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Astrophysical Analogue of One-Electron Rydberg Quasimolecules: One-Planet Binary Star Systems

N. KRYUKOV AND E. OKS

Physics Department, 206 Allison Lab., Auburn University, Auburn, AL 36849, USA

ABSTRACT: One-planet Binary Star Systems (OBSS) have a limited analogy to objects studied in atomic/molecular physics: One-electron Rydberg Quasimolecules (ORQ). Specifically, ORQ, consisting of two fully-stripped ions of the nuclear charges Z and Z' plus one highly-excited electron, are encountered in various plasmas containing more than one kind of ions. Classical analytical studies of ORQ resulted in the discovery of classical stable conic-helical electronic orbits. As for studies of planets in binary star systems, they are important because it was estimated that about the half of binary stars are capable of supporting habitable terrestrial planets within stable orbital ranges. Previously it was shown that despite several important distinctions between OBSS and ORQ, it is possible for OBSS to have stable or metastable conic-helical planetary orbits, whose axis of symmetry coincides with the interstellar axis. That study was performed in frames of the nonrelativistic classical mechanics. In the present paper we extend that study to the relativistic classical mechanics. We show that relativistic effects in OBSS can become significant for the situations where the mass of the planet is relatively small (such planets are so-called planetoids).

1. INTRODUCTION

One-electron Rydberg Quasimolecules (hereafter, ORQ), consisting of two fully-stripped ions of the nuclear charges Z and Z' plus one highly-excited electron, are encountered in various plasmas containing more than one kind of ions. Examples are (but not limited to) magnetic fusion plasmas, laser-produced plasmas, plasmas used for x-ray and VUV lasers, solar plasmas etc. In these plasmas, a fully-stripped ion of the nuclear charge Z' can come close to a hydrogenlike ion of the nuclear charge Z and form a short-lived molecule (i.e., quasimolecule). Conversely, a fully-stripped ion of the nuclear charge Z can come close to a hydrogenlike ion of the nuclear charge Z' and form a quasimolecule. Such quasimolecules are a very useful playground for theoretical and experimental studies of charge exchange, which is a physical process of the primary importance for many areas of physics (e.g., for the areas listed above).

Classical analytical studies of ORQ were first presented in papers [1, 2] and later in the book [3]. The latest works were presented in the review [4]. The primary result was the discovery of classical stable electronic orbits of the shape of a helix on the surface of a cone.

In astrophysics, One-planet Binary Star Systems (hereafter, OBSS) have a limited analogy to ORQ studied in atomic/molecular physics. It should be emphasized that studies of planets in binary star systems are especially important for the ongoing search of an extraterrestrial life. This is because it was estimated that about the half of binary stars are capable of supporting habitable terrestrial planets within stable orbital ranges (see, e.g., [5-7]). Therefore, the stability of planets around binary stars is the subject of a continuing interest.

In paper [8] one of us showed that stable (or metastable) conic-helical orbits are possible for OBSS. The overwhelming majority of papers on planetary orbits around binary stars considered only various types of the planetary motion in the orbital plane of the two stars (see, e.g., [5-7, 9-14] and references therein) – to the best of our knowledge. In distinction, in paper [8] there was presented the motion of a planet, roughly speaking, in the

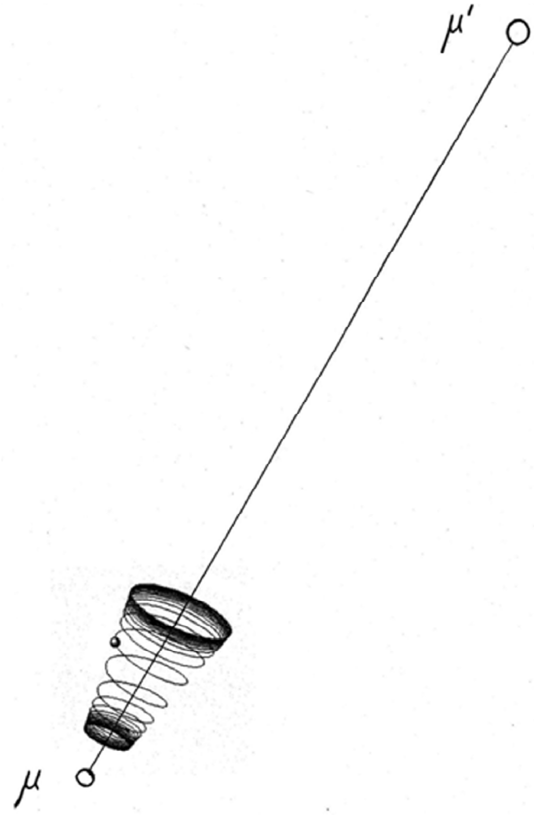

Figure 1: Sketch of the conic-helical motion of the planet in the model binary star system where the stars are stationary. We stretched the trajectory along the interstellar axis to make its details better visible.

plane perpendicular to the interstellar axis – more rigorously, the planet orbit is a helix on a conical surface, whose axis of symmetry coincides with the interstellar axis (Fig. 1). From a general point of view, paper [8] presented a new chapter in the centuries-old classical three-body problem.

It should be emphasized that there is only a limited similarity between ORQ and OBSS because of the following distinctions between these two physical systems. First, in ORQ the attractive centers (nuclei Z and Z') can be stationary and still be stable, while in binary star systems the rotation of the stars is necessary for the stability.

Second, in ORQ the attractive centers can engage in oscillations (called vibrations) and be stable without any rotation. This is not the case for OBSS.

Third, in ORQ the electronic degree of freedom has a much larger characteristic frequency and energy than the nuclear degrees of freedom. This is the basis for the standard Born-Oppenheimer approximation, where the primary contribution to the energy of the system can be obtained by freezing the nuclear motion. If necessary, the nuclear motion can be later taken into account by perturbation theory.

The Born-Oppenheimer approximation is a particular case of the general analytical method for system that can be separated into rapid and slow subsystems. For ORQ, this method is applicable because the ratio of frequencies (and energies) of the electronic motion, the vibrational nuclear motion, and the rotational nuclear motion is 1: $(m_e/M_n)^{1/2}$: (m_e/M_n). Here m_e is the electron mass and M_n is the total mass of the two nuclei, so that $m_e/M_n < 1/3600$ and the separation of the slow and rapid subsystems is justified "automatically". This is, generally speaking, not the case for OBSS.

Indeed, for OBSS the analogue of molecular oscillations (vibrations) is a periodic change of the separation between the two stars, which occurs for eccentric stellar orbits. One distinction from ORQ is that both oscillations and rotations have the same frequency: the Kepler frequency ω of the two stars orbiting their barycenter. Another distinction from ORQ is that the primary frequency Ω of the helical motion of the planet is not "automatically" much greater than the stars Kepler frequency ω.

