Name: ______________

Mathematical Statistics Preliminary Examination

August 17, 2019, 1:00pm - 5:00pm, Parker Hall, Room 250

Directions:

1. This is a closed-book in-class exam.
2. Your proctor will determine your seat in the room where you take the exam.
3. You may not use a calculator.
4. The proctor will provide as many blank sheets of paper as you need.
5. Work any five out of the eight problems. You may submit solutions for at most five problems.
6. You need to start each problem on a new page. Clearly label each problem and write only on one side of each page.
7. To get full credit you need to properly document your solutions.
8. Each problem is worth 10 points.
9. You need to turn in the typeset pages that were given to you along with your solutions.

Please mark the five problems you are submitting for grading in the table below.

Problem	1	2	3	4	5	6	7	8
Submit for grading								

1. Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample from a distribution with density function

$$f(x|\alpha) = \frac{1+\alpha x}{2} \text{ for } -1 \le x \le 1 \text{ and } -1 \le \alpha \le 1.$$

   (a) Show that the method of moments estimator of α is $\alpha_M = \frac{3}{n}\sum_{i=1}^{n} X_i$.

   (b) Show that α_M is an unbiased estimator of α.

   (c) Show that $Var(\alpha_M) = (3 - \alpha^2)/n$.

   (d) Suppose $n = 25$ and $\alpha = 0$. Estimate $P(|\alpha_M| > 0.5)$ using the Central Limit Theorem. Express your answer in terms of the cumulative distribution function Φ of the standard normal distribution.

2. Suppose X and Y have joint density

$$f(x, y) = 2 \text{ for } 0 \le x \le y \le 1.$$

   (a) Find $Cov(X, Y)$.

   (b) Find $E[Y|X = x]$.

   (c) Find the density function of $E[Y|X]$.

3. Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample from a distribution with density function

$$f(x|\theta) = \frac{\theta}{(1+x)^{\theta+1}} \text{ for } 0 < \theta < \infty \text{ and } 0 \le x < \infty.$$

   (a) Use the Factorization Theorem to find a sufficient statistic for θ.

   (b) Use the Lehmann-Scheffé Criteria to find a minimal sufficient statistic for θ.

   (c) Use the Exponential Family Theorem to find a complete statistic for θ.

4. Suppose X is a discrete random variable with the following distribution:

X	0	1	2	3
$f_X(x)$	$\frac{2}{3}\theta$	$\frac{1}{3}\theta$	$\frac{2}{3}(1-\theta)$	$\frac{1}{3}(1-\theta)$

where $0 \le \theta \le 1$. Ten independent observations of X yield the following data: 0, 1, 3, 0, 2, 2, 1, 3, 1, 2.

(a) Find the maximum likelihood estimate of θ.

(b) Calculate the Bayes' estimate of θ assuming that the prior distribution of Θ is uniform on $[0, 1]$. Hint: The following identity may help:

$$\int_0^1 s^a(1-s)^b ds = \frac{a!b!}{(a+b+1)!}$$

5. Suppose that $X_1, X_2, \ldots, X_n$ are independent random variables from the Poisson distribution with parameter λ. Recall that the probability mass function of the Poisson distribution with parameter λ is $f(x|\lambda) = \frac{\lambda^x}{x!}e^{-\lambda}$ and that the expected value and the variance of this distribution are both λ.

(a) Calculate the mean squared error of $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$.

(b) Compute the Cramer-Rao lower bound for the estimator $\overline{X}$ of λ.

(c) Is $\overline{X}$ the UMVUE of λ? Explain.

6. Suppose a certain die rolls a six with probability θ. This die is rolled until six occurs and the total number of rolls X is recorded.

(a) Find the distribution of X.

(b) Derive the likelihood ratio test for testing $H_0 : \theta = 1/6$ versus $H_1 : \theta = 1/5$ using n independent outcomes of X.

(c) Suppose a test rejects the null hypothesis when $X \le 2$. Note that in this scenario $n = 1$. Find the size and the power of this test.

(d) Is the test in (c) UMP for its level? Explain.

7. A single observation X is taken from a population with density function

$$f(x|\theta) = \theta e^{-\theta x} \text{ for } x > 0 \text{ and } \theta > 0.$$

(a) For $c > 0$ consider the test that rejects $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$ when $X \leq c$. Determine c so that this is a size α test where $0 < \alpha < 1$.

(b) Invert the test in (a) to obtain a $1 - \alpha$ confidence set for θ.

(c) Find a pivot for θ and then use this pivot to construct the shortest confidence interval for θ of size $1 - \alpha$.

8. The density function of the Pareto distribution is given by

$$f(x|x_0, \theta) = \theta x_0^{\theta} x^{-\theta-1}, \quad x \geq x_0, \quad \theta > 0.$$

Assume that $x_0 > 0$ is given and that $X_1, \ldots, X_n$ is an i.i.d. sample.

(a) Find the maximum likelihood estimator $\hat{\theta}$ of θ.

(b) Find an approximate distribution of $\hat{\theta}$.

(c) Find an approximate $100(1-\alpha)\%$ lower-bound confidence interval for θ, $0 < \alpha < 1$.