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What's more chaotic than chaos itself?
Brownian Motion - before, after, and beyond.

Math Graduate Seminar

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Outline

- 1 Bits of history
- 2 Luis Bachelier
- 3 Norbert Wiener
- 4 Rademacher chaos
- 5 Chaos representation
- 6 Spectral theory

Selected historical moments

1820 **Robert Brown** - volatile movements of a pollen particle
(not really the first)

1853 **Jules Regnault** - a 'growing circle' model of BM

1880 **Thorvald Thiele** - a 'computational' construction of BM

1900 **Louis Bachelier** - BM derived from Random Walk and Central Limit Theorem and Heat Equation, modeled upon price fluctuations at the Paris Stock Exchange ('the one')

1905-1906 **Albert Einstein** and **Marian von Smoluchowski** - a physical explanatory model (good for physicists, maybe)

1938 **Norbert Wiener** - multidimensional BM (also 'the one')

Bachelier's PhD Thesis "Théorie de la spéculation"

(modeling stock prices by a stochastic process in time)

Random walk: independent ± 1 hits, summed up, centered and time-scaled, a continuous time random walk in the limit (later: **Donsker's Theorem**).

Alternatively, **Chapman-Kolmogorov equations** yield the Gaussian probability distribution of the process.

A random quantity $Q(t)$ changes linearly with a value-and-time dependent coefficient $P(Q(t) \in [x, x + dx]) \approx p_{x,t} dx$

Bachelier's CK-equations

$$p_{z,t_1+t_2} = \int_{-\infty}^{\infty} p_{x,t_1} \cdot p_{z-x,t_2} dx$$

Solution (the functions is a probability, hence the normalizing constant π): $p_{x,t} = f(t) e^{-\pi f^2(t)x^2}$, where

$$f(t_1 + t_2) = \frac{f(t_1)f(t_2)}{\sqrt{f^2(t_1) + f^2(t_2)}}.$$

Solution: $f(t) = \frac{c}{\sqrt{t}}$

Today we call $Q(t)$ the **Brownian motion**.

Continuous homogenous chaos

"Among the simplest and most important ensembles of physics are those which have a spatially homogeneous character. Among these are the homogeneous gas, the homogeneous liquid, the homogeneous state of turbulence. "

Axioms will follow...

Discrete homogeneous chaos

Wiener considered nested binary partitions of \mathbb{R}^n ...

"Let us require that the probability that a set be empty depend only on its measure, and that the probability that two non-overlapping sets be empty be the product of the probabilities that each be empty. Let us assume that both empty and occupied sets exist. Let every set contained in an empty set be empty, while if a set be occupied, let at least one-half always be occupied. We thus get an infinite class of schedules of emptiness and occupiedness"

Wiener's axioms of chaos

Multidimensional Chaos: Any additive random quantity $\mathfrak{F}(A)$ defined for measurable $A \subset \mathbb{R}^n$.

Additivity:

$\mathfrak{F}(A \cup B) = \mathfrak{F}(A) + \mathfrak{F}(B)$, when A and B are disjoint.

(For processes on \mathbb{R} , it is an analog of the increment

$X(s, t] = X(t) - X(s)$, rising to finite unions, then limits...)

Metrical transitivity: $\mathfrak{F}(A + \mathbf{y}) \stackrel{\mathcal{D}}{=} \mathfrak{F}(A)$

(A weak analog of stationary increments)

Axioms, continued

Ergodicity: $E |\mathfrak{F}(A)| \log |\mathfrak{F}(A)| < \infty$

Birkhoff's Ergodic Theorem allows a construction of a chaos. Wiener did not use the Kolmogorov's axiomatic probability theory, with tools of measure theory. Every needed time the "randomness" had to be constructed from scratch.

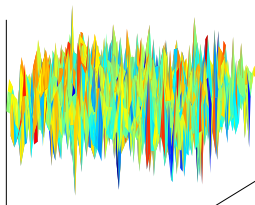
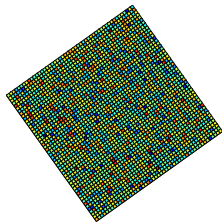
Implicitly, the additivity entails **σ -additivity**, a "true" random measure.

For the Gaussian chaos, $E |\mathfrak{F}(A)|^p < \infty$ for every p .

Theorem 1.1. Let \mathfrak{F} be a random measure on a measurable space (X, \mathcal{F}) .

A picture, finally!

Wiener's first "multidimensional chaos" was in fact one-dimensional, because \mathbb{R} and \mathbb{R}^n are indistinguishable from the point of view of the basic measure theory (through "Borel isomorphism").



Wiener's "*pure multi-D chaos*":

1. Non-deterministic random chaos;
2. Homogeneity;
3. Independent scatter;
4. Hilbertian L^2 -condition: $E |\mathfrak{F}(A)|^2 < \infty$, $E \mathfrak{F}(A) = 0$.