In paper [8] it was shown that there are ranges of parameters where actually Ω >> ω. Further it was demonstrated that there are ranges of parameters where the planetary motion is stable for a model case of stationary stars and that there are ranges of parameters where the planetary motion is stable (or metastable) for the real case of stars rotating in circular or elliptical orbits. The analysis in paper [8] was based on the non-relativistic classical mechanics.

In the present paper we extend the study from paper [8] to the relativistic classical mechanics. We show that relativistic effects in OBSS can become important for the situations where the mass of the planet is relatively small (such planets are so-called planetoids).

The paper is organized as follows. In Sect. 2 we reiterate the main results from the "non-relativistic" paper [8]. In Sect. 3 we present the relativistic study of OBSS and of the stability of planetoids in conic-helical orbits. In Sect. 4 we present conclusions.

2. THE MAIN RESULTS OF THE NON-RELATIVISTIC STUDY FROM PAPER [8]

In paper [8], first, there was considered a model system consisting of two immobile stars of masses μ and μ′, and a planet of a unit mass moving around a circle in the plane perpendicular to the interstellar axis, on which the circle is centered. The mass μ is at the origin and the O*z* axis is directed to the mass μ′ located at $z = R$. After introducing notations

$$Z = G\mu, \; Z' = G\mu' \tag{1}$$

where G is the gravitational constant, the Hamilton function of the system is written in the cylindrical coordinates (z, ρ, φ) as

$$H = \frac{1}{2}\left(p_z^2 + p_\rho^2 + \frac{p_\phi^2}{\rho^2} \right) + U(z,\rho) \tag{2}$$

where the potential energy is

$$U(z,\rho) = -\frac{Z}{\sqrt{z^2+\rho^2}} - \frac{Z'}{\sqrt{(R-z)^2+\rho^2}} \tag{3}$$

The relation between the momenta and the corresponding velocities follows from the Hamiltonian equations of the motion:

$$\frac{dz}{dt} = \frac{\partial H}{\partial p_z} = p_z, \;\; \frac{d\rho}{dt} = \frac{\partial H}{\partial p_\rho} = p_\rho, \;\; \frac{d\phi}{dt} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{\rho^2} \tag{4}$$

Since H does not depend on φ, the corresponding momentum is conserved:

$$p_\phi = \rho^2 \frac{d\phi}{dt} = M = \text{const} \tag{5}$$

Physically, the separation constant M is a projection of the planet angular momentum on the interstellar axis. Thus, the z- and ρ-motions can be determined separately from the φ-motion. Then the φ-motion can be found from the ρ-motion via Eq. (5).

The Hamilton function for the z- and ρ-motions can be represented in the form

$$H = \frac{p_z^2 + p_\rho^2}{2} + U_{eff}(z,\ \rho) \tag{6}$$

where an effective potential energy (EPE) is:

$$U_{eff}(z,\rho) = \frac{M^2}{2\rho^2} + U(z,\ \rho) \tag{7}$$

After introducing scaled (dimensionless) variables w and v, a scaled projection of the angular momentum m, as well as a ratio of the star masses b,

$$w \equiv \frac{z}{R},\ v \equiv \frac{\rho}{R},\ m \equiv \frac{M}{\sqrt{ZR}},\ b \equiv \frac{\mu'}{\mu} \tag{8}$$

the EPE can be re-written as

$$U_{eff} = \frac{Z}{R} u_{eff}(w,\ v,\ m,\ b),\ u_{eff}(w,\ v,\ m,\ b)$$

$$= \frac{m^2}{2v^2} - \frac{1}{\sqrt{w^2 + v^2}} - \frac{b}{\sqrt{(1-w)^2 + v^2}} \tag{9}$$

By equating to zero the derivative of the EPE with respect to w

$$\frac{\partial U_{eff}}{\partial w} = \frac{Z}{R}\frac{\partial u_{eff}}{\partial w} = 0 \tag{10}$$

a relation was found

$$\frac{b}{\left((1-w)^2 + v^2\right)^{3/2}} = \frac{w}{(1-w)\left(w^2 + v^2\right)^{3/2}} \tag{11}$$

which determines a line $v_0(w)$ in the plane $(w,\ v)$, where the equilibrium points of the EPE are located:

$$v_0(w,\ b) = \sqrt{\frac{w^{2/3}(1-w)^{4/3} - b^{2/3}w^2}{b^{2/3} - w^{2/3}(1-w)^{-2/3}}} \tag{12}$$

For $b < 1$, the equilibrium value of v exists for $0 \le w < b/(1 + b)$ and for $1/(1 + b^{1/2}) \le w \le 1$.

For $b > 1$, the equilibrium value of v exists for $0 \le w \le 1/(1 + b^{1/2})$ and for $b/(1 + b) < w \le 1$.

For $b = 1$, the equilibrium value of v exists for the entire range of $0 \le w \le 1$. Below we refer to these intervals as the "allowed ranges" of w.

By equating to zero the derivative of the EPE with respect to v

$$\frac{\partial U_{eff}}{\partial v} = \frac{Z}{R}\frac{\partial u_{eff}}{\partial v} = 0 \tag{13}$$

and then substituting $v_0(w,\ b)$ from Eq. (12) instead of v, another relation was found:

$$m = \frac{v_0^2(w,\ b)}{\sqrt{1-w}\left(w^2 + v_0^2(w,\ b)\right)^{3/4}} \equiv m_0(w,\ b) \tag{14}$$

While deriving Eq. (14), we used Eq. (11) to eliminate an explicit dependence on b, so that b enters Eq. (14) only implicitly–as an argument of the function $v_0(w, b)$. In a number of subsequent derivations, we will also use Eq. (11) for the same purpose without a further notice.

For each set of (w, b), where w belongs to the allowed ranges, Eq. (14) determines an equilibrium value of $m_0(w, b)$–in addition to the equilibrium value of $v_0(w, b)$ determined by Eq. (12). Then for some value of b, there was considered a set of equilibrium values (w_i, v_{0i}, m_{0i}) and the EPE u_{eff} was expanded in terms of δw and δv, where

$$\delta w \equiv w - w_i,\ \delta v \equiv v - v_{0i} \tag{15}$$

The expansion has the form

$$u_{eff} \approx u_0 + u_{ww}\frac{\delta w^2}{2} + u_{vv}\frac{\delta v^2}{2} + u_{wv}\delta w \delta v,\ u_0 \equiv u_{eff}\left(w_i,\ v_{0i},\ m_{0i}\right) \tag{16}$$

In the subsequent formulas the suffix i is dropped for brevity.

