Endomorphism Rings of Bimodules

Ulrich Albrecht
Rüdiger Göbel

Auburn University
Universität Duisburg-Essen

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All rings have units and are associative; and all functions are written on the left. Let $R$ and $S$ be rings. A left $S$-module $M$, which also is a right $R$-module, is a $S-R$-bimodule if $(sx)r = s(xr)$ for all $r \in R$, $s \in S$ and $x \in M$. A submodule $U$ of $M_R$ which is also a submodule of $_SM$ is a two-sided submodule of $M$.

Every right $R$-module $M$ is a $E$-$R$-bimodule where $E = \text{End}_R(M_R)$ is the endomorphism ring of $M$. In contrast to the commutative case, $E$ need not carry a left or right $R$-module structure. If $M$ is an $R$-$R$-bimodule, then left multiplication by the elements of $R$ induces a ring-morphism $\phi : R \to E$ which induces an $R$-$R$-bimodule structure on $E$. 
If $R$ is an integral domain, then the endomorphism ring of a torsion-free $R$-module is torsion-free. This work was motivated by the question if this remains true if $R$ is not commutative.

The singular submodule of a right $R$-module $M$ is $Z(M) = \{x \in M | xI = 0 \text{ for some essential right ideal } I \text{ of } R\}$. The module $M$ is non-singular (singular) if $Z(M) = 0$ ($Z(M) = M$). The ring is right non-singular if $R_R$ is non-singular. The $S$-closure $U^{cl}$ of a submodule $U$ of a non-singular module $M$ is the submodule $V$ of $M$ containing $U$ such that $V/U = Z(M/U)$. 
Every right non-singular ring $R$ has a \textit{maximal right ring of quotients} $Q^r(R)$ which is right self-injective and von Neumann regular. $Q^r(R)$ is constructed by extending the ring structure of $R$ to the injective hull of $R_R$. The ring $Q^r(R)$ usually does not coincide with the classical ring of quotients $Q(R)$ which is constructed similar to the field of quotients of an integral domain whenever $R$ satisfies the right Ore-condition. As we see, there are rings for which $R = Q(R)$, but $R \neq Q^r(R)$.

A module $M$ has \textit{finite Goldie dimension}, if it contains no infinite direct sums of non-zero submodules, and we write $\text{dim}M < \infty$ in this case. A right non-singular ring $R$ has finite right Goldie dimension if $\text{dim}R_R < \infty$. This occurs exactly if $Q^r(R)$ is semi-simple Artinian.
$R$ is semi-prime if $I^2 \neq 0$ for all non-zero right (left, two-sided) ideals of $R$. A semi-prime right finite dimensional right non-singular ring $R$ (called a semi-prime right Goldie-ring) has a semi-simple Artinian classical right ring of quotients which is also its maximal right ring of quotients.

Finally, an element $c$ of $R$ is regular if $cr = 0$ or $rc = 0$ implies $r = 0$. If $R$ is a semi-prime right Goldie ring, then $M_R$ is non-singular if and only if $xc = 0$ implies $x = 0$ for all $x \in M$ and all regular elements $c$ of $R$. 
Example

Consider the ring $R = \mathbb{Z}[x]$, and let $F$ be its field of quotients. For $I = xR$, $R/I$ has a free additive group and $R/I \otimes_{\mathbb{Z}} F$ is an $R$-$R$-bimodule which is torsion on the left, but torsion-free on the right.

To overcome this inherent non-symmetry of bimodules, an $R$-$R$-bimodule $M$ \textit{centrally symmetric} if $rx = xr$ for all $x \in M$ and all $r \in C(R)$ where $C(R)$ is the center of $R$. 
Let $M$ be an $R$-$R$-bimodule which is non-singular as a right $R$-module. To decide whether $E_R$ is non-singular, consider $\alpha \in E$. If $I$ is an essential right ideal of $R$ with $\alpha I = 0$, then $\alpha(IM) = 0$. Thus, $\alpha = 0$ provided that $IM_R$ is essential in $M_R$.

