MAXIMAL $\mathcal{S}$-CLOSED SUBMODULES

Abstract

The submodule $Z(M) = \{x \in M \mid xI = 0$ for an essential right ideal I of $R\}$ of a right module M over a ring R is the *singular submodule* of M. The module M is *non-singular* if $Z(M) = 0$; it is *singular* if $Z(M) = M$. The ring R is *right non-singular* if $Z(R_R) = 0$. Every right non-singular ring has a right self-injective regular maximal right ring of quotients Q^r. Whenever U is an essential submodule of M, then M/U is singular, and the converse holds if M is non-singular. A submodule U of a module M is *$\mathcal{S}$-closed* if M/U is non-singular. The *$\mathcal{S}$-closure* of a submodule U of a non-singular module M is the submodule U^* of M containing U such that $U^*/U = Z(M/U)$.

A right R-module M has *finite Goldie-dimension* if every direct sum of non-zero submodules of M is finite. The ring Q^r is semi-simple Artinian if and only if R_R has finite Goldie-dimension. A non-zero module is *uniform* if all its non-zero submodules are essential. Clearly, a proper $\mathcal{S}$-closed submodule U of a non-singular module M is a maximal $\mathcal{S}$-closed submodule if and only if M/U is uniform. When working with torsion-free modules M over an integral domain, one often considers submodules of M having co-rank 1, i.e. maximal $\mathcal{S}$-closed submodules of M . Since every non-singular right R-module contains a maximal $\mathcal{S}$-closed submodule if R is a right non-singular right Goldie-ring, the question arises naturally if a right non-singular ring R has to be a right Goldie-ring if all non-singular right R-modules contain maximal $\mathcal{S}$-closed submodules. We show in this talk that this is not the case, and presents various characterizations of right non-singular rings having this property.