Since generally $u_{wv} \neq 0$, a rotation of the reference frame is required in order to transform the EPE to so-called "normal" coordinates, diagonalizing the matrix of the second derivatives of the EPE [15, 16]:

$$\delta w' = \delta w\cos\alpha + \delta v\sin\alpha,\ \delta v' = -\delta w\sin\alpha + \delta v\cos\alpha \tag{17}$$

where

$$\text{tg}\ 2\alpha = \frac{2u_{wv}}{u_{ww} - u_{vv}} = \frac{(1-2w)v_0}{P} \tag{18}$$

so that

$$\cos\alpha = \sqrt{\frac{1+P/Q}{2}},\ \sin\alpha = \sqrt{\frac{1-P/Q}{2}}\ \text{sign}\ (1-2w) \tag{19}$$

Here

$$P \equiv w(1-w) + v_0^2,\ Q \equiv \sqrt{\left(w^2 + v_0^2\right)\left((1-w)^2 + v_0^2\right)} \tag{20}$$

In the normal coordinates, the EPE takes the form

$$u_{eff} \approx u_0 + \delta w'^2\frac{\omega_-^2}{2} + \delta v'^2\frac{\omega_+^2}{2} \tag{21}$$

where

$$\omega_\pm \equiv \frac{1}{\left(w^2 + v_0^2\right)^{3/4}}\sqrt{\frac{1}{1-w} \pm \frac{3w}{Q}} \tag{22}$$

The scaled (dimensionless) frequency ω_+ of small oscillations around the equilibrium in the direction of the normal coordinate $\delta v'$ is always real. According to the notations from paper [8], any frequency F and its scaled (dimensionless) counterpart f are related as follows:

$$f \equiv \sqrt{\frac{R^3}{Z}}F \tag{23}$$

As for the quantity ω_-, it is real if

$$v_0\left(w,b\right) \geq \sqrt{w(1-w) - \frac{1}{2} + \sqrt{9w^2\left(1-w\right)^2 - w(1-w) + \frac{1}{4}}} \equiv v_{crit}\left(w\right) \tag{24}$$

Physically, under the condition (24), the quantity ω_- is the frequency of small oscillations around the equilibrium in the direction of the normal coordinate $\delta w'$.

Thus, if $v_0(w, b) > v_{crit}(w)$, the EPE has a two-dimensional minimum at the equilibrium values of w and $v = v_0(w, b)$, so that the equilibrium is stable. After introducing a scaled (dimensionless) time

$$\tau \equiv \sqrt{\frac{Z}{R^3}}t \tag{25}$$

the following final expression for the small oscillations around the stable equilibrium was obtained:

$$\begin{gathered}\delta w(\tau) = a_w \cos(\omega_-\tau + \psi_w)\cos\alpha + a_v\cos(\omega_+\tau + \psi_v)\sin\alpha \\ \delta v(\tau) = a_w \cos(\omega_-\tau + \psi_w)\sin\alpha - a_v\cos(\omega_+\tau + \psi_v)\cos\alpha\end{gathered} \tag{26}$$

Here amplitudes a_w, a_v and phases ψ_w, ψ_v are determined by initial conditions; $\sin\alpha$ and $\cos\alpha$ are given by Eq. (19). Compared to the corresponding Eq. (28) from [8], here in Eq. (26) we corrected a typographic error in signs.

The solution for the φ-motion turned out to be

$$\varphi\left(\tau\right) \approx f_p \tau - \\ -2\frac{\frac{1}{\omega_-}a_w\left(\sin\left(\omega_-\tau + \psi_w\right) - \sin\psi_w\right)\sin\alpha + \frac{1}{\omega_+}a_v\left(\sin\left(\omega_+\tau + \psi_v\right) - \sin\psi_v\right)\cos\alpha}{\sqrt{1-w}\left(w^2 + v_0^2\right)^{3/4} v_0} \tag{27}$$

where

$$f_p \equiv \frac{1}{\sqrt{1-w}\left(w^2 + v_0^2\right)^{3/4}} \tag{28}$$

is a scaled (dimensionless) primary frequency of the φ-motion. Equations (27), (28) show that the φ-motion is a rotation about the internuclear axis with the frequency f_p, slightly modulated by oscillations of the scaled radius of the orbit v at the frequencies ω_+ and ω_-. In other words, for the stable motion, the planetary trajectory is a helix on the surface of a cone, with the axis coinciding with the interstellar axis. In this *conic-helical* state, the planet, while spiraling on the surface of the cone, oscillates between two end-circles which result from cutting the cone by two parallel planes perpendicular to its axis (Fig. 1).

Further in paper [8] there was a study of effects of the stars rotation and the eccentricity of their orbits on the conic-helical orbit of the planet for the situations where the Kepler frequency

$$\omega = \sqrt{\frac{G\left(\mu + \mu'\right)}{R^3}} \tag{29}$$

of the two stars orbiting their barycenter (which is also the frequency of oscillations of the interstellar distance in the case of eccentric stellar orbits) is much smaller than the primary frequency of the revolution of the planet around the interstellar axis. This situation allows applying the standard method of the separation of rapid and slow subsystems.

According to Eq. (23), the scaled, dimensionless counterpart of the Kepler frequency is

$$f_s = \sqrt{\frac{R^3}{Z}}\omega = \sqrt{1+b} \tag{30}$$

The ratio of the scaled primary frequency f_p of the planetary motion (given by Eq. (28)) to the scaled Kepler frequency f_s of the stars is

$$k(w,b) = \frac{1}{\sqrt{(1+b)(1-w)}\left(w^2 + v_0^2(w,b)\right)^{3/4}} \tag{31}$$

This ratio becomes sufficiently large if the projection of the planetary orbit on the interstellar axis is either close to the star of the smaller mass ($w \ll 1$) or close to the star of the larger mass ($(1 - w) \ll 1$). Those are the ranges of parameters where the separation into the rapid and slow subsystems is justified.

In the reference frame rotating together with the stars with the Kepler frequency ω, there is an additional force (see, e.g., [15-17]):

$$\mathbf{F}_1 = 2\mathbf{v} \times \boldsymbol{\omega} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{32}$$

Since $v \sim \Omega\rho$, where Ω is the primary frequency of the planetary motion and ρ is the average radius of the planetary orbit, and because $\Omega \gg \omega$, then the additional force is approximately

$$\mathbf{F}_1 \approx 2\mathbf{v} \times \boldsymbol{\omega} \tag{33}$$

Expression (33) has a clear physical meaning for the quantal counterpart-problem of ORQ. Namely, it is a "Lorentz electric field" $\mathbf{v} \times \mathbf{B}_{\text{eff}}/c$ in the effective magnetic field $\mathbf{B}_{\text{eff}} = c\boldsymbol{\omega}$ in atomic units (or $\mathbf{B}_{\text{eff}} = m_e c\boldsymbol{\omega}$ in the CGS units, m_e being the electron mass).