An $R$-$R$-bimodule $M$ such that $R_M$ and $M_R$ are non-singular has the right essentiality property if $IM_R$ is essential in $M_R$ for all essential right ideals $I$ of $R$. However, centrally symmetric bimodules need not have the right essentiality property as the next example shows:
Let $R$ be the lower triangular $2 \times 2$-matrix ring over a field $F$ whose right and left maximal ring of quotients is $Q = \text{Mat}_2(F)$. Let $e_1$ and $e_2$ be the canonical orthogonal idempotents having 1. Finally, $f$ is the element of $R$ whose only non-zero element is a 1 in the lower left hand corner. Then, $e_1f = fe_2 = f^2 = 0$ while $fe_1 = e_2f = f$. Moreover, every element $r$ of $R$ can be written as $r = ae_1 + be_2 + cf$ for uniquely determined $a, b, c \in F$. The right ideal $M = e_2Q$ of $R$ also is a left ideal. $R$ is the endomorphism ring of $M_R$ while $Q$ is the endomorphism ring of $R M$. 
Moreover, $M$ is injective as a right $R$-module since it is a direct summand of $Q$. On the other hand, if it were injective as a left $R$-module too, then there would be an idempotent $e$ of $R$ such that $M = Re$. It is easy to see that $e = xf + ye_2$ for some $x, y \in F$. But, $(ae_1 + be_2 + cf)(xf + ye_2) = be$ shows that $Re = Fe$ has dimension 1 as a $F$-vector space, while $M$ has dimension 2. Therefore, $RM$ cannot be injective. Finally, the left ideal $I = Re_1 = Fe_1 + Ff$ of $R$ is two-sided since $le_2 = 0$. If $r \notin I$, then $r = ae_1 + be_2 + cf$ with $b \neq 0$. But then, $rf = be_2f = bf$ is a non-zero element of $I$, so that $I$ is essential as a right ideal of $R$. However, $IM = Re_1e_2R = 0$. 
A right and left non-singular ring $R$ is a right and left Utumi-ring if $Q^\ell = Q^r$. Every semi-prime right and left Goldie ring is a right and left Utumi-ring. For such rings, a right and left non-singular $R$-$R$-bimodule $M$ has the right essentiality property exactly if $cM$ is essential in $M_R$ for all regular elements $c$ of $R$. 
Lemma

Let $R$ be a semi-prime, right and left Goldie ring with classical ring of quotients $Q(R)$, and $M$ an $R$-$R$-bimodule $M$ such that $_RM$ and $_MR$ are non-singular.

a) (i) $cM$ is essential in $M_R$ for all regular elements $c \in R$ iff (ii) $M \otimes_R Q$ is injective as a left $R$-module iff, (iii) for all $x \in M$ and all regular elements $c \in R$, there exists $y \in M$ and $d \in R$ regular such that $cy = xd$.

b) Let $U$ be a two-sided submodule of $M$ such that $U$ is essential in $M$ as a right and left submodule. If $cM$ is essential in $M_R$, then $cU$ is essential in $U_R$.

c) If $U$ is a two-sided submodule of $M$ such that $(M/U)_R$ and $_R(M/U)$ are non-singular, then $cM$ is essential in $M_R$ if and only if $cU$ is essential in $U$ and $c(M/U)$ is essential in $M/U$. 
Proof: a) i) ⇒ iii): Let $c$ be a regular element of $R$. Since $cM$ is an essential submodule of $M_R$, the module $M/cM$ is singular. Therefore, there is a regular element $d \in R$ such that $xd = cy$ for some $y \in M$ as desired.

