Exploding endpoints and Erdős spaces

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Definitions

A topological space X is:

- *hereditarily disconnected* if every connected subset of X is degenerate (either empty or consisting of exactly one point).
- *totally disconnected* if for every two points $x, y \in X$ there is a clopen set containing x and missing y.
- *almost zero-dimensional* provided X has a basis of open sets whose closures are intersections of clopen sets. This is equivalent to saying every point $x \in X$ has arbitrarily small neighborhoods which are intersections of clopen sets. Almost zero-dimensional spaces are totally disconnected, and have dimension at most 1.
- *zero-dimensional* if X has a basis of clopen sets.

$$\text{ZD} \implies \text{AZD} \implies \text{TD} \implies \text{HD}$$

- *cohesive* provided each point $x \in X$ has a neighborhood which contains no non-empty clopen set.

Erdős spaces

- Almost zero-dimensional spaces of positive dimension include:

  $\mathfrak{E} = \{x \in \ell^2 : x_i \in \mathbb{Q}$ for each $i < \omega\}$;

  $\mathfrak{E}_c = \{x \in \ell^2 : x_i \in \{0\} \cup \{1/n : n = 1, 2, 3, ...\}$ for each $i < \omega\}$.

- $\mathfrak{E}$ and $\mathfrak{E}_c$ are cohesive. Their bounded open sets contain no non-empty clopen sets, so actually $\mathfrak{E} \cup \{\infty\}$ and $\mathfrak{E}_c \cup \{\infty\}$ are connected (Erdős 1940)

- Another cohesive almost zero-dimensional space is the *stable complete Erdős space*, the ω-power of $\mathfrak{E}_c$.

  $$\mathfrak{E}_c^\omega \not\simeq \mathfrak{E}_c,$$

  despite $\mathfrak{E}^\omega \simeq \mathfrak{E}$. Erdős spaces are universal in the sense that all almost zero-dimensional spaces embed into them. And every complete almost zero-dimensional space is homeomorphic to a closed subspace of $\mathfrak{E}_c^\omega$. (Dijkstra and van Mill 2004 & 2010)

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- $\mathfrak{E}^n \simeq \mathfrak{E}$ for all $n \leq \omega$;
- $E \times \mathfrak{E} \simeq \mathfrak{E}$ for every complete AZD space E;
- $\mathfrak{E}_c \times \mathbb{Q}^\omega \simeq (\mathfrak{E}_c \times \mathbb{Q})^\omega \simeq \mathfrak{E}$;
- $\mathfrak{E}_c \simeq \{x \in \ell^2 : x_i \notin \mathbb{Q}$ for each $i < \omega\}$ (Oversteegen, Tymchatyn, Kawamura 1996)
- $\mathfrak{E}_c \simeq \{x \in \ell^1 : x_0 = 0$ and $x_n \in \{0, 1/n\}$ for each $n \geq 1\}$
- Identify C with the Cantor set $(\{0\} \cup \{1/n : n = 1, 2, 3, ...\})^\omega$. Define $\eta : C \to [0, 1]$ by $\eta(x) = 1/(1 + \|x\|)$, where $1/\infty = 0$. Let $L_0^\eta = \{\langle x, t \rangle : 0 \leq t \leq \eta(x)\}$. Then ∇L_0^η is also a Lelek fan. And $\nabla\{\langle x, \eta(x)\rangle : x \in \mathfrak{E}_c\} \simeq \mathfrak{E}_c$.