In paper [8] the Ox axis was chosen along vector **ω**, which is obviously perpendicular to the interstellar axis chosen as the Oz axis. Since the planetary velocity **v** is primarily in the xy-plane perpendicular to the interstellar axis, then the additional force $\mathbf{F}_1$ is primarily along the interstellar axis.

Representing $\mathbf{r}/\rho = \mathbf{e}_x \cos \Omega t + \mathbf{e}_y \sin \Omega t$, so that $\mathbf{v}/\rho = (-\Omega \sin \Omega t)\mathbf{e}_x + (\Omega \cos \Omega t)\mathbf{e}_y$, and using $\boldsymbol{\omega} = \omega\mathbf{e}_x$, Eq. (32) yields:

$$\mathbf{F}_1 = \rho(-2\omega\Omega\cos\Omega t\mathbf{e}_z + \omega^2 \sin\Omega t\mathbf{e}_y) \tag{34}$$

In the ranges of parameters where $\Omega \gg \omega$, Eq. (34) becomes

$$\mathbf{F}_1 \approx -2\rho\omega\Omega\cos\Omega t\mathbf{e}_z \tag{35}$$

Thus, for the $z\rho$-motion (which in the scaled coordinates is wv-motion), the situation represents a two-dimensional oscillator, having the eigenfrequencies ω_+ and ω_- defined by Eq. (22), that is driven by the force $\mathbf{F}_1$ oscillating at the frequency Ω. Using the well-known formulas for driven oscillators (see, e.g. [17]), the solution in the coordinates w', v' rotated by the angle α (defined by Eq. (18)) compared to the coordinates w, v (i.e., in the coordinates, where the two oscillators are decoupled) can be immediately written. Then coming back to the original scaled coordinates w, v, one obtains:

$$\delta w(\tau) = 2v_0(w,b)\omega_s f_p \left(\frac{\cos^2 \alpha}{f_p^2 - \omega_-^{\,2}} + \frac{\sin^2 \alpha}{f_p^2 - \omega_+^{\,2}} \right) \cos f_p \tau$$

$$\delta v(\tau) = 2v_0(w,b)\omega_s f_p \sin\alpha \ \cos\alpha\left(\frac{1}{f_p^2-\omega_-^2}-\frac{1}{f_p^2-\omega_+^2}\right)\cos f_p\tau \tag{36}$$

Using the relation $(\omega_+^2 + \omega_-^2)/2 = f_p^2$ and Eq. (19), the latter formulas can be finally re-written in the following simple form:

$$\delta w(\tau) = \frac{2v_0(w,b)\omega_s f_p \cos 2\alpha}{\omega_+^2-\omega_-^2}\cos f_p\tau$$

$$\delta v(\tau) = \frac{2v_0(w,b)\omega_s f_p \sin 2\alpha}{\omega_+^2-\omega_-^2}\cos f_p\tau \tag{37}$$

We note that in Eqs. (32)–(37) we corrected some typographic errors compared to the corresponding equations from paper [8].

Equation (37) shows that the trajectory of the planet is conic-helical. The forced oscillations of the "circular" planetary orbit are primarily along the interstellar axis. For the quantal counterpart-problem of ORQ this should have been expected. Indeed, the electron in a circular orbit is like a charged "ring": so, in the monochromatic electric (Lorentz) field perpendicular to the axis of the charged ring, the ring should oscillate along its axis.

Figure 2 shows the scaled amplitude $\delta w_0 = 2v_0(w, b)\omega_s f_p(\cos 2\alpha)/|(\omega_+^2 - \omega_-^2)|$ of the oscillations of the planetary orbit along the interstellar axis versus the scaled projection $w = z/R$ of the average plane of the planetary orbit on the interstellar axis, for three values of the ratio b of the stellar masses. It is seen that $\delta w_0 << 1$ (i.e., the amplitude of the oscillations is much smaller than the interstellar distance) for b greater or of the order of 10. For the validity condition $\Omega >> \omega$ of this result to be satisfied with a large margin of "safety", the average plane of the planetary orbit should be very close to the star of the smaller mass (and the closer it is in the range of $w < 0.05$, the smaller is the amplitude δw_0, as seen from Fig. 2). As long as the primary frequency Ω of the planet revolution about the interstellar axis exceeds by many orders of magnitude the Kepler frequency ω of stars rotation, the conic-helical planetary orbit would remain stable for a very long time. (Rigorously speaking, the planet is in a metastable state,

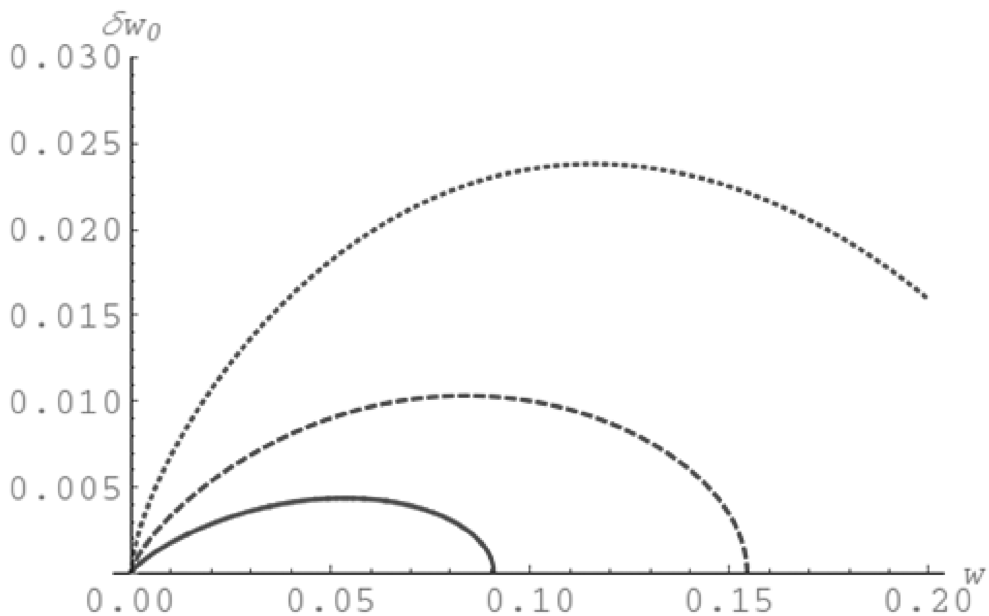

Figure 2: The scaled amplitude δw_0 of the oscillations of the planetary orbit along the interstellar axis versus the scaled projection $w = z/R$ of the average plane of the planetary orbit on the interstellar axis, for three values of the ratio b of the stellar masses: b = 100 (solid line), b = 30 (dashed line), and b = 10 (dotted line).

which is yet another analogy with atomic/molecular systems: they have metastable states living by many orders of magnitude longer than other states of the system.)