iii) ⇒ ii): Let $x \otimes rc^{-1} \in M \otimes_R Q$ for some $x \in M$, $r \in R$ and a regular element $c \in R$. Select a regular element $d$ of $R$ and $y \in M$ such that $xd = cy$ as in b), and observe

$$x \otimes rc^{-1} = xd \otimes d^{-1}rc^{-1} = cy \otimes d^{-1}rc^{-1} \in c(M \otimes_R Q).$$

Thus, $R(M \otimes_R Q)$ is a non-singular left $R$-module which is divisible in the classical sense. Since $M \otimes_R Q \cong_R Q^{(I)}$ for some index-set $I$, it is injective.
ii) \implies i) \text{ Let } c \text{ be a regular element of } R. \text{ For every } x \in M, \text{ there exist } y \in M, r \in R \text{ and a regular element } d \in R \text{ such that } c(y \otimes rd^{-1}) = x \otimes 1_Q \text{ since } M \otimes_R Q \text{ is divisible as a left } R\text{-module. We obtain } cyr \otimes 1_Q = xd \otimes 1_Q. \text{ Thus, } xd = cyr \text{ in view of the fact that the embedding } M \to M \otimes_R Q \text{ is one-to-one because } M \otimes_R Q \text{ is the injective hull of } M_R. \text{ Therefore, } M/cM \text{ is singular, from which we obtain that } cM \text{ is essential in } M \text{ because } M \text{ is non-singular.}
b) Suppose that \( c \in R \) is regular. Since \( _RM \) is non-singular, left multiplication by \( c \) is an isomorphism \( _RM \rightarrow c_M \). Hence, \( cM/cU \cong M/U \) is singular. However, \( M/cM \) is singular too since \( cM \) is essential in \( M \). Therefore, \( M/cU \) is singular, and \( cU \) is essential in \( U \).

c) Suppose that \( cM \) is essential in \( _RM \) for a regular element \( c \) of \( R \). For every \( u \in U \), we can find \( y \in M \) and a regular element \( d \) of \( R \) such that \( ud = cy \) by a). Then, \( y + U \in Z(R(M/U)) = 0 \), and \( y \in U \). Thus, \( U/cU \) is singular. Moreover, \( (M/U)/c(M/U) \cong M/[cM + U] \) is singular as an image of \( M/cM \). Since \( M/U \) is non-singular, \( c(M/U) \) is essential in \( M/U \).
Conversely assume that $cU$ is essential in $U$ and that $c(M/U)$ is essential in $M/U$ for some regular element $c$ of $R$. Consider the short-exact sequence

$$0 \rightarrow [U + cM]/cM \rightarrow M/cM \rightarrow M/[U + cM] \rightarrow 0.$$ 

Since $_R(M/U)$ is non-singular, $cM \cap U = cU$ and $[U + cM]/cM \cong U/cU$ is singular since $cU$ is essential in $U$. Moreover, $M/[U + cM]$ is isomorphic to the singular module $(M/U)/c(M/U)$. Therefore, $M/cM$ has to be singular too.
A ring $R$ is right bounded if every essential right ideal of $R$ contains a two-sided ideal which is essential as a right ideal.

**Theorem**

The following are equivalent for a ring $R$ with a maximal right and left ring of quotients $Q(R)$:

a) $R$ is semi-prime.

b) Every two-sided submodule $M$ of the $R$-$R$-bimodule $Q(R)$ has the right essentiality property.

c) An $R$-$R$-bimodule $M$ such that $RM$ and $MR$ are non-singular and finite dimensional has the right essentiality property.

Moreover, for such a ring, $R^X$ has the right essentiality property for all index-sets $X$ if and only if $R$ is right bounded.
Proof: Observe that a right and left non-singular right and left Utumi-ring $R$ has finite right Goldie dimension if and only if it has finite left Goldie dimension.

\( a \Rightarrow c \): If $R$ is a semi-prime right and left Goldie ring, then every essential right ideal $I$ of $R$ contains a regular element $c$. Since $M$ is non-singular as a left $R$-module, $cM \cong M$ as a right $R$-module. In particular, $\dim cM = \dim M < \infty$ yields that $cM$ is essential in $M_R$.

