Trigonometric Approximation in Lie Groups

Wayne Lawton

Linear Algebra Seminar, March 29, 2016

Department of Mathematics and Statistics
Auburn University, Auburn, Alabama

Abstract

Weierstrass proved that every continuous function from the circle group R/Z to the Lie group R^n can be uniformly approximated by functions with trigonometric polynomial entries. This property also holds for semisimple Lie groups. We discuss approximation methods and applications to optics, robotics, and wavelets.

Orthonormal Wavelet Bases

Orthonormal wavelets bases for $L^2(R)$

$$\{2^{J/2}\psi(2^J x-k): J,k\in Z\}$$

$$\psi(x)=2\sum_{k\in Z} d(k)\,\varphi(2x-k)$$

$$\varphi(x)=2\sum_{k\in Z} c(k)\,\varphi(2x-k)$$

where c and d satisfy: $\sum_{k\in Z} c(k)=1$

$$\sum_{k\in Z} c(k)\,\overline{c(k-2l)}=\sum_{k\in Z} d(k)\,\overline{d(k-2l)}=\tfrac{1}{2}\delta_{0,l},\ l\in Z$$

$$\sum_{k\in Z} c(k)\,\overline{d(k-2l)}=0,\ l\in Z$$

Fourier Transforms

$$z = e^{-2\pi i\theta} \in T$$

$$C(z) = \sum_{k\in Z} c(k) z^k \qquad D(z) = \sum_{k\in Z} d(k) z^k$$

satisfy: $C(1) = 1$

$$|C(z)|^2 + |C(-z)|^2 = |D(z)|^2 + |D(-z)|^2 = 1$$

$$C(z)D(z) + C(-z)D(-z) = 0$$

$$\Leftrightarrow M(1) = I, M(z) = \begin{bmatrix} C(z) & D(z) \\ C(-z) & D(-z) \end{bmatrix}$$

$$= \begin{bmatrix} C(z) & -\overline{C(-z)} \\ C(-z) & \overline{C(z)} \end{bmatrix} \in SU(2), z \in T$$

Polyphase Matrix

$$M(z) = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} P(z^2) \begin{bmatrix} 1 & 0 \\ 0 & -\bar{z} \end{bmatrix}$$

where $P \in C(T, SU(2))$.

Question Can P (so M) be approximated by $\widetilde{P} \in C_{pol}(T, SU(2))$, loops with Laurent; trigonometric polynomial entries in $z; \theta$.

Answer: Yes, we refine the sketchy proof in given in Ref. L99. We may assume that P is smooth (in fact real analytic).

Hopf Fibration

$$SU(2) \ni g \to \{ \begin{bmatrix} w & u+iv \\ u-iv & -w \end{bmatrix} \to g \begin{bmatrix} w & u+iv \\ u-iv & -w \end{bmatrix} g^{-1} : \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in R^2 \}$$

gives the exact sequence

$$\{\pm I_2\} \to SU(2) \xrightarrow{h} SO(3) \to \{I_3\}$$

and fibration

$$S^3 \approx SU(2) \ni g \xrightarrow{H} h(g)\, e_3 \in S^3$$

whose fibers are the right cosets

$$\{ g \begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix} : z \in T \},\ g \in T.$$

Sard's Theorem

Since the entries of P are smooth

$\exists\, g \in SU(2)$ such that $\{e_3, -e_3\}$

is disjoint from the image of

$$H(gP) = h(gP)\,e_3 = h(g)h(P)\,e_3 = h(g)H(P)$$

$$Q = gP = \begin{bmatrix} Q_e & -\overline{Q}_o \\ Q_o & \overline{Q}_e \end{bmatrix} \Leftrightarrow \{\pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\}$$

is disjoint from the image of $Q\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}Q^{-1} =$

$$\begin{bmatrix} |Q_e|^2 - |Q_o|^2 & 2Q_e\overline{Q}_o \\ 2Q_o\overline{Q}_e & |Q_o|^2 - |Q_e|^2 \end{bmatrix} \Leftrightarrow \begin{matrix} Q_e(z) \neq 0,\ z \in T \\ Q_o(z) \neq 0,\ z \in T \end{matrix}$$

Winding Numbers

$$Q = \begin{bmatrix} Q_e & -\overline{Q}_o \\ Q_o & \overline{Q}_e \end{bmatrix} = \begin{bmatrix} \alpha \, | Q_e | & -\overline{\beta} \, | Q_o | \\ \beta \, | Q_o | & \overline{\alpha} \, | Q_e | \end{bmatrix}$$

where $\alpha, \beta \in C(T,T)$

$$Q(z^2) = \begin{bmatrix} \gamma(z) & 0 \\ 0 & \overline{\gamma(z)} \end{bmatrix} \begin{bmatrix} |Q_e(z^2)| & -|Q_o(z^2)| \\ |Q_o(z^2)| & |Q_e(z^2)| \end{bmatrix} \begin{bmatrix} \delta(z) & 0 \\ 0 & \overline{\delta(z)} \end{bmatrix}$$

$$\Leftrightarrow \gamma(z)^2 = \alpha(z^2)\beta(z^2), \delta(z)^2 = \alpha(z^2)\overline{\beta(z^2)}$$

winding numbers of right sides are even so

$\exists \gamma, \delta \in C(T,T)$ satisfying these equations.