Figure 3 shows the scaled amplitude $\delta v = 2v_0(w, b)\, \omega_s f_p (\sin 2\alpha)/|\omega_+^2 - \omega_-^2|$ of the oscillations of the planetary orbit in the plane perpendicular to the interstellar axis versus the scaled projection $w = z/R$ of the average plane of the planetary orbit on the interstellar axis, for three values of the ratio b of the stellar masses. It is seen that for the range of b greater or of the order of 10 (required for having $\delta w_0 << 1$), we have also $\delta v << 1$.

Finally, the effect of the eccentric orbits of the stars was studied in paper [8]. In the reference frame rotating together with the stars with the Kepler frequency ω, a non-zero eccentricity ε of the stars orbits results in the oscillation of the interstellar distance R with the frequency ω. In the ranges of parameters where $\Omega >> \omega$, the oscillation of the interstellar distance is an adiabatic "perturbation" of the planetary motion. According to the principle of adiabatic invariance, the planetary motion will adjust to the slowly varying R while keeping as the constant the average nonzero projection M of the planetary angular momentum on the interstellar axis. The projection of the planetary angular momentum on the interstellar axis undergoes small oscillations (caused by stars rotation) around the nonzero average M, the latter being the adiabatic invariant.

Particularly, for the average plane of the planetary orbit close to the star of the smaller mass, it was shown in paper [8] that the eccentricity ε of the stars orbit would not affect the stability of the planetary motion if ε does not exceed some critical value $\varepsilon_c(b)$ given by Eq. (53) of paper [8].

3. RELATIVISTIC EFFECTS

The relativistic force acting on the planet (scaled by its mass, i.e., the force divided by the mass of the planet) is given by the formula

$$\mathbf{F} = \gamma_0 \left(\mathbf{a} + \mathbf{V} \frac{\gamma_0^2}{c^2} \mathbf{V} \cdot \dot{\mathbf{V}} \right) \tag{38}$$

where $\mathbf{V}$ is the velocity of the planet and $\gamma_0 = (1 - V^2/c^2)^{-1/2}$. The additional acceleration given by (32) can be substituted into (38) as the last factor, with $\mathbf{V}$ as the time derivative of the position in the rotating reference frame

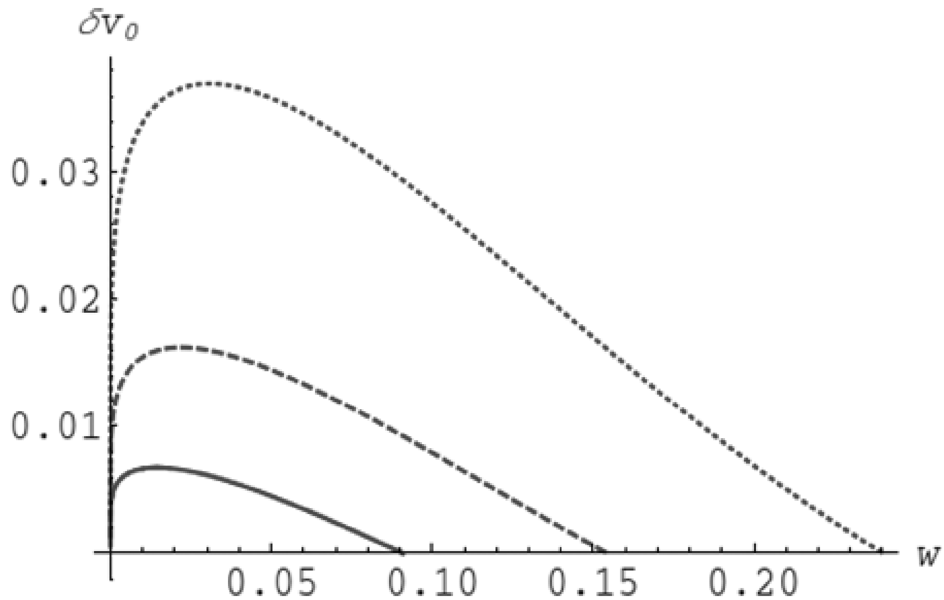

Figure 3: The scaled amplitude δv_0 of the oscillations of the planetary orbit along the radial coordinate versus the scaled projection $w = z/R$ of the average plane of the planetary orbit on the interstellar axis, for three values of the ratio b of the stellar masses: b = 100 (solid line), b = 30 (dashed line), and b = 10 (dotted line).

$$\mathbf{r} = \rho\cos\Omega t\mathbf{e_x} + \rho\sin\Omega t\mathbf{e_y} \tag{39}$$

After the calculation, we obtain the following values for the components of the force:

$$F_x = -\gamma_0\omega V\frac{\gamma_0^2\beta^2}{2}\frac{\omega}{\Omega}\sin 2\Omega t\sin\Omega t$$

$$F_y = \gamma_0\omega V\frac{\omega}{\Omega}\left(\sin\Omega t + \frac{\gamma_0^2\beta^2}{2}\sin 2\Omega t\cos\Omega t\right)$$

$$F_z = -2\gamma_0\omega V\cos\Omega t \tag{40}$$

(here $\beta = V/c$). As $\omega << \Omega$, the dominant term is the z-projection of the force (given by the third equation in (40)). By analogy with the non-relativistic case in Sect. 2, we derive the small oscillations about the equilibrium due to the dominant force term:

$$\delta w(\tau) = \frac{2v\tilde{\omega}\tilde{\Omega}\cos 2\alpha}{\left(\omega_+{}^2 - \omega_-{}^2\right)\sqrt{1-\beta^2}}\cos\tilde{\Omega}\tau,\ \ \delta v(\tau) = \frac{2v\tilde{\omega}\tilde{\Omega}\sin 2\alpha}{\left(\omega_+{}^2 - \omega_-{}^2\right)\sqrt{1-\beta^2}}\cos\tilde{\Omega} \tag{41}$$

where α was given by Eq. (19), $v = \rho/R$ (its equilibrium value given by (12)) and the tilde above means scaling by multiplying by $(R^3/Z)^{1/2}$. We can now find the conditions, under which the amplitude of the oscillations is small while the condition $\omega << \Omega$ stays valid. In the relativistic case, the Hamiltonian, divided by the mass m of the planet $h = H/m$, for the OBSS case is given by

$$h = c\sqrt{m^2c^2 + p_z^2 + p_\rho^2 + \frac{p_\phi^2}{\rho^2}} - \frac{Z}{\sqrt{z^2+\rho^2}} - \frac{Z'}{\sqrt{(R-z)^2+\rho^2}} - mc^2 \tag{42}$$