Since $R$ is right and left Utumi and has finite right and left Goldie dimension, $c \Rightarrow b$ is obvious.
b) ⇒ a): To show that $R$ is semi-prime, let $N$ be a two-sided ideal of $R$ with $N^2 = 0$. Select a right ideal $I$ of $R$ which is maximal with respect to the property $N \cap I = 0$. Then $N \oplus I$ is an essential right ideal of $R$ such that $(N \oplus I)N = N^2 + IN \subseteq IN = 0$. By the right essentiality property, $IN$ is essential in $N$, so that $N = 0.$
Finally, let $R$ be a semi-prime right and left Goldie ring. Suppose that $R$ is right bounded, and let $X$ be any index-set. Consider a regular element $c$ of $R$, and choose a two-sided ideal $I$ of $R$ such that $I \subseteq cR$ and $I_R$ is essential in $R_R$. Since $R$ is a semi-prime right Goldie ring, $I$ contains a regular element $d$. If $(r_x)_{x \in X} \in R^X$, then $r_xd \in I \subseteq cR$ for all $x \in X$, say $r_xd = cs_x$. Then, $(r_x)_{x \in X}d = c(s_x)_{x \in X} \in cR^X$, and $cR^X$ is essential in $R^X$. 
Conversely, let $c$ be a regular element of $R$. Since $c(R^R)$ is essential in $R^R$, there is a regular element $d \in R$ such that $(r)_{r \in R}d \in cR^R$. Therefore, for every $r \in R$, there is $s_r \in R$ such that $rd = cs_r \in cR$. Thus, $Rd \subseteq cR$, and $cR$ contains the two-sided ideal $RdR$ which is essential as a right and as a left ideal of $R$ since it contains the regular element $d$. □
Let $R$ be a semi-prime right and left bounded, right and left Goldie ring. The following are equivalent if $R^+$ is torsion-free:

- **a)** $\mathbb{Q}R = \mathbb{Q} \otimes \mathbb{Z} R$ is semi-simple Artinian.
- **b)** If $M$ is an $R$-$R$-bimodule, then $M_R$ is singular if and only if $R^+_M$ is singular.
- **c)** If $M$ is an $R$-$R$-bimodule, then $Z(M_R) = Z(R^+_M)$.
- **d)** If $M$ is an $R$-$R$-bimodule, then $M_R$ is non-singular if and only if $R^+_M$ is non-singular.
- **e)** Every $R$-$R$-bimodule $M$ such that $M_R$ and $R^+_M$ are non-singular has the right and left essentiality property.
Proof: \( a) \Rightarrow b): \) If \( I \) is an essential right ideal of \( R \), then \( \mathbb{Q}I \) is an essential right ideal of the semi-simple Artinian ring \( \mathbb{Q}R \). Since \( \mathbb{Q}I = \mathbb{Q}R \), there is a non-zero integer \( n \) with \( nR \subseteq I \). Hence, an \( R \)-module is singular if and only if its additive group is torsion. Thus, \( b) \) obviously holds.

\( b) \Rightarrow c):\) By symmetry, it suffices to show \( Z(M_R) \subseteq Z(RM) \). If \( x \in Z(M_R) \), then there is an essential right ideal \( I \) of \( R \) such that \( xI = 0 \). Thus, \( (rx)I = 0 \) for all \( r \in R \), and \( Z(M_R) \) is a submodule of \( RM \) too. However, \( Z(M_R)_R \) is singular yields that \( _RZ(M_R) \) is singular by \( b) \). Hence, \( Z(M_R) \subseteq Z(RM) \).