Figure: Cantor fan and Lelek fan

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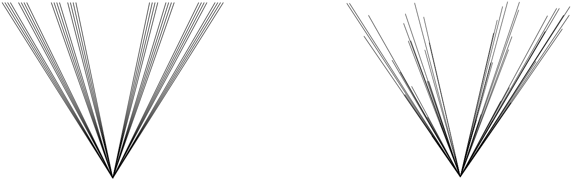


Figure: Cantor fan and Lelek fan

- $\exists$ homogeneous AZD space of positive dimension that is not cohesive;
- $\exists$ rigid cohesive AZD space

Endpoints of Julia sets

- For each $a \in (-\infty, -1)$ define $f_a : \mathbb{C} \to \mathbb{C}$ by $f_a(z) = e^z + a$.
- The Julia set $J(f_a)$ is a *Cantor bouquet* consisting of an uncountable union of pairwise disjoint rays, each joining a finite endpoint to the point at infinity. Let $E(f_a)$ be the set of finite endpoints these rays.

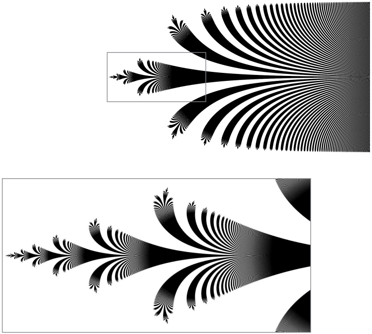


Figure: Images of $J(f_{-2})$

- $E(f_a) \cup \{\infty\}$ is connected, even though $E(f_a)$ is totally disconnected (Mayer 1990).
- The one-point compactification $J(f_a) \cup \{\infty\}$ is a Lelek fan (Oversteegen & Aarts 1991). A **Lelek fan** is a smooth fan with a dense set of endpoints. Every two Lelek fans are homeomorphic, so $E(f_a) \simeq \mathfrak{E}_c$.
- Let $\dot{E}(f_a)$ be the set of **escaping endpoints** of $J(f_a)$. Then $\dot{E}(f_a) \cup \{\infty\}$ is connected. The even smaller set of **fast escaping endpoints** $\ddot{E}(f_a)$ also has the property that its union with $\{\infty\}$ is connected. (Alhabib and Rempe-Gillen 2017)

Embedding results

Theorem 1

For every cohesive almost zero-dimensional space X, there is a dense homeomorphic embedding $h : X \hookrightarrow \mathfrak{E}_c$ such that $h[X] \cup \{\infty\}$ is connected.

Corollary 2

Every cohesive almost zero-dimensional space has a one-point connectification in the Cantor fan.

Corollary 3

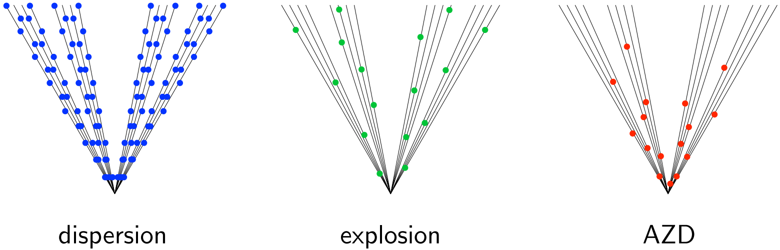
Every cohesive almost zero-dimensional subset of $C \times \mathbb{R}$ is nowhere dense.

Proof.

Suppose X is cohesive AZD dense in $C \times \mathbb{R}$. By Theorem 1 and Lavrentiev's Theorem, there is a *complete* cohesive AZD X' with $X \subseteq X' \subseteq C \times \mathbb{R}$. Then there exists $c \in C$ such that $\overline{X' \cap \{c\} \times \mathbb{R}} = \{c\} \times \mathbb{R}$. Let O be a convex open subset of the plane such that $x \in W := O \cap X'$ and W contains no non-empty clopen subset of X'. Let $x_1 = \langle c, r_1 \rangle$ and $x_2 = \langle c, r_2 \rangle$ be points in W such that $r_1 < r < r_2$.

... □

Rim-type



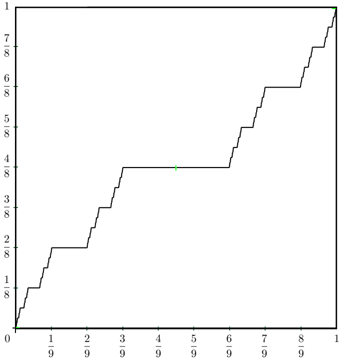
In the Cantor fan:

- $\exists$ rim-discrete connected G_δ-set with a *dispersion point*; and
- $\exists$ non-Borel rim-discrete connected set with an *explosion point*.