In a circular state, $p_z = p_\rho = 0$ and $|p_\varphi|/m = \text{const} = L$. Using the scaling

$$\ell = \frac{L}{cR},\ \varepsilon = -\frac{R}{Z}E,\ r = \frac{Z}{L^2}R \tag{43}$$

we write the scaled energy for the circular state:

$$\varepsilon = \frac{1}{\sqrt{w^2+v^2}} + \frac{b}{\sqrt{(1-w)^2+v^2}} + \frac{c^2R}{Z}\left(1-\sqrt{1+\frac{\ell^2}{v^2}}\right) \tag{44}$$

For the relativistic motion,

$$L = \frac{mV\rho}{\sqrt{1-\frac{V^2}{c^2}}} \tag{45}$$

and, using the scaling $\rho = vR$ from (8) and the first formula in (43), we can find the speed of the planet in the relativistic OBSS case in the units of the speed of light:

$$\beta = \frac{1}{\sqrt{1+\frac{p}{\ell^2}}} \tag{46}$$

where $p = v^2$. From the first and the third formulas in (43), $\ell^2 = \alpha/r^2$, where $\alpha = (Z/(cL))^2$. Thus,

$$\beta = \frac{1}{\sqrt{1+\frac{pr^2}{\alpha}}} \tag{47}$$

The equilibrium values for p and r can be obtained from differentiating (44) with respect to w and v. The first differentiation gives the same relation as in the non-relativistic case, so the equilibrium value of p is the squared right-hand side of (12). The second differentiation (with respect to v), with the later substitution of the equilibrium value of v (or p), yields the equilibrium value of ℓ, which is related to r by $\ell^2 = \alpha/r^2$ mentioned above. Using the substitution [4]

$$\gamma = \left(\frac{1}{w} - 1\right)^{1/3} \tag{48}$$

which significantly simplifies the formulas in the two-Coulomb-center problem, we find the speed of the planet in the circular state (in the units of the speed of light):

$$\beta = \sqrt{\frac{\alpha\left(\gamma^4 - b^{2/3}\right)^3}{\left(\gamma^3 - 1\right)^3\left(\gamma^3 + 1\right)}} \tag{49}$$

From (45) with the substituted value of $V = \Omega\rho$, the first relation in (43), the second relation in (8) and using $\ell^2 = \alpha/r^2$, we find

$$\Omega = \frac{c}{R}\frac{\sqrt{\alpha}}{pr}\sqrt{1-\beta^2} \tag{50}$$

and

$$\tilde{\Omega} = c\sqrt{\frac{R}{Z}}\frac{\sqrt{\alpha}}{pr}\sqrt{1-\beta^2} \tag{51}$$

As given in (1), and measuring now the mass of the star μ in the units of the mass of the Sun (in distinction to the nonrelativistic case in Sect. 2, where the star masses were measured in units of the mass of the planet), we get

$$Z = GM_{\odot}\mu \tag{52}$$

where $M_{\odot} = 1.989 \times 10^{33}$ g. Substituting (52) into (51), we obtain

$$\tilde{\Omega} = \sqrt{\frac{R}{\mu s}}\frac{\sqrt{\alpha}}{pr}\sqrt{1-\beta^2} \tag{53}$$

where the quantity $s = GM_{\odot}/c^2$ (which is one half of the Schwarzschild radius of the Sun and is approximately equal to 147700 cm).

As defined above, $\alpha^{1/2} = Z/(cL)$, and substituting the scaling relation for L from (43), $\ell^2 = \alpha/r^2$ and (52), we have

$$\alpha = \frac{\mu s}{R} r \tag{54}$$

We substitute (54) into (49), and then substitute the resulting equation for β and the solution for α into (53), obtaining the equation for the scaled frequency of the revolution of the planet that only depends on the ratio $R/(\mu s)$ and γ (or *w*–see (48)). Finally, from (41), and because $\tilde{\omega} = (1 + b)^{1/2}$ from (29) and (30), the amplitudes of the small oscillations about the equilibrium on the *w*-axis and *v*-axis are

$$\delta w_0 = \frac{2\sqrt{p}\sqrt{1+b}\cos 2\alpha}{\omega_+{}^2 - \omega_-{}^2}\tilde{\Omega}$$

$$\delta v_0 = \frac{2\sqrt{p}\sqrt{1+b}\sin 2\alpha}{\omega_+{}^2 - \omega_-{}^2}\tilde{\Omega} \tag{55}$$

where α is given by Eqs. (17) – (19), and substituting the equation for the scaled frequency of the revolution of the planet obtained previously, the equilibrium values for *p* and *r*, the equation for *b* and the frequencies from (22), we derive the amplitudes of the small oscillations of the planet on the *w*-axis and the *v*-axis in the relativistic case:

$$\delta w_0 = \frac{\gamma^{3/2}\sqrt{1+b}\left(\gamma^3 - 1\right)^{7/4}\sqrt{\gamma^4 - b^{2/3}}\left(b^{2/3} + \gamma\right)}{3\left(b^{2/3}\gamma^2 - 1\right)^{9/4}\left(\gamma^3 + 1\right)^{9/4}}$$

$$\delta v_0 = \frac{\sqrt{\gamma}\sqrt{1+b}\left(\gamma^3 - 1\right)^{7/4}\left(\gamma^4 - b^{2/3}\right)}{3\left(b^{2/3}\gamma^2 - 1\right)^{7/4}\left(\gamma^3 + 1\right)^{9/4}} \tag{56}$$

The amplitude in the relativistic case is the same as in the non-relativistic case.

We checked that the ratio of the frequencies of the revolution of the planet Ω and the Kepler frequency ω of the revolution of the stars about their barycenter is much greater than 1. By using the previously found values of Ω and ω and substituting the equilibrium values of *p*, *r*, α and β, we derive the formula for the ratio *k* of the frequencies depending on the axial coordinate for the given interstellar distance *R* and the star masses μ and μ′:

$$k = \frac{\left(b^{2/3}\gamma^2 - 1\right)^{3/4}\left(\gamma^3 + 1\right)^{5/4}}{\sqrt{1+b}\,\gamma^{3/2}\left(\gamma^3 - 1\right)^{3/4}}\sqrt{1 - \frac{\mu s}{R}\frac{\left(\gamma^4 - b^{2/3}\right)\sqrt{\left(b^{2/3}\gamma^2 - 1\right)\left(\gamma^3 + 1\right)}}{\gamma\left(\gamma^3 - 1\right)^{3/2}}} \tag{57}$$

From Eq. (57) it can be found out that *k* in the relativistic case is equal to the non-relativistic k_{NR} multiplied by $(1 - \beta^2)^{1/2}$. Figure 4 shows the ratio k/k_{NR} versus the scaled radius of the orbit *v*, for the masses of the stars μ = 1 and μ′ = 100 (in the units of the mass of the Sun) and the interstellar distance *R* = 100 a.u. It is seen that the relativistic effects become significant when $v \sim 10^{-9}$ or smaller.

