Preliminary Exam Algebra

Spring 2010

Attempt a total of at least 8 problems. Moreover, a minimum of two problems from each group has to be attempted.

Part 1: Group Theory

1. State the three Sylow Theorems, and prove the first.
2. Show that a finite group G has odd order if and only if every element of G is a square, i.e. for every $g \in G$ there is $h \in G$ such that $g = h^2$.
3. Let G be a finite group. If H is a normal subgroup of G and P is a p-Sylow subgroup of H, then $G = N_G(P)H$ where $N_G(P)$ denotes the normalize of P in G.
4. Determine the number of non-isomorphic groups of order 1800.

Part 2: Commutative Ring Theory

1. State and give an outline of the proof of Gauss' Lemma.
2. Argue that any torsion module M over a pid R can be factored into the direct sum of the maximal primary submodules of M.
3. Argue that if H is a finitely generated p-primary abelian group, then, for any element x of maximal order in H, $< x >$ is a direct summand of H.
4. Let F be a free abelian group of finite rank. Show that if G is a subgroup of F for which F/G has no elements of finite order, then there is a subgroup H of $\underline{G}$ for which F is internal direct sum of G and H.

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Part 3: Non-Commutative Rings

1. State the Artin-Wedderburn Theorem.
2. Every finitely generated torsion-free module over a division ring D has a basis.
3. Let R be a right Noetherian ring. Show that every right R-module contains a maximal injective submodule.
4. Show that submodules of projective modules are projective if and only if R is right hereditary.

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Part 4: Galois Theory

1. State the main first Main Theorem of Galois Theory.
2. Give an outline of the proof that every field K has an algebraic closure.
3. Let F be an extension field of K. If $u \in F$ is algebraic over K of odd degree, then u^2 is algebraic over K and $K(u) = K(u^2)$.
4. Find the Galois group of $f(x) = x^4 + 2x^2 - 2$ over Q.