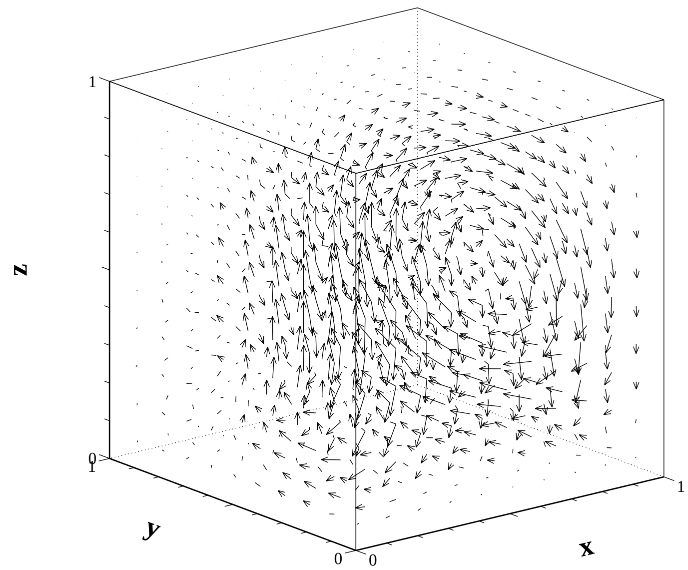


FIG. 4.4. *Velocity.*

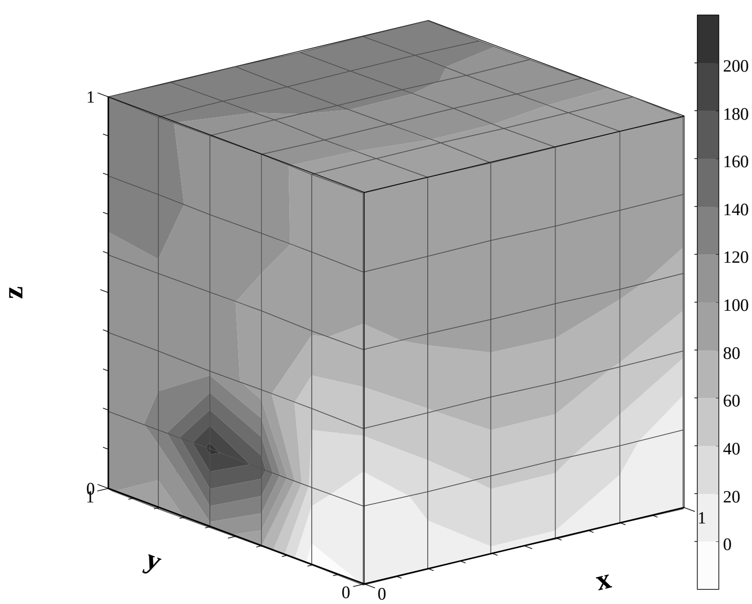


FIG. 4.5. *Pressure.*

and ten iterations to achieve the desired accuracy (that is, a distance in $\mathbf{Y} \times M$ of less than $10^{-8}$ between consecutive iterates). Of course, global convergence of the scheme is guaranteed only under a small-data assumption, and it is exceedingly difficult to pinpoint "how small is small enough" (our a priori estimates are clearly quite pes-