Dirichlet and Fejer Kernels

$$D_N(z) = \sum_{-N}^{N} z^n = \frac{\sin(2N+1)\pi\theta}{\sin \pi\theta}$$

$$K_N(z) = \tfrac{1}{N+1}\sum_{0}^{N} D_N(z) = \tfrac{1}{N+1}\left(\frac{\sin(N+1)\pi\theta}{\sin \pi\theta}\right)^2$$

For every $f \in C(T, C)$ the convolution

$$(K_N * f)(z) = \int_0^1 K_N(e^{2\pi i\theta}) f(e^{-2\pi i\theta} z) d\theta$$

is a Laurent polynomial of degree N and

$$\lim_{N \to \infty} K_N * f = f.$$

Riesz-Fejer Spectral Factorization Lemma

If $f \in C_{pol}(T,[0,\infty))$ is a Laurent polynomial of degree N then there exists and algebraic polynomial $g \in C_{pol}(T,C)$ of degree N with $f = |g|^2$. We observe that there exist Laurent polynomials $Q_{e,N}, Q_{o,N}$ with

$$K_N * |Q_e|^2 = |Q_{eN}|^2 \quad K_N * |Q_o|^2 = |Q_{o,N}|^2$$

$$|Q_{e,N}|^2 + |Q_{e,N}|^2 = 1$$

Back to Winding Numbers

$$Q_N = \begin{bmatrix} Q_{e,N} & -\overline{Q}_{o,N} \\ Q_{o,N} & \overline{Q}_{e,N} \end{bmatrix} = \begin{bmatrix} \alpha_N \,|Q_{e,N}| & -\overline{\beta}_N \,|Q_{o,N}| \\ \beta_N \,|Q_{o,N}| & \overline{\alpha}_N \,|Q_{e,N}| \end{bmatrix}$$

where $\alpha_N, \beta_N \in C(T,T)$

$$Q_N(z^2) = \begin{bmatrix} \gamma_N(z) & 0 \\ 0 & \overline{\gamma_N(z)} \end{bmatrix} \begin{bmatrix} |Q_{e,N}(z^2)| & -|Q_{o,N}(z^2)| \\ |Q_{o,N}(z^2)| & |Q_{e,N}(z^2)| \end{bmatrix} \begin{bmatrix} \delta_N(z) & 0 \\ 0 & \overline{\delta_N(z)} \end{bmatrix}$$

$$\Leftrightarrow \gamma_N(z)^2 = \alpha_N(z^2)\beta_N(z^2), \delta_N(z)^2 = \alpha_N(z^2)\overline{\beta_N(z^2)}$$

winding numbers of right sides are even so

$\exists \gamma, \delta \in C(T,T)$ satisfying these equations.

Milestone

$$Q(z^2) = \lim_{N\to\infty} \begin{bmatrix} \gamma(z) & 0 \\ 0 & \overline{\gamma(z)} \end{bmatrix} \begin{bmatrix} |Q_{e,N}(z^2)| & -|Q_{o,N}(z^2)| \\ |Q_{o,N}(z^2)| & |Q_{e,N}(z^2)| \end{bmatrix} \begin{bmatrix} \delta(z) & 0 \\ 0 & \overline{\delta(z)} \end{bmatrix}$$

$$= \lim_{N\to\infty} \Gamma_N(z) \begin{bmatrix} Q_{e,N}(z^2) & -Q_{o,N}(z^2) \\ Q_{o,N}(z^2) & Q_{e,N}(z^2) \end{bmatrix} \Delta_N(z) \quad \text{where}$$

$$\Gamma_N(z) = \begin{bmatrix} \gamma(z)\overline{\gamma_N(z)} & 0 \\ 0 & \overline{\gamma(z)}\gamma_N(z) \end{bmatrix}$$

$$\Delta_N(z) = \begin{bmatrix} \delta(z)\overline{\delta_N(z)} & 0 \\ 0 & \overline{\delta(z)}\delta_N(z) \end{bmatrix}$$

Finish Line

It suffices to show that every diagonal loop

$$\begin{bmatrix} f & 0 \\ 0 & \overline{f} \end{bmatrix} \in C(T, SU(2))$$

can be approximated

by elements in $C_{pol}(T, SU(2))$. Approximate

$f \approx f_N = K_N * f$, observe that $1 - | f_N |^2 > 0$ and

choose $g_N \in C_{pol}(T, C)$ so $1 - | f_N |^2 = | g_N |^2$.

Then

$$\begin{bmatrix} f & 0 \\ 0 & \overline{f} \end{bmatrix} \approx \begin{bmatrix} f_N & -\overline{g}_N \\ g_N & \overline{f}_N \end{bmatrix} \in C_{pol}(T, SU(2)).$$

Conjugate Quadrature Filters

$\mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{T}$ integer, real, complex, unit circle

$C(\mathbb{T})$ $\mathbb{C}$ - valued, continuous functions

$P(\mathbb{T})$ Laurent polynomials

$m \geq 2$ fixed integer

$\omega \equiv e^{i 2\pi / m}$ primitive m-th root of unity

$P_Q(\mathbb{T}), C_Q(\mathbb{T})$ conjugate quadrature filters

$$\sum_{k=0}^{m-1} | F(\omega^k z) |^2 = 1, \quad z \in \mathbb{T}$$

Applications and Requirements

CQF's are used to construct paraunitary filter banks and orthonormal wavelet bases

$P_Q(\mathrm{T})$ FIR filters, compactly supp. wavelets

$C_Q(\mathrm{T}) \setminus P_Q(\mathrm{T})$ linear phase filters

Factor $U(z) \equiv [(1-z^m)/(1-z)]^d$ for regularity

needed for stable filterbanks & smooth wavelets

Design Approaches

Much more difficult to design polynomial CQF's

Jorgensen describes an approach based on factorizing their polyphase representations (Notices of the AMS, 50(8)(2003),880-894)

We describe an alternate approach that is based on approximating elements in $C_Q(\mathrm{T})$ by elements in $P_Q(\mathrm{T})$