The range of $v = (1 \div 2) \times 10^{-9}$ for *R* = 100 a.u. corresponds to the range of the radius of the planetary orbit ρ = (15 ÷ 30) km. Since the radius of the planet should be smaller than ρ, in this case it should be a planetoid.

Figure 5 show the dependence *k*(*v*) for the example where the mass of the lighter star μ = 1 and the heavier star μ′ = 100 (in the units of the mass of the Sun) and for the interstellar distance *R* = 100 a.u. (wide binary system). It is seen that the ratio of the frequencies is greater than 10^{12} for $v < 2 \times 10^{-9}$, which means that such a system would be stable for a very long time.

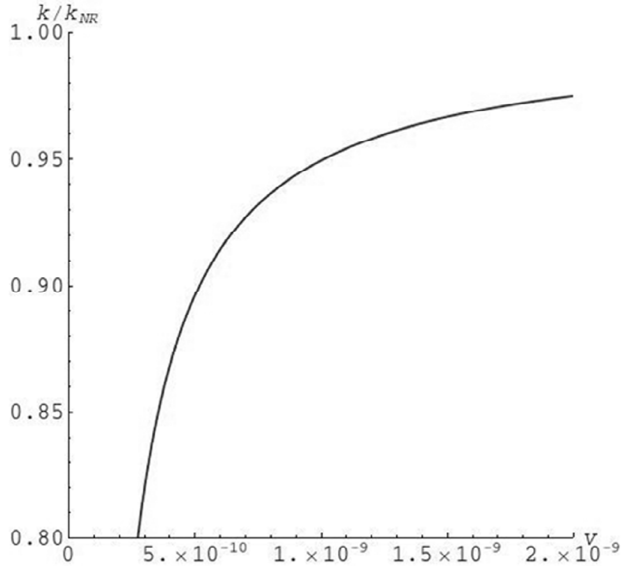

Figure 4: The ratio of the relativistic and non-relativistic *k* (which is the ratio of the frequencies of the revolution of the planet Ω and the Kepler frequency ω of the revolution of the stars about their barycenter) versus the scaled radius of the planetary orbit *v* = ρ/*R*, for the masses of the stars μ = 1 and μ′ = 100 (in the units of the mass of the Sun) and the interstellar distance *R* = 100 a.u.

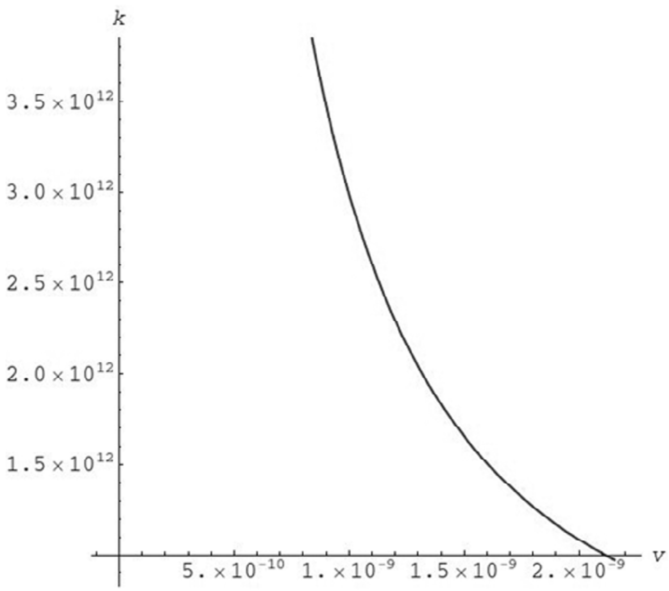

Figure 5: The ratio of the revolution frequency of the planet and the Kepler frequency of the rotation of the stars versus the scaled orbit radius *v* = ρ/*R*, for the example where the mass of the lighter star μ = 1 and the heavier star μ′ = 100 (in the units of the mass of the Sun) and the interstellar distance *R* = 100 a.u. in the relativistic case.

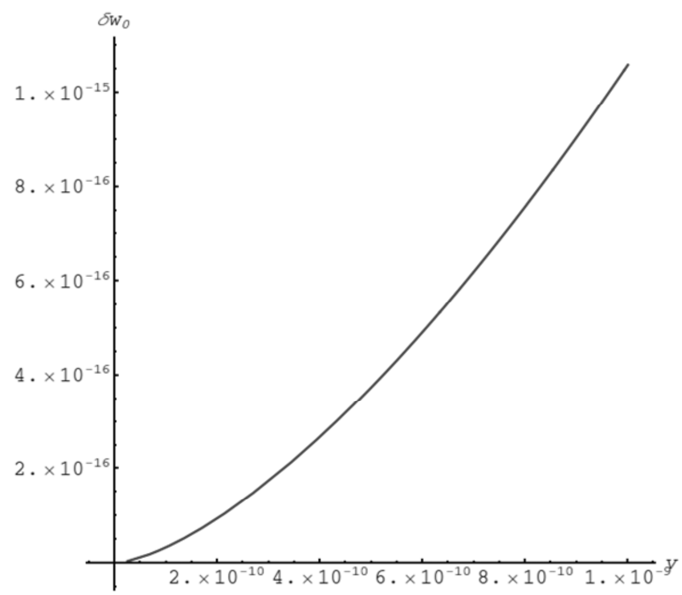

Figure 6: The scaled amplitude δw_0 of the oscillations of the planetary orbit along the interstellar axis versus the scaled orbit radius *v* = ρ/*R* for the example where the mass of the lighter star μ = 1 and the heavier star μ′ = 100 (in the units of the mass of the Sun).

In Fig. 6 we present the plot of the scaled amplitude δw_0 of the axial oscillations of the planetary orbit from Eq. (56) versus the scaled radial coordinate *v* for the mass of the lighter star μ = 1 and the heavier star μ′ = 100, in the units of the mass of the Sun. It is seen that the amplitude of oscillations is really much smaller – about quadrillion (!) times smaller – than the interstellar distance.

