Conformal Minors, Solid Graphs and Pfaffian Orientations

Nishad Kothari
University of Vienna

A connected graph, on at least two vertices, is *matching covered* if each edge lies in some perfect matching. For several problems in matching theory, such as those related to counting the number of perfect matchings, one may restrict attention to matching covered graphs. These graphs are also called 1-*extendable*, and there is extensive literature relating to them; Lovász and Plummer (1986). Special types of minors, known as *conformal minors*, play an important role in the theory of matching covered graphs — somewhat similar to that of *topological minors* in the theory of planar graphs.

Lovász (1983) proved that each nonbipartite matching covered graph either contains K_4, or contains the triangular prism $\overline{C_6}$, as a conformal minor. This leads to two problems: (i) characterize K_4-free graphs, and (ii) characterize $\overline{C_6}$-free graphs. In a joint work with Murty (2016), we solved each of these problems for the case of planar graphs. We first showed that it suffices to solve these problems for special types of matching covered graphs called *bricks* — these are nonbipartite and 3-connected. Thereafter, we proved that a planar brick is K_4-free if and only if its unique planar embedding has precisely two odd faces. Finally, we showed that apart from two infinite families there is a unique $\overline{C_6}$-free planar brick — which we call the *Tricorn*. Our proofs utilize the brick generation theorem of Norine and Thomas (2008). It remains to characterize K_4-free nonplanar bricks, and to characterize $\overline{C_6}$-free nonplanar bricks.

A graph is *solid* if its perfect matching polytope may be described by non-negativity and degree constraints. A classical result of Birkhoff and of von Neumann states that every bipartite graph is solid. The converse is not true, and it suffices to characterize solid bricks in order to characterize all solid nonbipartite graphs. Carvalho, Lucchesi and Murty (2006) showed that the only solid planar bricks are the odd wheels. In a joint work with Carvalho, Lucchesi and Murty (2017), we showed that except for the Petersen graph, a nonplanar brick G is $\overline{C_6}$-free if and only if G is solid.

Kasteleyn (1963) introduced *Pfaffian orientations*, and proved that every planar graph admits a Pfaffian orientation. A graph is *Pfaffian* if it admits a Pfaffian orientation, and for such a graph one may compute the number of perfect matchings in polynomial time. Little (1975) showed that a bipartite graph G is Pfaffian if and only if G is $K_{3,3}$-free. In order to characterize all Pfaffian nonbipartite graphs, it suffices to characterize Pfaffian bricks. I conjecture that every K_4-free brick is Pfaffian. I have computational evidence, and several other reasons, to believe this conjecture. The smallest K_4-free nonplanar brick has 14 vertices, and it appears in my work with Murty (2016) wherein we refer to it as the *Trellis*.