This approach can preserve specified factors

Two-Step Approximation Method

Problem given $U \in P(\mathrm{T}), H \in C(\mathrm{T}) \ni UH \in C_Q(\mathrm{T})$

construct $P \in P(\mathrm{T}) \ni UP \in P_Q(\mathrm{T}), P \approx H$

Solution

Step One

construct $Q \in P(\mathrm{T}) \ni UQ \in P_Q(\mathrm{T}), |Q| \approx |H|$

Step Two

construct $P \in P(\mathrm{T}) \ni UP \in P_Q(\mathrm{T}), P \approx H$

Table of Contents

Polyphase Representations

Theorem The functors

$$\mathrm{T} \xrightarrow{\ \mathrm{f}\ } \mathrm{Y} \Rightarrow \mathrm{T} \xrightarrow{\ \tau \mathrm{f}\ } \mathrm{Y} \quad (\tau \mathrm{f})(\mathrm{z}) \equiv \mathrm{f}\,(\omega z), \quad z \in \mathrm{T}$$

$$\mathrm{T} \xrightarrow{\ \mathrm{h}\ } \mathrm{Y} \Rightarrow \mathrm{T} \xrightarrow{\ \sigma \mathrm{h}\ } \mathrm{Y} \quad (\sigma \mathrm{h})(\mathrm{z}) \equiv \mathrm{h}\,(z^m), \quad z \in \mathrm{T}$$

satisfy $\tau \mathrm{f} = \mathrm{f} \Leftrightarrow \exists \mathrm{h} : \mathrm{T} \rightarrow Y \ni \mathrm{f} = \sigma \mathrm{h}$

Corollary $\mathrm{v} : \mathrm{T} \rightarrow \mathrm{C}^{\mathrm{m}}$ is a <u>modulation vector</u>

$\mathrm{v}(\mathrm{F}) \equiv [\mathrm{F}, \tau \mathrm{F}, \ldots, \tau^{m-1} \mathrm{F}]^{\mathrm{T}}$ for $\mathrm{F} \in \mathrm{C}(\mathrm{T}) \Leftrightarrow \mathrm{Cv} = \tau \mathrm{v}$

where

$$C \equiv \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \cdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is the m x m <u>circulant matrix</u>

Polyphase Representations

Proposition $\forall \mathrm{w(F)} : \mathrm{T} \to \mathrm{C}^{\mathrm{m}}$ is polyphase vector for $\mathrm{F} \in \mathrm{C(T)} \Leftrightarrow \mathrm{v(F)} = \Omega\, \Lambda\, \sigma\, \mathrm{w(F)}$ where

$\Omega \equiv$ (Fourier transform)

$$\frac{1}{\sqrt{m}}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{-2} \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega \end{bmatrix}$$

$\Lambda(z) \equiv$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & z & 0 & \cdots & 0 \\ 0 & 0 & z^2 & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & z^{m-1} \end{bmatrix}$$

Corollary $\mathrm{F(z)} = \sum_{k=1}^{m} z^{k-1} w(F)_k (z^m),$

$$\mathrm{F} \in \mathrm{P(T)} \Leftrightarrow w(F)_k \in \mathrm{P(T)},$$

$$\mathrm{F} \in \mathrm{C}_Q(\mathrm{T}) \Leftrightarrow w(F) : \mathrm{T} \to S^{2m-1} \subset C^m$$

Winding Number

Definition The winding number of $f: T \to T$

$$W(f) \equiv \frac{1}{i2\pi}\int_0^{2\pi} \frac{df(e^{i\theta})}{f(e^{i\theta})}$$ if f is differentiable

$$W(f) \equiv W(\tilde{f})$$ $\tilde{f}$ is differentiable and $\| f - \tilde{f} \| < 2$

Remark W(f) is well defined, takes values in Z, is a continuous function of f, is a special case of the Brouwer degree of a map of sphere to itself

Lemma Given f:T→T there exists h:T→iR with

$f = \exp(h)$ iff $W(f) = 0$

Homotopy and Matrix Extension

Definition Maps $f_i : S^n \to S^n, i = 0,1$ are homotopic iff $\exists\, F : [0,1] \times S^n \to S^n \ni F(j,\cdot) = f_j,\ j = 0,1$

Theorem (H. Hopf) Map of a sphere into itself are homotopic iff their Brouwer degrees are equal

Corollary f is homotopic to constant iff W(f)=0

Proposition

$$\forall\, f : T \to S^{2m-1}, \exists\, g : T \to SU(m) \ni g_{*,1} = f$$

Proof Let $e_1 \equiv [1,0,...,0]^T$ then $g \to p(g) \equiv g e_1$ is a fiber bundle, and hence a fibration $p : SU(m) \to S^{2m-1} = SU(m)/SU(m-1)$ and the result follows from the homotopy lifting property

Definition g is a polyphase matrix for f

Algebra and Matrix Extension

Proposition If entries $f : T \rightarrow C^m$ in $P(T)$ and have no common zeros in $C \setminus \{0\}$ then

$\exists\, g : T \rightarrow SL(m),$ with entries in $P(T)$ and $g_{*,1} = f$

Proof Follows from the Smith form for f

Proposition If entries $f : T \rightarrow S^{2m-1}$ in $P(T)$ then

$\exists\, g : T \rightarrow SU(m),$ with entries in $P(T)$ and $g_{*,1} = f$

Proof Follows from the factorization theorem for m x 1 paraunitary matrices

Loop Groups

Remark Elements in $C(T) \otimes C^{m \times m}$, called loops, may be regarded as matrix-valued functions on T or as matrices having values in C(T)

Definition Loop groups

$$G \equiv C(T) \otimes SU(m)$$

$$G^{\infty} \equiv C^{\infty}(T) \otimes SU(m)$$

$$G_{pol} \equiv P(T) \otimes SU(m)$$

their Lie algebras

$$\mathcal{G} \equiv C(T) \otimes su(m)$$

$$\mathcal{G}^{\infty} \equiv C^{\infty}(T) \otimes su(m)$$

$$\mathcal{G}_{pol} \equiv P(T) \otimes su(m)$$

Exponential Function

Proposition Let $O \subset \mathrm{su(m)}$ be matrices whose spectral radius $< \pi$ Then $\exp : \mathrm{su(m)} \rightarrow \mathrm{SU(m)}$ is a real-analytic diffeomorphism of O onto an open neigborhood O of $\mathrm{I} \in \mathrm{SU(m)}$