\( c) \Rightarrow d):\) If \( M_R \) is non-singular, then \( 0 = Z(M_R) = Z(RM) \) by \( c) \).
Let $M$ be an $R$-$R$-bimodule such that $0 = Z(M_R) = Z(RM)$. Suppose that there is an essential right ideal $I$ of $R$ such that $IM$ is not essential in $M_R$. Since $M_R$ is non-singular, the $S$-closure $(IM)^{cl}$ of $IM$ in $M_R$ is a proper submodule of $M_R$. Because $R$ is right bounded, $I$ contains a two-sided ideal $J$ which is essential as a right ideal. Observe that $J$ contains a regular element $d$. But, $Rd \subseteq J$ yields that $J$ is also essential as a left ideal of $R$ since $R$ is a semi-prime left Goldie ring. Let $N$ be the $S$-closure of $JM$ in $M_R$. Since $N \subseteq (IM)^{cl}$, we obtain that $M/N$ is a non-zero, non-singular right $R$-module.
To see that $N$ is also a submodule of $RM$, let $x \in N$, and $r \in R$, and select an essential right ideal $K$ of $R$ such that $xK \subseteq JM$. Since $J$ is a two-sided ideal of $R$, we obtain $(rx)K \subseteq R(JM) = JM$. Thus, $rx \in (JM)^{cl} = N$. Consequently, $M/N$ is an $R$-$R$-bimodule. Since $(M/N)_R$ is non-singular, the same holds for $R(M/N)$ by d). Since $(IM)^{cl}$ is a proper submodule of $M$, we can find $x \in M \setminus (IM)^{cl}$. Observe that $x + N$ is a non-zero element of $R(M/N)$ such that $dx \in JM \subseteq N$. Hence, $Rdx \subseteq N$. Since $d$ is a regular element of $R$, $Rd$ is an essential left ideal of $R$. Thus, $0 \neq x + N \in Z(R(M/N)) = 0$, a contradiction.
e) ⇒ a): Let $L$ be a left $R$-module and $N$ a right $R$-module. The Abelian group $L \otimes \mathbb{Z} N$ carries a right and left $R$-module structure defined by $[x \otimes y]r = x \otimes (yr)$ and $s[x \otimes y] = sx \otimes y$ for all $x \in L$, $y \in N$ and $r, s \in R$. Since
\[
r([x \otimes y]s) = r[x \otimes (ys)] = (rx) \otimes (ys) = [(rx) \otimes y]s = (r[x \otimes y])s,
\]
$L \otimes \mathbb{Z} N$ is an $R$-$R$-bimodule.
It suffices to show that \((R/I)^+\) is torsion for all essential left ideals \(I\) of \(R\). Then, \(Q\) will be the classical ring of quotients of \(R\). If \(I\) is an essential left ideal of \(R\) such that \(R/I\) is not torsion, then we may assume that \((R/I)^+\) is a non-zero torsion-free Abelian group since we can replace \(I\) with its \(\mathbb{Z}\)-purification \(I_*\) if necessary. Here, the \(\mathbb{Z}\)-purification \(U_*\) of a subgroup \(U\) of a torsion-free Abelian group \(G\) is the smallest pure subgroup of \(G\) containing \(U\). Let \(Q(R)\) be the classical right and left ring of quotients of \(R\). Since \(Q(R)^+\) is torsion-free, \(M = (R/I) \otimes_{\mathbb{Z}} Q(R) \neq 0\) is a non-zero \(R\)-\(R\)-bimodule as was shown in the last paragraph.
Since right multiplication by a regular element of \( R \) is a \( \mathbb{Z} \)-automorphism of \( Q(R) \), \( X \otimes \mathbb{Z} Q(R) \) is a non-singular right \( R \)-module for all choices of \( X \). Moreover, left multiplication by a regular element \( d \) of \( R \) is a \( \mathbb{Z} \)-monomorphism \( R \rightarrow R \). Since \( R^+ \) is torsion-free, it is flat as an Abelian group, and left multiplication by \( d \) is a \( \mathbb{Z} \)-monomorphism from \( R \otimes \mathbb{Z} Q(R) \) to \( R \otimes \mathbb{Z} Q(R) \). Therefore, \( R \otimes \mathbb{Z} Q(R) \) is non-singular as a left \( R \)-module too. In the induced exact sequence
\[
0 \rightarrow I \otimes \mathbb{Z} Q(R) \rightarrow R \otimes \mathbb{Z} Q(R) \rightarrow M \rightarrow 0,
\]
all maps are right and left \( R \)-module homomorphisms. Because \( M_R \) is non-singular as a right \( R \)-module, \([I \otimes \mathbb{Z} Q(R)]_R \) cannot be essential in \([R \otimes \mathbb{Z} Q(R)]_R \).
Since $R$ is a left bounded ring, there is a two-sided ideal $J$ of $R$ such that $J \subseteq I$ and $J$ is essential in $R \oplus R$. As before, $J$ contains a regular element of $R$, and, therefore, is essential as a right ideal too. Observe that $J[R \otimes \mathbb{Z} Q(R)] \subseteq I \otimes \mathbb{Z} Q(R)$ since, for all $x \in J$, $r \in R$ and $q \in Q(R)$, we have $x[r \otimes q] = xr \otimes q \in I \otimes \mathbb{Z} Q(R)$ since $xr \in JR = J \subseteq I$. Since $J[R \otimes \mathbb{Z} Q(R)]_R$ is essential in $(R \otimes \mathbb{Z} Q(R))_R$ by e), we obtain that $(I \otimes \mathbb{Z} Q(R))_R$ is essential in $(R \otimes \mathbb{Z} Q(R))_R$, which contradicts what has already been shown. \qed
Corollary