4. CONCLUSIONS

We extended the nonrelativistic study from paper [8] to the relativistic classical mechanics (and also corrected some printing errors from paper [8]). We showed that relativistic effects in OBSS can become significant for the situations where the mass of the planet is relatively small (such planets are so-called planetoids). In this case the conic-helical trajectory of the planetoid encloses the star of the lighter mass, the average plane of the planetary orbit practically coinciding with the position of the star of the lighter mass. It is counterintuitive that a distant heavier star can cause such a modification of the planetoid orbit around the lighter star. We showed that in this case the frequency of the rapid subsystem (i.e., the frequency of the planetoid revolution around the lighter star) exceeds the frequency of the slow subsystem (i.e., the Kepler frequency of the stars rotation around their barycenter) by more than a trillion times. This means that the standard analytical method of separating rapid and slow subsystems, which we used, is well justified and that this configuration will remain stable for a very long time.

NOTE ADDED IN THE PROOF

A fiction writer, Greg Egan, made comments about the (nonrelativistic) results from paper [8] in some blogs and similar non-scientific outlets that do not engage the peer-review process (do not send a submission to referees and do not verify the content). However, even within Egan's model, based on a tilted orbital plane of the planet with respect to the interstellar axis (the model having zero or little relevance to the scope of paper [8], where the orbital plane of the planet was not tilted), his calculations are erroneous for the following reason.

He tried to calculate the projection of the planetary angular momentum $\mathbf{M} = \mathbf{R} \times \mathbf{P}$ on the interstellar axis *in the rotating frame*, i.e., in the frame rotating with the angular velocity $\boldsymbol{\omega}_s$ of the stars rotation. He used formula $\mathbf{M} = \mathbf{R} \times \mathbf{V}$, assuming that the linear momentum of the planet of unit mass is $\mathbf{P} = \mathbf{V}$. However, it is well-known that in a rotating frame the linear momentum has the form $\mathbf{P} = \mathbf{V} - \mathbf{R} \times \boldsymbol{\omega}_s$ (see, e,g, Eq. (39.10) of book [15]). Therefore, his results are incorrect even within his model.

It is also worth noting that Egan, while trying to show that the projection of the planetary angular momentum on the interstellar axis has a time dependence, actually forces an open door. Neither in subsection 2.2 of paper [8] (devoted to the analytical solution for the three-dimensional motion) nor in the entire section 3 of paper [8] (devoted to the effects of stars rotation and of the eccentricity of their orbits) did it say that the projection of the planetary angular momentum on the interstellar axis does not have any time dependence. Using Eq. (51) of paper [8], describing oscillations of the scaled radius $v(\tau)$ of the planetary orbit and of the scaled projection $w(\tau)$ of the planetary orbit on the interstellar axis (τ being the scaled time), it is easy to obtain the following expression for the scaled projection m of the planetary angular momentum on the interstellar axis

$$m(\tau) = [1 + 2\,\delta v_0 \cos(f_p\tau) + O(\delta v_0^2)]f_p, \qquad (*)$$

where

$$\delta v_0 = [2v_0\omega_s f_p/|\omega_+^2 - \omega_-^2|]\sin 2\alpha. \qquad (**)$$

Here f_p is the scaled frequency of the rotation of the planet around the interstellar axis; the quantities $\omega_\pm$ and α are defined by Eqs. (23) and (20), respectively, from paper [8]. (The quantity α above has no relevance to the tilt angle α from Egan's model.) While deriving the above Eq. (*), the terms ~ ω_s were omitted due to the inequality $\omega_s << f_p$.

The above Eq. (*) shows that the projection of the planetary angular momentum on the interstellar axis oscillates around a *nonzero average value*. This is sufficient for the validity of the conclusions of paper [8].

In Egan's blog, he attempted also to do simulations. However, first, not all codes are properly verified and validated. Second, fully-numerical simulations are generally ill-suited for capturing so-called emergent principles and phenomena, such as conservation laws and symmetries. Third, as any fully-numerical method, they lack the physical insight.

A number of physicists warned about this. In 2005 Post and Votta – two leading experts in computational science – published a very insightful article (Physics Today, January 2005, p. 35), where they wrote, in particular:

> "*By verification we mean the determination that the code solves the chosen model correctly. Validation, on the other hand, is the determination that the model itself captures the essential physical phenomena with adequate fidelity. Without adequate verification and validation, computational results are not credible.*"

They described the underlying problems as follows:

"*Part of the problem is simply that it's hard to decide whether a code result is right or wrong. Our experience as referees and editors tells us that the peer review process in computational science generally doesn't provide as effective a filter as it does for experiment or theory. Many things that a referee cannot detect could be wrong with a computational-science paper. The code could have hidden defects, it might be applying algorithms improperly, or its spatial or temporal resolution might be inappropriately coarse.*

The existing peer review process for computational science is not effective. Seldom can a referee reproduce a paper's result."

Their paper caused lots of comments published in Physics Today, August 2005, p. 12. In one of the comments, J. Loncaric from Los Alamos wrote, in particular:

"*…components can be combined, but their combination could be wrong even though the components test well individually. A combination that is insensitive to minor component errors could still give invalid results. Each component has an unstated region of applicability that is often horribly complicated to describe, yet the combination could unexpectedly exceed individual component limits.*"

The bottom line is that simulations cannot be considered as the ultimate test of an analytical theory. The ultimate tests of the analytical theory are comparisons with observations/experiments and/or with a more rigorous analytical theory. Currently we are performing simulations of the conic-helical orbits in the nonrelativistic regime. The simulations are confirming the stability of these orbits (the simulations will be published elsewhere). The underlying code was properly verified and validated – in the meaning defined by Post and Votta. However, codes are black boxes, so that it would be virtually impossible for a third party to check our code. But this goes both ways. Namely, if somebody's (say, John Doe's) simulations would show an instability of the conic-helical orbits, this would not disprove the analytical results of paper [8]. Indeed, first, it could be that the code by John Doe was not properly verified and validated – in the meaning defined by Post and Votta – and there would be virtually no way for us to check John Doe's code. Second – but most importantly – the ultimate test of our analytical theory can be only the comparison with a more rigorous analytical theory (or with observations).

The last, but not least. In the direct communication with Egan, in order to test his simulations skills, one of us offered him to simulate a test case that has a known analytical solution. However, Egan's simulations of the test case failed to reproduce the known analytical solution.

In summary: Egan's comments contain some new analytical results and some correct statement. However, it is shown above that his new analytical results are incorrect, while his correct statement is not new (the statement that there is a time dependence of the projection of the planetary angular momentum on the interstellar axis). His simulations skills also seem very questionable since he failed to reproduce by simulations a test case that has a known analytical solution. Thus, *the comments by the fiction writer Mr. Egan seem to be his new work of fiction.*

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