However:

- $\nexists$ rim-countable connected set X with a point ∞ such that $X \setminus \{\infty\}$ is almost zero-dimensional.

Construction of rim-discrete examples.



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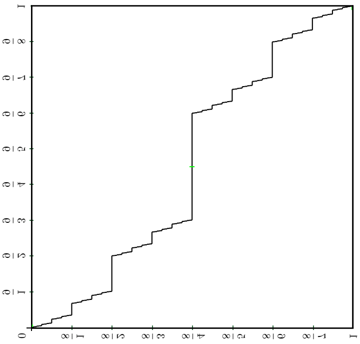

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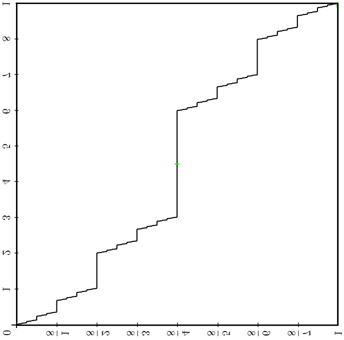

Let D be the "inverse" of the graph of the Cantor function. Let M be the set of midpoints of vertical arcs in D. There is a collection of pairwise disjoint sets $D_n \cong D$ such that $[0,1]^2 \setminus \bigcup\{D_n : n < \omega\}$ is zero-dimensional and $\pi_0[M_n] \subseteq \mathbb{P}$. Let

$$X = (\mathbb{P}\times[0,1])\setminus\bigcup\{D_n\setminus M_n : n < \omega\}.$$

Then ∇X is a rim-discrete connected G_δ-set with dispersion point $\langle \frac{1}{2}, 0\rangle$.

For an explosion point example, define the D_n's so that $\pi_0[M_i] \cap \pi_0[M_j] = \varnothing$ whenever $i \neq j$. Each non-vertical continuum $K \subseteq [0,1]^2$ either intersects some M_n, or $|\pi_0[K \cap X]| = \mathfrak{c}$. Well-order the set of all such continua $\{K_\alpha : \alpha < \mathfrak{c}\}$. Recursively select $y_\alpha \in K_\alpha$ so that $\pi_0(y_\alpha) \in \mathbb{P} \setminus \{\pi_0(y_\beta) : \beta < \alpha\}$ and $\pi_0(y_\alpha) \notin \pi_0[M_n]$ for any $n < \omega$. Put $Y = \bigcup\{M_n : n < \omega\} \cup \{y_\alpha : \alpha < \mathfrak{c}\}$. Then ∇Y is rim-discrete connected with explosion point $\langle \frac{1}{2}, 0\rangle$.

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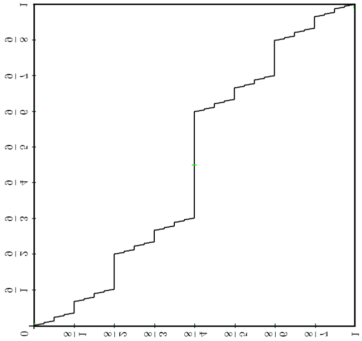

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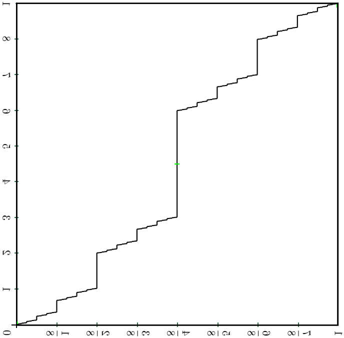