The following conditions are equivalent for a right and left non-singular, right and left Utumi-ring $R$ such that $R^+$ is torsion-free:

a) $R$ is right and left bounded ring such that $\mathbb{Q}R$ is semi-simple Artinian.

b) Every $R$-$R$-bimodule $M$, such that $M_R$ and $R_M$ are non-singular, has the right and left essentiality property.

Furthermore, every nonsingular $R$-$R$-bimodule has an endomorphism ring which is right and left non-singular as a right and left $R$-module in this case.
A ring $R$ satisfies the *central condition* if every essential right ideal of $R$ contains a central regular element of $R$. For instance, the group algebra $\mathbb{Z}[G]$ satisfies the central condition whenever $G$ is a finite group.

**Corollary**

Consider the following conditions for a ring $R$.

a) $R$ satisfies the central condition.

b) $R$ is a semi-prime, right and left bounded, right and left Goldie ring such that the following conditions hold for all centrally symmetric $R$-$R$-bimodules $M$:

i) $Z(RM) = Z(M_R)$.

ii) If $M_R$ is non-singular, then $RM$ is non-singular, and $M$ has the right and left essentiality property.

Then a) always implies b), and the converse holds if $R$ is prime.
Closely related to the essentiality property is the following: A right and left non-singular $R$-$R$-bimodule has the finite rank property provided every finite dimensional submodule of $M_R$ is contained in a finite dimensional submodule of $R M$ and vice-versa.

**Theorem**

Let $R$ be a semi-prime right and left Goldie ring.

a) Every bimodule with the finite rank property has the right and left essentiality property.

b) If $R^+$ is a torsion-free group, then every right and left non-singular $R$-$R$-bimodule has the finite rank property if and only if $R^+$ has finite rank.
However, there exist torsion-free rings $R$ of infinite rank such that $\mathbb{Q}R$ is a division algebra. Consequently, there exist rings $R$ such that all $R$-$R$-bimodules which are non-singular as a right and as a left module have the right and left essentiality property, but not all such modules have the finite rank property. Moreover, considering centrally symmetric bimodules instead of arbitrary bimodules will only replace the condition that $R^+$ has finite rank by the requirement that $R$ has finite dimension as a module over its center similar to the situation in Corollary 6.