

School Bowl Solution

Answer An (m, n) rectangle can be tiled with L tiles if and only if:

- 1) Neither m nor n equals 1.
- 2) At least one of the numbers m, n is a multiple of 3.
- 3) If one of the numbers m, n equals 3, the other must be even.

Why are these conditions necessary for a tiling?

- 1) If either m or n is 1, there's no room for any L tiles at all.
- 2) Each tile has area 3, and so the total area mn must be a multiple of 3.
- 3) If, say, $n = 3$, the only way to tile the left end of the rectangle is to begin by placing two L tiles in, forming a $(2, 3)$, and leaving an $(n-2, 3)$ rectangle left to tile; now just repeat this argument.

How do we tile if the conditions are true?

Note first that if an (m, n) rectangle can be tiled, so can an (n, m) one.

Case 1 m is even, n is a multiple of 3.

Tile the rectangle with $(2, 3)$ tiles, then tile each $(2, 3)$ with two L tiles.

Case 2 m is a multiple of 6.

Find x even, and y a multiple of 3, with $n = x + y$. Divide the (m, n) rectangle into an (m, x) rectangle and an (m, y) rectangle, each of which can be tiled by Case 1.

Case 3 m and n are both odd, m is a multiple of 3.

The hardest case is $(9, 5)$. Here is a way, with tiles A through O:

A	A	B	B	C	C	D	D	E
A	F	B	G	G	C	D	E	E
F	F	H	H	G	I	I	J	J
K	K	L	H	M	I	N	J	O
K	L	L	M	M	N	N	O	O

In general, divide the rectangle into a $(9, 5)$ rectangle, an $(m-9, 5)$ rectangle, and an $(m, n-5)$ rectangle. The second one can be tiled by Case 2, the third by Case 1. Divide and conquer!