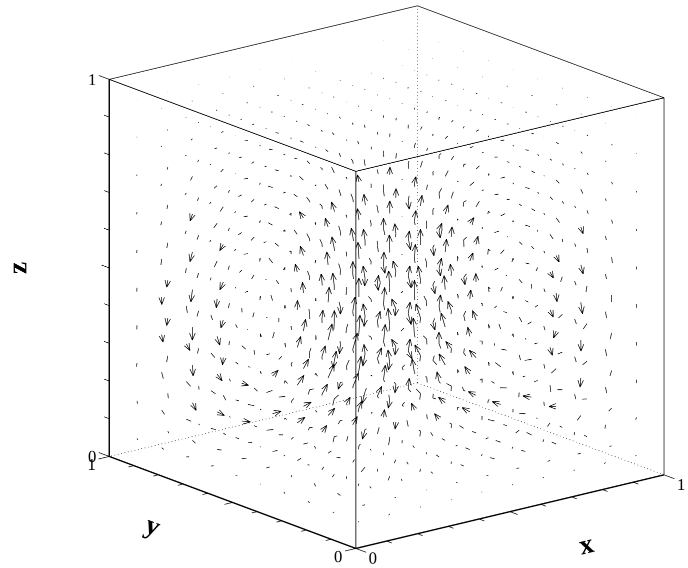


Fig. 4.6. *Velocity (magnetic field of external current neglected).*

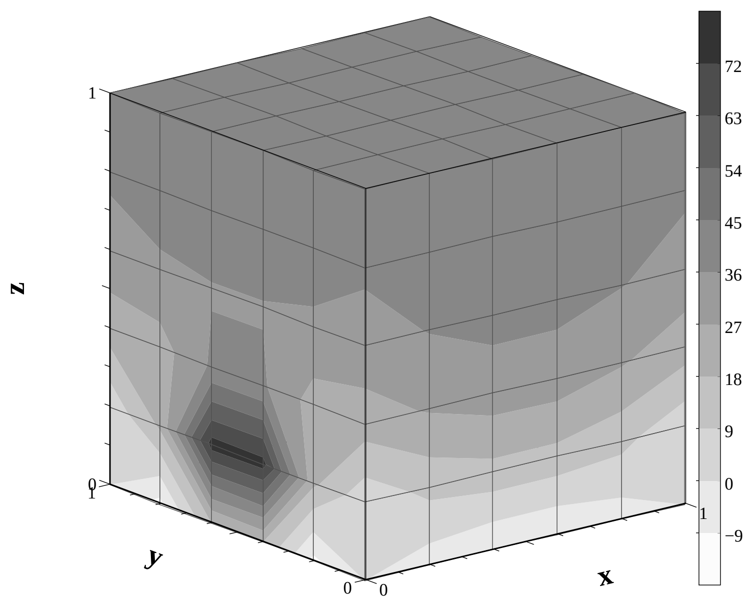


Fig. 4.7. *Pressure (magnetic field of external current neglected).*

simistic). In the experiments, data and parameters were roughly of order one (some data even larger), and no convergence problems were encountered. Obvious limitations of our method arise from the expected instability of steady flow in the case of

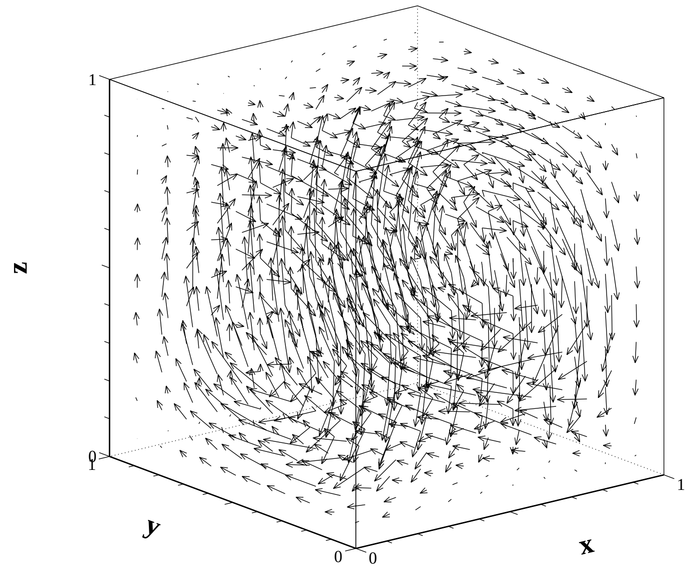


FIG. 4.8. *Velocity* (*with additional applied magnetic field*).

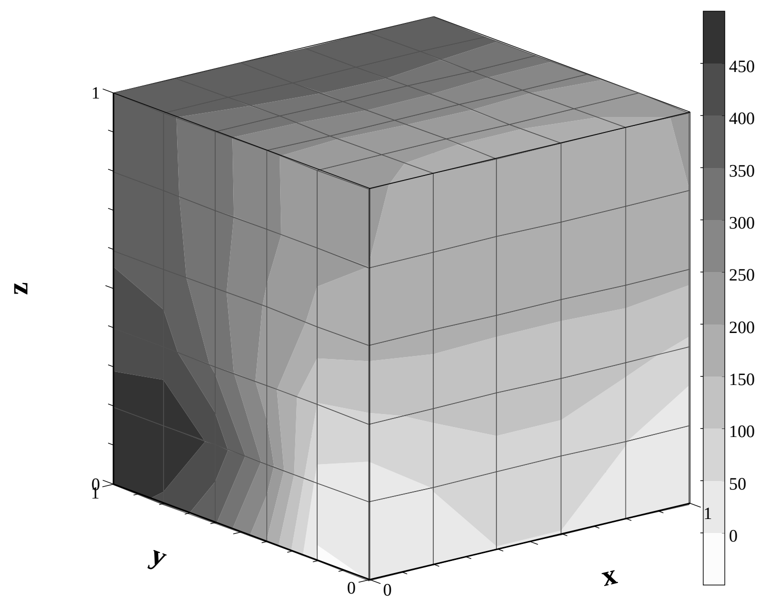


FIG. 4.9. *Pressure* (*with additional applied magnetic field*).

high Reynolds (and/or magnetic Reynolds) numbers.