Proposition (Trotter) Given $\mathrm{h}_1, \ldots, \mathrm{h}_\mathrm{M} \in G$

$$\lim_{\mathrm{L} \rightarrow \infty} \left[\exp\left(\tfrac{\mathrm{h}_1}{\mathrm{L}}\right) \cdots \exp\left(\tfrac{\mathrm{h}_\mathrm{M}}{\mathrm{L}}\right) \right]^\mathrm{L} = \exp\left(\mathrm{h}_1 + \cdots + \mathrm{h}_\mathrm{M}\right)$$

Furthermore, if $\mathrm{h}_1, \ldots, \mathrm{h}_\mathrm{M} \in G^\infty$ then convergence holds in the $C^\infty(\mathrm{T})$ topology

Magic Basis

Theorem For $n \geq 0,\ \rho \in \{1, i\}$ define

$$a(n,\rho,z) \equiv \begin{bmatrix} 0 & \rho\, z^{n} \\ -\overline{\rho}\, z^{-n} & 0 \end{bmatrix},\ b(n,\rho,z) \equiv \overline{a(n,\rho,z)}$$

$$c(n,\rho,z) \equiv \frac{1}{2}\begin{bmatrix} \rho\, z^{n} - \overline{\rho}\, z^{-n} & -\rho\, z^{n} - \overline{\rho}\, z^{-n} \\ \rho\, z^{n} + \overline{\rho}\, z^{-n} & -\rho\, z^{n} + \overline{\rho}\, z^{-n} \end{bmatrix}$$

$X \equiv \{c(0,i,z), a(0,i,z), a(0,i,z)\}$ is basis for $\mathrm{su}(2)$

$\mathrm{B}_2 \equiv X \cup \{a, b, c : n > 0, \rho = 1, i\}$ basis $\mathrm{P(T)} \otimes \mathrm{su}(2)$

leads to basis B for G_{pol} and $B \in \mathrm{B} \Rightarrow B^{2} = -I$

Density

Theorem $\mathcal{G}_{pol}$ is dense in $\mathcal{G}^{\infty}, \mathcal{G}$

Proof Euler's formula implies that

$$B \in \mathcal{B} \Rightarrow \exp \theta B = \cos \theta I + \sin \theta B \in \mathcal{G}_{pol}$$

Trotter's formula implies that every element in

$\exp G_{pol}$ is the limit of elements in $\mathcal{G}_{pol}$ and every

element in $\mathcal{G}$ is the product of elements in $\exp G_{pol}$

Corollary $\mathcal{P}_Q(\mathrm{T})$ is dense in $\mathcal{C}_Q(\mathrm{T})$

Proof Approximate polyphase matrix of $\mathrm{F} \in \mathcal{C}_Q(\mathrm{T})$

Spectral Factorization

Definition Let $H \in C_+(T)$ A function $F \in C(T)$ is a spectral factor of H if $|F|^2 = H$

Definition $P \in P(T)$ is minimal phase if all its roots have modulus ≥ 1

Theorem (L. Fejer and F. Riesz) Every $P \in P_+(T)$ has a minimal phase spectral factor

Definition $F \in C(T)$ is an outer function if $\exists c \in T, H \in C_+(T) \ni \log H \in L^1(T)$ and

$$F(z) = c \exp\left[\lim_{r \uparrow 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{is} + rz}{e^{is} - rz} \log H(e^{is}) \, ds \right]$$

Bezout Identities

Theorem If $\mathrm{U}_1,...,\mathrm{U}_m \in \mathrm{P}_+(\mathrm{T})$ have no common roots in $C\setminus\{0\}$ and $\mathrm{H}_1,...,\mathrm{H}_m \in C(\mathrm{T})$ satisfy the Bezout identity $\mathrm{U}_1\mathrm{H}_1+\cdots+\mathrm{U}_\mathrm{m}\mathrm{H}_m=1$ then

$$\forall\delta>0,\ \exists\,\mathrm{Q}_1,...,\mathrm{Q}_m\in\mathrm{P}(\mathrm{T})\ni$$

$$\mathrm{U}_1\mathrm{Q}_1+\cdots+\mathrm{U}_\mathrm{m}\mathrm{Q}_m=1,\quad \|H_k-Q_k\|<\delta,\ k=1,..,m$$

Proof Uses matrix extension in $\mathrm{P}(\mathrm{T})\otimes\mathrm{SL}(\mathrm{m})$ and Weierstrass approximation

Remark Extends the 1-dim version of a multi-dim result in W. Lawton and C. A. Micchelli, Bezout identities with inequality constraints, Vietnam Journal of Mathematics 28#2(2000),1-29

Step One

Theorem If $\mathrm{H} \in \mathrm{C}(\mathrm{T}), \mathrm{U} \in \mathrm{P}(\mathrm{T}), \mathrm{UH} \in \mathrm{C}_{\mathrm{Q}}(\mathrm{T})$

then $\forall \varepsilon > 0, \exists Q \in \mathrm{P}(\mathrm{T}) \ni Q$ has no zeros in T

$\mathrm{UQ} \in \mathrm{P}_{\mathrm{Q}}(\mathrm{T})$ and $\| \, |H| - |Q| \, \| < \varepsilon$

Proof Uses previous theorem

Modulation Matrices

Definition $\mathrm{V}:\mathrm{T}\rightarrow C^{m\times m}$ is a (unitary) modulation matrix if it maps T into U(m) and if $\mathrm{CV}=\tau\mathrm{V}$

Proposition $\mathrm{V}:\mathrm{T}\rightarrow C^{m\times m}$ is a modulation matrix iff $\exists\,\mathrm{W}:\mathrm{T}\rightarrow\mathrm{U(m)}\ni\mathrm{V}=\Omega\Lambda\sigma\mathrm{W}$

