Consistent Probability Systems

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Or how the simple entails the complex

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Outline

A family of random variables?

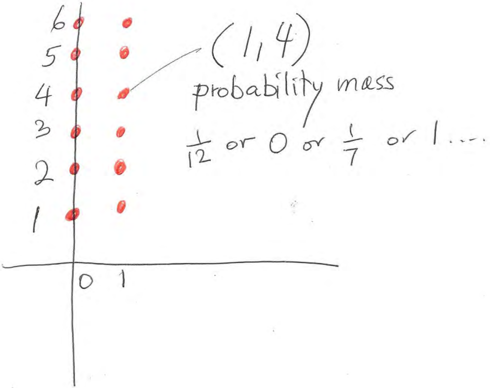
Flip a coin: $X = 0$ if heads, $X = 1$ if tails.

Roll a die, $Y \in \{ 1, ..., 6 \}$.

Does the pair $\{ X, Y \}$ make a family?

Is this a stochastic process?

A point set on the plane does the trick



The **stochastic process** here:

A sequence (X, Y),

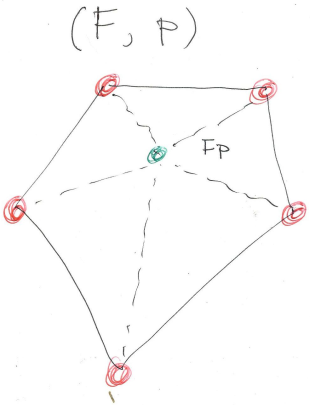
defined on the same probability space,

where all one- and two-dimensional distributions are specified.

In the same manner: (X, Y, Z) or $(X_1, \ldots, X_n)$.

But, an infinite sequence $(X_n, n \in \mathbb{N})$ or a function $(X_t : t \in T)$?

First, a loose pile of probability spaces



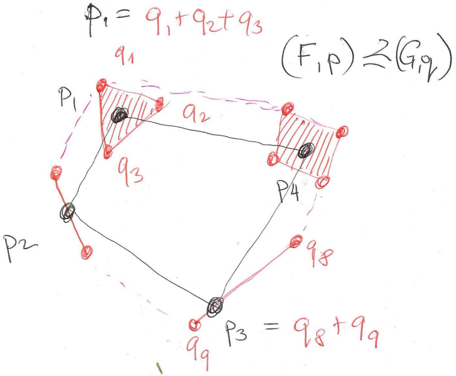
A consistent family

Write $P = (F, p) \preceq Q = (G, q)$ and say that Q extends P, if

each point of F lies in a convex envelope of some points of Q, and every component of p is a sum of some components of q.

A family equipped with this relations is called **consistent**.

For example...



What's a poset?

A partially ordered set, with the order being:

reflexive: $P \preceq P$;

antisymmetric: $P \preceq Q$ and $Q \preceq P$ implies $P = Q$;

transitive: $P \preceq Q$ and $Q \preceq R$ implies $P \preceq R$.

What's a lattice?

A poset such that every two elements admit:

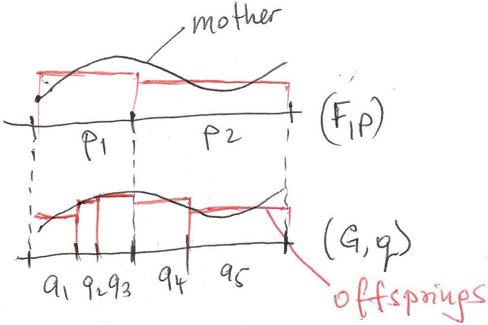
the **supremum** $P \vee Q$, or the least upper bound, and

the **infimum** $P \wedge Q$, or the greatest lower bound.

Every greater (finer) element can be projected onto smaller (cruder) one.

Our families are lattices, typically with no maximal element, unless...we extend the domination even more.

This is well known...



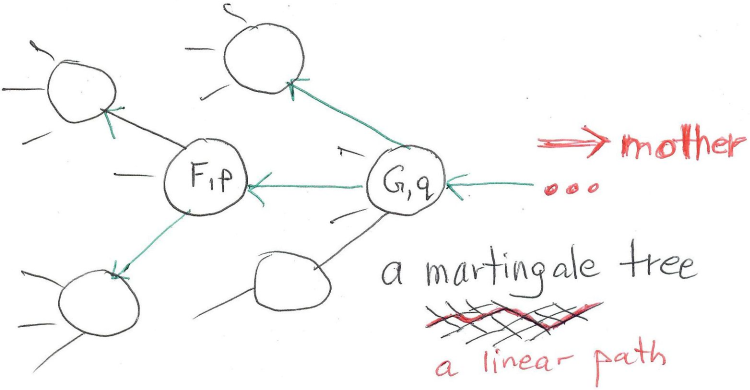
There is a mother of all this offspring

By Newton's Fundamental Theorem of Calculus, as long as the partition size increases and its mesh converges to 0, the limit will exist.

That is, all the patterns are projections of the original function.

But, what if have only a coset of patterns, does "the mother" exist?

This is a martingale



The Garden of Forking Paths

Jorge Luis Borges

El Jardín de senderos que se bifurcan

Consistent Cartesian products

Consider the space $\mathbb{R}^T = \{ x = x(t) | x : T \to \mathbb{R} \}$.