In terms of speed, the present implementation leaves much room for improvement, for example, through the use of iterative (rather than direct) solvers or multilevel methods. Also, the expensive computation of the induced magnetic field via evaluation of the Biot–Savart integral (1.5) could be speeded up by exploiting fast multiple or multilevel methods. A viable alternative to integrating the Biot–Savart formula may be to solve the div-curl system (1.7), for example, using a variational approach. Furthermore, it may be possible to devise more convenient pairs of finite-element spaces for the current density and electric potential. The pair presently employed satisfies the crucial LBB-condition (4.4) almost by definition but necessitates the use of somewhat nonstandard elements for the current density. Finally, parts of the code are inherently parallelizable—a feature that would have to be exploited in order to deal with industrial-strength applications. These issues, along with various extensions of our method, systematic performance tests of the numerical implementation, and more physically rooted computer experiments are the subject of ongoing research and will be discussed elsewhere.

## REFERENCES


[1] O. Besson, J. Bourgeois, P.-A. Chevalier, J. Rappaz, and R. Touzani, *Numerical modelling of electromagnetic casting processes*, J. Comput. Phys., 92 (1991), pp. 482–507.

[2] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.

[3] P. G. Ciarlet, *Basic error estimates for elliptic problems*, in Handbook of Numerical Analysis, Vol. II, Finite Element Methods, Part 1, P. G. Ciarlet and J. L. Lions, eds., North-Holland, Amsterdam, 1991, pp. 17–351.

[4] G. Duvaut and J. L. Lions, *Inéquations en thermoélasticité et magnétohydrodynamique*, Arch. Rational Mech. Anal., 46 (1972), pp. 241–279.

[5] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations, Theory and Algorithms*, Springer Series in Computational Mathematics, Vol. 5, Springer-Verlag, New York, 1986.

[6] M. D. Gunzburger, *Finite Element Methods for Viscous Incompressible Flows*, Academic Press, Boston, 1989.

[7] M. D. Gunzburger, A. J. Meir, and J. S. Peterson, *On the existence, uniqueness, and finite element approximation of solutions of the equations of stationary, incompressible magnetohydrodynamics*, Math. Comput., 56 (1991), pp. 523–563.

[8] W. F. Hughes and F. J. Young, *The Electromagnetodynamics of Fluids*, John Wiley, New York, 1966.

[9] O. A. Ladyzhenskaya and V. A. Solonnikov, *Solutions of some non-stationary problems of magnetohydrodynamics for a viscous incompressible fluid*, Trudy Mat. Inst. Steklov., 59 (1960), pp. 115–173 (in Russian).

[10] O. A. Ladyzhenskaya and V. A. Solonnikov, *The linearization principle and invariant manifolds for problems of magnetohydrodynamics*, J. Soviet Math., 8 (1977), pp. 384–422.

[11] A. J. Meir and P. G. Schmidt, *A velocity-current formulation for stationary MHD flow*, Appl. Math. Comp., 65 (1994), pp. 95–109.

[12] A. J. Meir and P. G. Schmidt, *Variational methods for stationary MHD flow under natural interface conditions*, Nonlinear Anal., 26 (1996), pp. 659–689.

[13] J. Rappaz and R. Touzani, *On a two-dimensional magnetohydrodynamic problem*, I. *Modelling and analysis*, RAIRO Modél. Math. Anal. Numér., 26 (1992), pp. 347–364.

[14] J. Rappaz and R. Touzani, *On a two-dimensional magnetohydrodynamic problem*, II. *Numerical analysis*, RAIRO Modél. Math. Anal. Numér., 30 (1996), pp. 215–235.

[15] J. E. Roberts and J.-M. Thomas, *Mixed and hybrid methods*, in Handbook of Numerical Analysis, Vol. II, Finite Element Methods, Part 1, P. G. Ciarlet and J. L. Lions, eds., North-Holland, Amsterdam, 1991, pp. 523–639.

[16] V. N. Samokhin, *Existence of a solution of a modification of a system of equations of magnetohydrodynamics*, Math. USSR-Sb., 72 (1992), pp. 373–385.

[17] V. N. Samokhin, *On stationary problems in magnetohydrodynamics of non-Newtonian media*, Siberian Math. J., 33 (1992), pp. 654–662.

[18] L. R. Scott and S. Zhang, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp., 54 (1990), pp. 483–493.

[19] M. Sermange and R. Temam, *Some mathematical questions related to the MHD equations*,

Comm. Pure Appl. Math, 36 (1983), pp. 635–664.

[20] J. A. Shercliff, *A Textbook of Magnetohydrodynamics*, Pergamon Press, Oxford, 1965.

[21] V. A. Solonnikov, *Some stationary boundary value problems of magnetohydrodynamics*, Trudy Mat. Inst. Steklov., 59 (1960), pp. 174–187 (in Russian).

[22] R. Stenberg, *On some three-dimensional finite elements for incompressible media*, Comput. Methods Appl. Mech. Engrg., 63 (1987), pp. 261–269.

[23] R. Stenberg, *Error analysis of some finite element methods for the Stokes problem*, Math. Comp., 54 (1990), pp. 495–508.

[24] Z. Yoshida and Y. Giga, *On the Ohm-Navier-Stokes system in magnetohydrodynamics*, J. Math. Phys., 24 (1983), pp. 2860–2864.