Furthermore $\mathrm{V}_{\mathrm{i,j}}\in\mathrm{P(T)}\Leftrightarrow\mathrm{W}_{\mathrm{i,j}}\in\mathrm{P(T)}$ and

$\mathrm{F}\in\mathrm{C}_{\mathrm{Q}}(\mathrm{T})\Rightarrow\exists\,\text{modulation matrix }\mathrm{V}\ni\mathrm{V}_{1,1}=\mathrm{F}$

and if $\mathrm{F}\in\mathrm{P}_{\mathrm{Q}}(\mathrm{T})$ we may choose $\mathrm{V}\ni\mathrm{V}_{\mathrm{i,j}}\in\mathrm{P(T)}$

Proof Follows directly from previous results

Stabilizer Subgroups

Definition Subroups $\mathrm{S}_{\mathrm{r}} \equiv \sigma \mathrm{G}, \quad \mathrm{S}_{\ell} \equiv \Omega \Lambda \mathrm{S}_{\mathrm{r}} \Lambda^{-1} \Omega^{-1}$

Lie algebras

$$S_{\mathrm{r}} \equiv \{h \in G : \exp h \in \mathrm{S}_{\mathrm{r}}\}, \quad S_{\ell} \equiv \{h \in G : \exp h \in \mathrm{S}_{\ell}\}$$

Subroups $\mathrm{S}_{\mathrm{r}}^{\infty} \equiv \mathrm{S}_{\mathrm{r}} \cap \mathrm{G}^{\infty}, \quad \mathrm{S}_{\ell}^{\infty} \equiv \mathrm{S}_{\ell} \cap \mathrm{G}^{\infty}$

$$\Rightarrow \mathrm{S}_{\mathrm{r}}^{\infty} = \sigma \mathrm{G}^{\infty}, \quad \mathrm{S}_{\ell}^{\infty} = \Omega \Lambda \mathrm{S}_{\mathrm{r}}^{\infty} \Lambda^{-1} \Omega^{-1}$$

Lie algebras $S_r^{\infty} = S_r \cap G^{\infty}, \quad S_{\ell}^{\infty} = S_{\ell} \cap G^{\infty}$

Corollary $\mathrm{V} : \mathrm{T} \rightarrow C^{m \times m}$ a modulation matrix $g \in \mathrm{G}$

$\mathrm{g} \in \mathrm{S}_{\ell} \Leftrightarrow \mathrm{g}\,\mathrm{V}$ is a modulation matrix $\Leftrightarrow \mathrm{C}\mathrm{g}\mathrm{C}^{-1} = \tau\, \mathrm{g}$

$\mathrm{g} \in \mathrm{S}_{r} \Leftrightarrow \mathrm{V}\mathrm{g}$ is a modulation matrix $\Leftrightarrow \mathrm{g} = \tau\, \mathrm{g}$

Analogous statements hold for C^{∞} functions

Bases for Stabilizer Subgroups

Corollary $\sigma \mathrm{B}$ is a basis for $S_r \cap G_{pol}$ and

$\Omega \Lambda \sigma \mathrm{B} \Lambda^{-1} \Omega^{-1}$ is a basis for $S_\ell \cap G_{pol}$

Furthermore, $B^2 = -I$ if B is in either basis

Corollary $\mathrm{S_r} \cap \mathrm{G_{pol}}$ is dense in $\mathrm{S_r^\infty}$ and in $\mathrm{S_r}$

$\mathrm{S}_\ell \cap \mathrm{G_{pol}}$ is dense in S_ℓ^∞ and in S_ℓ

Proof Follows from density theorem and the fact that $\forall h \in G, \exp \sigma h = \sigma \exp h$

Structure of Left Stabilizer Algebra

Proposition If $h \in G$ then $h \in S_\ell \Leftrightarrow$

$$h = \begin{bmatrix} h_1 & h_2 & h_3 & \cdots & h_m \\ \tau\, h_m & \tau\, h_1 & \tau\, h_2 & \cdots & \tau\, h_{m-1} \\ \tau^2\, h_{m-1} & \tau^2\, h_m & \tau^2\, h_1 & \cdots & \tau^2\, h_{m-2} \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ \tau^{m-1}\, h_2 & \tau^{m-1}\, h_3 & \tau^{m-1}\, h_4 & \cdots & \tau^{m-1}\, h_1 \end{bmatrix}$$

where $h_1, \ldots, h_m \in G$ satisfy Structure Equations

$$w(h_1)_1 = 0, \quad h_1 \in i\,\mathrm{R} \qquad m = 2n \Rightarrow h_{n+1} = -\overline{\tau^n h_{n+1}}$$

$$2n - 1 \le m \le 2n \Rightarrow h_{m+2-j} = -\overline{\tau^{m+1-j} h_j}, \qquad j = 2, \ldots, n$$

Diagonal Stabilizer Subgroups

Definition $\mathrm{D} \equiv \{\mathrm{g} \in \mathrm{G} : \mathrm{g} \text{ is a diagonal matrix}\}$

$$\mathrm{D}^{\infty} \equiv \mathrm{D} \cap \mathrm{G}^{\infty}$$

$$D \equiv \{d \in G : \exp(d) \in \mathrm{D}\}$$

$$D^{\infty} \equiv \{d \in G : \exp(d) \in \mathrm{D}^{\infty}\}$$

Lemma $D = \{h \in G : h \text{ is a diagonal matrix}\}$

$$D^{\infty} \equiv D \cap G^{\infty}$$

Proposition $h \in D \cap S_r \Rightarrow$

$$h = i\sigma \,\mathrm{diag}\,[b_1, \ldots, b_m], \; b_j \in \mathrm{C(T)} \text{ real}, \; \sum_{\mathrm{j=1}}^{\mathrm{m}} b_j = 0$$

$$h \in D \cap S_{\ell} \Rightarrow$$

$$h = i\,\mathrm{diag}\,[a, \tau a, \ldots, \tau^{m-1} a], \; a \in \mathrm{C(T)} \text{ real}, \; \mathrm{w(a)}_1 = 0$$