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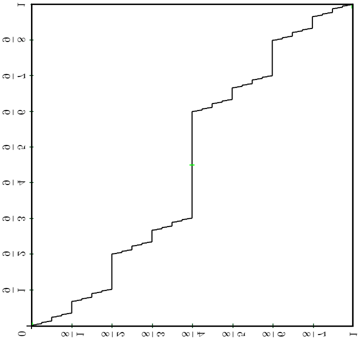

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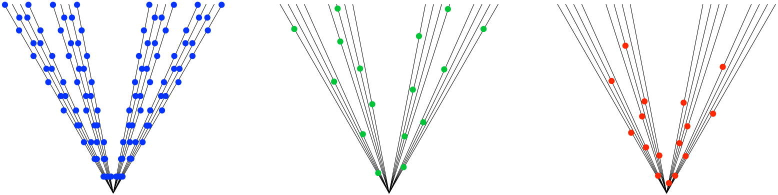

dispersion explosion AZD

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Theorem 4 (Taras Banakh)

Every non-empty bounded open subset of $\mathfrak{E}$ (resp. $\mathfrak{E}_c$) has uncountable boundary.

Proof.

Let U be a non-empty open subset of $\mathfrak{E}$ with $\|U\| \leq N$. For $i \in \{0, 1\}$ consider the closed subspace $X_0 = \{(x_n)_{n\in\omega} \in X : \forall n \in \omega \; x_{2n} = 0\}$ and observe that $|U \cap X_0| = \mathfrak{c}$. We establish a one-to-one function from $U \cap X_0$ into ∂U.

Let $(e_i)_{i\in\omega}$ be the standard orthonormal basis for ℓ_2. Inductively define two sequences of rationals $(x_{2i})_{i\in\omega}$ and $(x'_{2i})_{i\in\omega}$ such that for every $n \in \omega$ we have $|x'_{2n} - x_{2n}| < N/2^n$;

$$u + \sum_{i=0}^{n} x_{2i} e_{2i} \in U; \text{ and}$$

$$u + \sum_{i=0}^{n-1} x_{2i} e_{2i} + x'_{2n} e_{2n} \notin U.$$

The function $u \mapsto u + \sum_{i=0}^{\infty} x_{2i} e_{2i} \in \partial U$, $u \in U \cap X_0$, is injective. □

An intersection of clopen sets is called a C-**set**.

Lemma 5

In an almost zero-dimensional space, every closed σ-C-set is a C-set.

Theorem 6

Every rim-σ-compact almost zero-dimensional space is zero-dimensional.

Proof of Lemma 5.

Suppose $A = \bigcup\{A_i : i < \omega\}$ where each A_i is a C-set, and A is closed. To prove A is a C-set, it suffices to show that for every $x \in X \setminus A$ there is an X-clopen set B such that $x \in B \subseteq X \setminus A$.

By the Lindelöf property and the fact that X has a neighborhood basis of C-sets, it is possible to write the open set $X \setminus A$ as the union of countably many C-sets whose interiors cover $X \setminus A$. The property of being a C-set is closed under finite unions, so in fact there is an increasing sequence of C-sets $C_0 \subseteq C_1 \subseteq \ldots$ with $x \in C_0$ and

$$\bigcup\{C_i : i < \omega\} = \bigcup\{C_i^{\circ} : i < \omega\} = X \setminus A.$$

For each $i < \omega$ there is an X-clopen set B_i such that $C_i \subseteq B_i \subseteq X \setminus A_i$. Let $B = \bigcap\{B_i : i < \omega\}$. Apparently B is closed, $x \in B$, and

$$B \subseteq X \setminus A.$$

Further, if $y \in B$ then there exists $j < \omega$ such that $y \in C_j^{\circ}$. The open set $C_j^{\circ} \cap \bigcap\{B_i : i < j\}$ witnesses $y \in B^{\circ}$. This shows B is open. □

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Proof of Theorem 6.