Finite sets K entail the natural poset $\{ \mathbb{R}^K \}$, ordered by the inclusion of sets, or, projections:

$$\pi_{K,L} : \mathbb{R}^K \to \mathbb{R}^L, \quad L \subset K, \qquad \pi_{K,L} x_K = x_k \Big|_L, \; x_k \in \mathbb{R}^K.$$

Consistent FDD's

Consider a consistent family (a poset, a lattice) $\mathcal{X} = \{ (\mathbb{R}^K, \mu_K) \}$ of probability measures, a.k.a. **finite dimensional distributions.**

Let $\mathcal{C}_T$ denote the "cylindrical" σ-algebra, generated by open intervals with finite dimensional bases.

$\mathcal{X}$ has a "mother"

Kolmogorov Extension Theorem

There exists a probability measure μ on $\mathcal{C}_T$ that extends all given consistent μ_K. That is, every μ_K is a projection of μ:

$$\mu_K = \pi_K \mu, \qquad \pi_K = x\Big|_K.$$

In other words:

There is a stochastic process $(X(t) : t \in T)$ with the given FDD.

What about Brownian Motion

FDD's of every zero-mean (even complex valued) Gaussian $X(t)$ process are uniquely determined by the **covariance function** $C(s,t) = \mathrm{E}\,\overline{X(s)}\,X(t)$.

"Covariance function" and "**positive definite function**" are synonims

$$\sum_j \sum_k C(t_j, t_k)\,\overline{z_j}\,z_k \geq 0, \quad \{\, t_j \,\} \subset T, \quad \{\, z_j \,\} \subset \mathbb{C}$$

For Brownian Motion on $T = [0, \infty)$

$$C(s,t) = s \wedge t.$$

KET says a BM exists, but...

A problem: KET does not guarantee the continuity of its trajectories.

In fact, there are Gaussian processes with the BM's covariance $s \wedge t$ whose trajectories are discontinuous at every point.

A version with continuous paths must and can be constructed, but it is not easy - another long story.

It is easy and convenient to take it for granted.

Positive definite functions again

More familiar examples:

1. Positive definite $n \times n$ matrices: $T = \{1, \ldots, n\}$;
2. An inner product $\left\langle u, v \right\rangle$ in a (complex) Hilbert space:

$$\sum_j \sum_k \left\langle u_j, u_k \right\rangle \overline{z_j}\, z_k = \left\| \sum_j z_j u_j \right\|^2 \geq 0$$

3. A homomorphic embedding $\phi : G \to H$ of a group into a Hilbert space entails

$$C(g, h) = \left\langle \phi(f), \phi(g) \right\rangle$$

GNS Theorem

Gelfand-Neumark-Segal Theorem admits a short powerful proof based on KET.

For a positive definite function $C(s,t)$, there exists a Hilbert space H and a mapping $\Lambda : T \to H$ such that

1. *The linear span of the range $\Lambda(T)$ is dense in H;*
2. $C(s,t) = \Big\langle \Lambda(s), \Lambda(t) \Big\rangle$.

The only example...

The **Gelfand pair** (H, Λ) is unitarily unique, i.e., for another pair (H', Λ') that satisfies conditions 1 and 2 there is a unitary operator $U : H \to H'$ that closes the diagram

$$\begin{array}{ccccc} & & T & & \\ & \Lambda \swarrow & & \searrow \Lambda' & \\ H & & \xrightarrow{U} & & H' \end{array}$$

Exercise: Prove Schur Lemma

An entrywise product of positive definite matrices is positive definite.

Hint: Use Gauss distribution.

Unitary operators?

Unitary operators in a complex Hilbert - or just Euclidean - space extend the role of rigid linear transformations - isometries - of a real Hilbert - or Euclidean - space.

They change merely the reference frame yet leave the essential properties invariant.

The simplest description comes from the setting of eigenvalues and eigenvectors through the spectral theory.

Classical versus quantum probability

Classical: events, probabilities, random variables, independence

In the language of matrices, a discrete probability can be placed on the diagonal, and then the matrix may be modified unitarily. The result: a positive definite matrix P with the unit trace.

An event: either a random point is there, or is not: 0 or 1. As above, it entails a projection E.

Translation

probability

$$D = \begin{bmatrix} p_1 & & \\ & \ddots & \\ & & p_n \end{bmatrix} \quad \longmapsto \quad \begin{bmatrix} P = UDU^* \end{bmatrix}$$

event

$$R = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \quad \longmapsto \quad \begin{bmatrix} E = URU^* \end{bmatrix}$$

"state"

random variable

$$X = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \quad \longmapsto \quad \begin{bmatrix} UXU^* \end{bmatrix}$$

"observable"

Computations?

The probability of an event E: $\mathrm{tr} PE$.

Real random variable: real eigenvalues, a Hermitian matrix X.

The expectation with respect to a probability P: $\mathsf{E}_P X = \mathrm{tr} PX$.

The k^{th} moment: $\mathsf{E}_P X^k$,

The Fourier transform: $\mathsf{E}_P e^{itX}$.

etc.

Conclusion

The classical probabilistic triples,

$$(\Omega, \mathcal{F}, \mathsf{P})$$

seen as dynamic consistent systems, can be replaced by or rather extended to richer and more powerful consistent systems of bounded linear operators acting on complex Hilbert spaces.

But this is yet another story...

t.b.c.