Phase Transformations

Corollary V modulation matrix $f\!:\mathrm{T} \rightarrow \mathrm{T}, W(f) = 0$

$$\Rightarrow \exists\, d_\ell \in D \cap S_\ell,\, d_r \in D \cap S_r \ni$$

$$((\exp d_\ell)\, V\, (\exp d_r))_{1,1} = f\, V_{1,1}$$

Proof Since $W(f) = 0$ $\exists\, h : \mathrm{T} \rightarrow \mathrm{iR} \ni \exp h = f$

Construct

$d_\ell \equiv i\,\mathrm{diag}\,[a, \tau a, ..., \tau^{m-1} a]$ $d_r \equiv i\,\sigma\,\mathrm{diag}\,[b_1, ..., b_m]$

where and $ib_1 \equiv w(h)_1$

$ia \equiv h - \sigma\, w(h)_1$ $b_2 \equiv -b_1$

hence $w(a)_1 = 0$ $b_3, ..., b_m \equiv 0$

Factor Preserving Transformations

Definition $\mathrm{M}_{\mathrm{r}} \equiv \{\, g : \mathrm{T} \to \mathrm{C}^{m \times m} : U \mid g_{\mathrm{i},1},\ \mathrm{i} \geq 2 \,\}$

$\mathrm{M}_{\ell} \equiv \{\, g : \mathrm{T} \to \mathrm{C}^{m \times m} : U \mid g_{1,\mathrm{j}},\ \mathrm{j} \geq 2 \,\}$

Subroups $\mathrm{U}_{\mathrm{r}} \equiv \mathrm{G} \cap \mathrm{M}_{\mathrm{r}}$ $\mathrm{U}_{\ell} \equiv \mathrm{G} \cap \mathrm{M}_{\ell}$ $\mathrm{U}_{\mathrm{r}}^{\infty}$ $\mathrm{U}_{\ell}^{\infty}$

Lemma The Lie algebras

$$U_{\mathrm{r}} \equiv \{h \in G : \exp h \in \mathrm{U}_{\mathrm{r}}\} = G \cap \mathrm{M}_{r} \quad U_{r}^{\infty} = G^{\infty} \cap \mathrm{M}_{r}$$

$$U_{\ell} \equiv \{h \in G : \exp h \in \mathrm{U}_{\ell}\} = G \cap \mathrm{M}_{\ell} \quad U_{\ell}^{\infty} = G^{\infty} \cap \mathrm{M}_{\ell}$$

Proposition If $V : T \to C^{m \times m}$ and $U \mid V_{1,1}$

then $\mathrm{g} \in \mathrm{U}_{\ell} \Rightarrow U \mid (\mathrm{g}\,\mathrm{V})_{1,1}$ and $\mathrm{g} \in \mathrm{U}_{r} \Rightarrow U \mid (\mathrm{V}\,\mathrm{g})_{1,1}$

Definitions and assertions hold for C^{∞} functions

Proof Follows directly from the equations

$$(\mathrm{g}\,\mathrm{V})_{1,1} = \sum_{k=1}^{m} g_{1,k}\, \mathrm{V}_{k,1} \qquad (\mathrm{V}\,\mathrm{g})_{1,1} = \sum_{k=1}^{m} \mathrm{V}_{1,k}\, g_{k,1}$$

Jets

Definition $C^\infty(T)$ space of infinitely differentiable complex-valued functions on T with topology of uniform convergence of N-derivatives for any N

$$D_z : P(T) \rightarrow P(T), \quad D_z f \equiv \partial f / \partial z$$

$$D_\theta : C^\infty(T) \rightarrow C^\infty(T), \quad D_\theta f \equiv \partial f(e^{i\theta}) / \partial \theta = izD_z f$$

For $U(z) = \prod_{j=1}^{s} (z - \mu_j)^{d_j}, \, d_j \geq 0, \; d \equiv \sum_{j=1}^{s} d_j$

define U-jet maps $J_z : P(T) \rightarrow C^d, J_\theta : C^\infty(T) \rightarrow C^d$

$$J_z f \equiv [\, f(\mu_1), \ldots, D_z^{d_1 - 1} f(\mu_1), f(\mu_2), \ldots, D_z^{d_s - 1} f(\mu_s) \,)$$

$$J_\theta f \equiv [\, f(\mu_1), \ldots, D_\theta^{d_1 - 1} f(\mu_1), f(\mu_2), \ldots, D_\theta^{d_s - 1} f(\mu_s) \,)$$

Parameterization of Jets

Lemma $\mathrm{P(T)} \subset \mathrm{C}^{\infty}\mathrm{(T)}$ and $\exists$ linear isomorphism

$\mathrm{L} : \mathrm{C}^{\mathrm{d}} \rightarrow \mathrm{C}^{\mathrm{d}} \ni \quad \mathrm{J}_{\theta} f = L\, \mathrm{J}_{\mathrm{z}} f, \quad f \in P(T)$

Proof Follows from $D_{\theta} = izD_{\mathrm{z}}$

Proposition $\ker(\mathrm{J}_{\mathrm{z}}) = U\, \mathrm{P(T)}$ is an ideal in $\mathrm{P(T)}$

and $\ker(\mathrm{J}_{\theta}) = U\, \mathrm{C}^{\infty}\mathrm{(T)}$ is an ideal in $\mathrm{C}^{\infty}\mathrm{(T)}$