Let X be rim-σ-compact AZD. Let $x \in X$ and let U be any open set containing x. Let V be an open set with $x \in V \subseteq \overline{V} \subseteq U$ and ∂V is σ-compact. Since X is totally disconnected, every compact subset of X is a C-set. Thus ∂V is a σ-C-set. By Lemma 5, ∂V is a C-set. Thus there is a clopen set A with $\partial V \subseteq A \subseteq X \setminus \{x\}$. Then $B := V \setminus A$ is an X-clopen set with $x \in B \subseteq U$. □

Lemma 5 also implies:

Theorem 7

Cohesive almost zero-dimensional space is nowhere rational.

Theorem 8

If X is almost zero-dimensional and $X \cup \{\infty\}$ is connected, then every σ-compact separator of $X \cup \{\infty\}$ contains the point ∞.

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Subsets of curves

Recall HD $\Longleftarrow$ TD $\Longleftarrow$ AZD $\Longleftarrow$ ZD.

By a classical result,

$$\mathsf{HD} \overset{(1)}{\Longrightarrow} \mathsf{TD} \overset{(2)}{\Longrightarrow} \mathsf{AZD} \overset{(3)}{\Longrightarrow} \mathsf{ZD}$$

for subsets of **hereditarily locally connected** continua.

By results of S.D. Iliadis and E.D. Tymchatyn, the rim-discrete examples show (1) and (2) are generally false for subsets of **rational** curves.

The implication (3) extends to the larger class of subsets of **rational** curves.

Question 1

Can $\mathfrak{E}_c$ be embedded into a Suslinian continuum?

Question 2

Is $\mathfrak{E}_c \cup \{\infty\}$ σ-connected?

Homeomorphism types of endpoint sets

- Define $I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty\}$.
- Define the maximum modulus function

$$M(r) := M(r, f) := \max\{|f(z)| : |z| = r\}$$

for $r \geq 0$. Choose $R > 0$ sufficiently large that $M^n(R) \to +\infty$ as $n \to \infty$ and define

$$A_R(f) := \{z \in \mathbb{C} : |f^n(z)| \geq M^n(R) \text{ for all } n \geq 0\}.$$

The *fast escaping set* for f is the increasing union of closed sets

$$A(f) = \bigcup_{n \geq 0} f^{-n}[A_R(f)].$$

- For $a \in (-\infty, -1)$ and $f_a = \exp + a$, define

$$\dot{E}(f_a) = I(f_a) \cap E(f_a); \text{ and}$$
$$\ddot{E}(f_a) = A(f_a) \cap E(f_a)$$

Theorem 9

$\dot{E}(f_a)$ and $\ddot{E}(f_a)$ are first category.

Proof.

For any transcendental entire function f, $I(f) \cap J(f)$ is first category. We have $I(f_a) \subseteq J(f_a)$, so $I(f_a)$ is first category. $\dot{E}(f_a)$ and $\ddot{E}(f_a)$ are dense subsets of $I(f_a)$. □

Theorem 10

Neither $\dot{E}(f_a)$ nor $\ddot{E}(f_a)$ is homeomorphic to $E(f_a)$.

Proof.

$E(f_a)$ is completely metrizable (recall $E(f_a) \simeq \mathfrak{E}_c$). □

Theorem 11

$\ddot{E}(f_a) \not\simeq \mathfrak{E}$.

Proof.

$\ddot{E}(f_a)$ is an absolute $G_{\delta\sigma}$-space because $A(f_a)$ and $E(f_a)$ are F_σ and G_δ subsets of $\mathbb{C}$, respectively. On the other hand, $\mathfrak{E}$ is not absolute $G_{\delta\sigma}$ because it has a closed subspace homeomorphic to $\mathbb{Q}^\omega$. □

Theorem 9

$\dot{E}(f_a)$ and $\ddot{E}(f_a)$ are first category.

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Question 3

(a) $\ddot{E}(f_a) \simeq \dot{E}(f_a)$?

(b) $\dot{E}(f_a) \simeq \mathfrak{E}$?

(c) $\ddot{E}(f_a) \simeq \mathbb{Q} \times \mathfrak{E}_c$?