Graph Theory Prelim – 2004

1. Prove or disprove:
    a. Every connected simple graph G has a spanning tree with the same maximum degree as G.
    b. Every connected simple graph G has a spanning tree with the same diameter as G (the diameter of G is the distance between two vertices that are farthest apart in G).

2. Let $\alpha(G)$ and $\alpha'(G)$ denote the independence and edge-independence numbers of G, respectively. Let $\chi(G)$ and $\chi'(G)$ denote the chromatic number and index of G, respectively. Let $n(G)$ and $e(G)$ denote the number of vertices and edges in G respectively.
    a. Show that $\chi(G) \geq n(G)/ \alpha(G)$.
    b. Find a similar relationship between $\chi'(G)$, $e(G)$, and $\alpha'(G)$, giving a reason for your answer.

3.
    a. State Hall's Theorem on matchings in bipartite graphs.
    b. Suppose that H is a bipartite graph with bipartition {X,Y} of its vertex set. Suppose each vertex in X has degree d, and each vertex in Y has degree at most d. Find the size of a maximum matching in H, and prove that your answer is true.

4. A graph is said to be partitionable if its vertices can be partitioned into 2 non-empty subsets such that each vertex has at least as many neighbors in its own subset as it does in the other subset. Show that:
    a. $K_{a,b}$ is partitionable if and only if a and b are even.
    b. Disconnected graphs are partitionable.
    c. There exists a connected non-bipartite partitionable graph on 7 vertices.

5. Let M and N be matchings in a graph, where M is maximal (that is, it is contained in no larger matching).
    a. Show that $|N| \leq 2|M|$.
    b. For all positive integers m, find a simple graph G that has a maximal matching M of size m and that has another matching N of size 2m.