$\Rightarrow$

$$\mathrm{J}_{\mathrm{z}}\mathrm{P(T)} \approx \mathrm{P(T)}/\,\mathrm{U\,P(T)}, \mathrm{J}_{\theta}\mathrm{C}^{\infty}\mathrm{(T)} \approx \mathrm{C}^{\infty}\mathrm{(T)}/\,\mathrm{U\,C}^{\infty}\mathrm{(T)}$$

$\exists$ linear injection $\Phi : \mathrm{C}^{\mathrm{d}} \rightarrow \mathrm{P(T)} \ni \ \mathrm{J}_{\mathrm{z}}\Phi\,\mathrm{v} = \mathrm{v}, \ \mathrm{v} \in \mathrm{C}^{\mathrm{d}}$

$\Phi\,\mathrm{C}^{\mathrm{d}} =$ space of algebraic polynomials of degree $< \mathrm{d}$

Proof First two assertions are standard algebra, Shilov's Linear Algebra proves third using CRT

Extended Jets

Definition The extended right and left jets

$$\mathrm{J_r} : G^\infty \to \mathrm{C}^{\mathrm{d(m-1)}} \quad \text{and} \quad \mathrm{J}_\ell : G^\infty \to \mathrm{C}^{\mathrm{d(m-1)}}$$

are C-linear maps of the loop algebra into $\mathrm{C}^{\mathrm{d(m-1)}}$

$$\mathrm{J_r} h \equiv [\mathrm{J}_\theta h_{2,1}, \ldots, \mathrm{J}_\theta h_{m,1}]^{\mathrm{T}}, \quad h \in G^\infty$$

$$\mathrm{J}_\ell h \equiv [\mathrm{J}_\theta h_{1,2}, \ldots, \mathrm{J}_\theta h_{1,m}]^{\mathrm{T}}, \quad h \in G^\infty$$

Lemma $U_r^\infty = \{h \in G^\infty : \mathrm{J}_r\, h = 0\}$

$$U_\ell^\infty = \{h \in G^\infty : \mathrm{J}_\ell\, h = 0\}$$

Lemma $V_r \equiv \mathrm{J}_r S_r^\infty = \mathrm{J}_r (S_r \cap G_{pol})$

$$V_\ell \equiv \mathrm{J}_\ell S_\ell^\infty = \mathrm{J}_\ell (S_\ell \cap G_{pol})$$

are R-linear subspaces of $\mathrm{C}^{\mathrm{d(m-1)}}$

Cross Sections and Hermite Interpolation

Lemma If $d_r \in D \cap S_r \cap G_{pol}$ there exists

$\Theta_{\mathrm{r}} : V_r \to S_r \cap G_{pol} \ni h \to \Theta_{\mathrm{r}}(h) - d_r$ is R-linear

and $\mathrm{diag}(\Theta_r(h)) = d_r, \quad h \in V_r$

and $\mathrm{J}_{\mathrm{r}}\Theta_{\mathrm{r}} : V_r \to V_r$ is the identity map on V_r

Analogous assertions hold for d_ℓ and Θ_ℓ

Theorem If $d_r \in D \cap S_r$ then $\exp(d_r)$

is in the closure of $\mathrm{U_r} \cap \mathrm{S_r} \cap \mathrm{G_{pol}}$

Analogous assertions hold for $d_\ell \in D \cap S_\ell \cap G_{pol}$

Proof Let $A_r : \Theta_r(V_r) \to \mathrm{G}_{pol}$ Trotter approx. exp

$\Rightarrow B_r \equiv \mathrm{J}_r \log A_r \Theta_r : V_r \to V_r$ approx. identity so

result follows by Brouwer degree argument

Step Two

Theorem If $\mathrm{H} \in \mathrm{C}(\mathrm{T}), \mathrm{U} \in \mathrm{P}(\mathrm{T}), \mathrm{UH} \in \mathrm{C}_{\mathrm{Q}}(\mathrm{T})$ then $\forall \varepsilon > 0, \exists\, P \in \mathrm{P}(\mathrm{T}) \ni P$ has no zeros in T $\mathrm{UP} \in \mathrm{P}_{\mathrm{Q}}(\mathrm{T})$ and $\| \, H - P \, \| < \varepsilon$

Proof Compute $\widetilde{\mathrm{H}} \in \mathrm{C}(\mathrm{T})$ with no zeros in T with $\widetilde{\mathrm{H}} \approx \mathrm{H}$ then compute Q using Step One and multiplication by an integer power of z so that $U\,Q \in \mathrm{P}_{\mathrm{Q}}(\mathrm{T}),\ |\,Q\,| \approx |\,H\,|,\ W(\mathrm{phase}(f)) = 0$ where $f \equiv \mathrm{phase}(\widetilde{H}/Q) : \mathrm{T} \rightarrow \mathrm{T}$

Now compute d_r, d_ℓ as in the Phase Modulation page and then apply the previous Theorem

References

BC92 S. Basu, Complete parameterization of 2-D non-separable orthonormal wavelets, IEE Int. Conf. on Time Frequency Analysis, Victoria, Canada, 1992.

B98 S. Basu, Multi-dimensional filter banks and wavelets - a systems theoretic perspective, J. Franklin Inst, 335B(8) (1998) 1367-1409.

D88 I. Daubechies, Orthonormal bases of compactly supported wavelets, Comm. Pure App. Math., 41 (1988) 909-996.

D92, I. Daubechies, Ten Lectures on Wavelets, SIAM, 1992.

E04 L. Ephremidze, An elementary proof of the polynomial matrix spectral factorization theorem, Proc. Roy. Soc. Edinburgh, Sect. A 144 (2014) 747-751.

ESL15 L. Ephremidze, E. Lagvilava and I. Spitkovsky, Rank-deficient spectral factorization and wavelets completion problem, International Journal of Wavelets, Multiresolution and Information Processing, 13(3) (2015), 1550013 (9 pages), World Scientific

L78 T.Y.Lam, Serre Conjecture, Lect. Notes Math, 635, Springer, 1978.

LR99 W. Lawton and H. L. Resnikoff, Multidimensional wavelet bases, Aware, Inc., Bedford, MA, Technical Report (40 pages), May 1993.

LLS96 W. Lawton, S. L. Lee and Z. Shen, An algorithm for matrix extension and wavelet construction, Mathematics of Computation, 65(214), (1996) 723--737.

LM97a W. Lawton and C. A. Micchelli, Design of conjugate quadrature filters having specified zeros, Vol. 3, p. 2069-2073, Proc. ICASSP97, Munich, Germany, April 21--24, 1997.

LM97b W. Lawton and C. A. Micchelli, Construction of conjugate quadrature filters with specified zeros, Numerical Algorithms, 14 (1997) 383--399.

LL99 W. Lawton and Z. Lin, Matrix completion problems in multidimensional systems, Proc. of IEEE Int. Symposium on Circuits and Systems, Orlando, Florida, May 30 - June 2, 1999.

LM00 W. Lawton and C. Micchelli, B\'ezout identities with inequality constraints, Vietnam J. of Math., 28(2) (2000) 1-29.

L99 W. Lawton, Conjugate quadrature filters, Advances in Wavelets, 103--119 (Ka-Sing Lau, ed.), Springer, Singapore, 1999.

L00 W. Lawton, Infinite convolution products and refinable distributions on Lie groups, Transactions of the American Mathematical Society, 352 (6), (2000), 2913--2936.

LN01 W. Lawton and L. Noakes, Computing the inertia operator of a rigid body, Journal of Mathematical Physics, 42(4), (2001) 1--11.

L02 W. Lawton, The inverse problem for Euler's equation on 2 and 3 dimensional Lie groups, ScienceAsia, 28(1) (2002) 61-70.

LL04 W. Lawton andY. Lenbury, Interpolatory properties of trajectories in Lie groups, p. 19-31 in Proceedings of Conference on Harmonic Analysis and its Applications in Osaka, Osaka, Japan, November 15--17, 2004.

L04 W. Lawton, Hermite interpolation in loop groups and conjugate quadrature filter approximation, Acta Applicandae Mathematicae, 84 (3), (2004) 315--349.

OMK04 P. Oswald, C. K. Madsen and R. L. Konsbruck, Analysis of scalable PMD compensators using FIR filters and wavelength-dependent optical power measurements, J. of Lightwave Technology, 22 (2) (2004) 647-657.

OS08 P. Oswald and T. Shingel, Splitting methods for SU(N) loop approximation, J. of Approximation Theory 161 (2009) 174–186.

OS10 P. Oswald and T. Shingel, Close-to-optimal bounds for SU(N) loop approximation, J. of Approximation Theory 162 (2010) 1511-1517.

S10 T. Shingel, Trigonometric approximation of SO(N) loops, Constructive Approximation, 32 (2010) 597-618.

PR03 J. A. Parker and M. A. Rieffel, Wavelet filter functions, the matrix completion problem, and projective modules over ***C(T^n),*** The Journal of Fourier Analysis and Appl., 9(2) (2003) 101-116.

PR04 J. A. Parker and M. A. Rieffel, Projective multi-resolution analyses for L^2(R^2), The Journal of Fourier Analysis and Appl., 9(2) (2003) 101-116.

RS95 A. Ron and Z. Shen, Frames and stable bases for shift-invariant subspaces of L2(R^d),Can.J. Math.74(5)(1995)1051-1094.
RS97a A. Ron and Z. Shen, Weyl-Heisenberg frames and Riesz-bases in L2(R^d), Duke Math. J. 89 (2) (1997) 237-282.
RS97b A. Ron and Z. Shen, Affine systems in L2(R^d): The analysis of the analysis operator, Journal of Functional Analysis, 148 (1997) 408-447.
DHRS04 I. Daubechies, B. Han, A. Ron and Z. Shen, Framelets: MRA-based constructions of wavelet frames, Applied and Computational Harmonic Analysis 14(1) (2003) 1–46.
FGS15 Z. Fan, H. Ji and Z. Shen, Dual Grammian analysis: duality principle and unitary extension principle, Mathematics of Computation, http://dx.doi.org/10.1090/mcom/2987
Article electronically published on June 23, 2015
V86 M. Vidyasagar, Control System Synthesis-A Factorization Approach. MIT Press, 1986.
YP84 D. C. Youla T. and Pickel, The Quillen-Suslin theorem and the structure of n-dimensional polynomial matrices, IEEE Transactions on Circuits and Systems 31(6) (1984) 513-518.

FZ95 S. J. Favier and R. Zalik, On the stability of frames and Riesz bases, Appl. Comput. Harm. Analysis 2 (1995) 160–173.

GZ95 N. K. Govil and R. Zalik, Perturbations of the Haar Wavelet, Proc. Amer. Math. Soc. 125 (1997) 3363–3370.

Z99 R. Zalik, Riesz Bases and Multiresolution Analyses, Appl. Comput. Harm. Analysis 7 (1999) 315–331.

GZ04 A. L. Gonzales and R. Zalik, Riesz bases, multiresolution analyses, and perturbation in “Wavelets, Frames, and Operator Theory” (D. Larson, P. E. T. Jorgensen and C. Heil, Eds.), 163–182. Contemporary Math., Vol. 345, AMS, Providence, RI, 2004.

Z07 R. Zalik, On MRA Riesz wavelets, Proc. American Math. Soc. 135 (2007), 777-785.

AZ12 A. San Antol´ın and R. Zalik, Matrix–valued wavelets and multiresolution analysis, J. Appl. Functional Anal. 7(2012)13–25.

AZ13 A. San Antol´ın andR. Zalik, A family of nonseparable scaling functions and compactly supported tight framelets, J. Mathematical Analysis and Applications 404 (2013), 201-211. DOI: 10.1016/j.jmaa.2013.02.040.