Two examples:

1. Gaussian \mathfrak{G} , mean 0.
2. Poisson \mathfrak{D} , centered: $\mathfrak{D} - E \mathfrak{D}$, or symmetrized $\mathfrak{D} - \mathfrak{D}'$, with independent copies.

Wiener integral

For step functions, $f = c_k$ on A_k and the Hilbert norm:

$$Wf = \int f d\mathfrak{F} = \sum_k c_k \mathfrak{F}(A_k); \quad \left\| \int f d\mathfrak{F} \right\|^2 = \|f\|^2,$$

extends to the quantity defined and the isometry held over L^2 .

(only orthogonal scatter is needed)

Wiener's polynomial chaos:

Let f be a step function, or alternatively,

$$f(x_1, \dots, x_n) = f_1(x_1) \cdot \dots \cdot f_n(x_1), \quad f_k \in L^2. \quad (1)$$

Then

$$\int \dots \int_{\mathbb{R}^n} f(x_1, \dots, x_n) \mathfrak{F}(dx_1) \dots \mathfrak{F}(dx_n)$$

is well defined. The definition extends to a linear combinations of functions (1), and then to their limits, e.g., when

$$|f(x_1, \dots, x_n)| \leq |f_1(x_1)| \cdot \dots \cdot |f_n(x_1)|$$

(the Wiener's assumption).

Gaussian polynomial chaos, examples

Let $\gamma = (\gamma_k)$ be i.i.d. standard Gaussian $N(0, 1)$.

2nd order chaos:
$$\sum_k c_k \gamma_k$$

1st order chaos:
$$\sum_k \sum_j c_{kj} \gamma_k \gamma_j$$

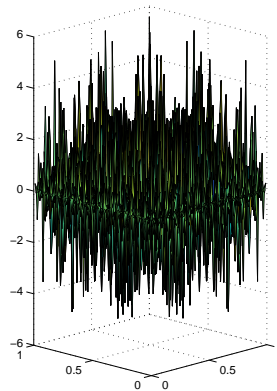
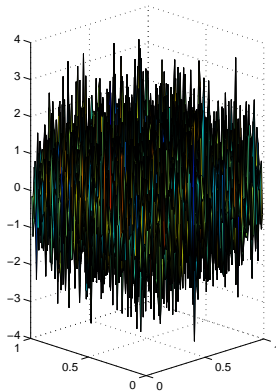
nth order chaos:
$$\sum_{j_1, \dots, j_n} c_{j_1, \dots, j_n} \gamma_{j_1} \cdots \gamma_{j_n} = \sum_{J_n} c_{J_n} \gamma^{J_n}$$

$(J_n = (j_1, \dots, j_n))$

polynomial chaos:
$$\sum_J c_J \gamma^J \quad (J \in \mathbb{N}^{\mathbb{N}})$$

chaos (diagonal-free):
$$\sum_d c_d \gamma^d \quad (d \in \{0, 1\}^{\mathbb{N}})$$

Independent scatter and polynomial chaos



Rademacher and Haar functions

Shift and scale the marks $D \mapsto S$ -square wave. Or, directly:

$$\rho_n(x) = \text{sign} \left(\sin(2^n \pi x) \right), \quad x \in [0, 1].$$

Haar function: a single double-tooth piece of a Rademacher function; a dyadic squeeze and shift of the “square bump” $2h(x) - 1$.

2^n of Haar functions of order n form a basis in the vector space \mathbb{D}_n of piecewise functions, constant on dyadic intervals

$$\left(\frac{k-1}{2^n}, \frac{k}{2^n} \right], \quad k = 1, \dots, 2^n$$

Walsh functions

Rademacher functions span only an n -dimensional subspace.

The products ("chaoses") are called **Walsh functions**:

$$w_{d_1, \dots, d_n} = \rho_1^{d_1} \cdots \rho_n^{d_n}, \quad d_1, \dots, d_n \in \{0, 1\}.$$

Since there are 2^n Walsh functions and they are independent, they also form a basis of \mathbb{D}_n .

Powers

Walsh functions can be ordered **lexicographically**

$$w_1 = w_1, w_2 = w_{01}, w_3 = w_{11}, w_4 = w_{001}, w_5 = w_{101}, w_6 = w_{011}, \dots$$

(skipping zeros on the right).

Indicate a sequence by a boldface font, e.g., $\mathbf{d} = (d_n)$.

Put $\mathcal{D} = \{ \mathbf{d} = (d_n) : d_n = 0 \text{ eventually} \}$.

Define the power

$$w_{\mathbf{d}} = \rho^{\mathbf{d}}, \quad \mathbf{d} \in \mathcal{D}.$$

Rademacher chaos - bras and kets

Walsh polynomials, a.k.a. **Rademacher random chaos**:

$$\langle x | \rho \rangle = \sum_{d \in \mathcal{D}} x_d \rho^d,$$

The **braket** indicates the interaction between the array $x = [x_d]$ of coefficients and the power-bearing sequence ρ .

The symbol can be split into two parts, called by Paul Dirac

$$\text{“bra”}: \langle x | \quad \text{and} \quad \text{“ket”}: | \rho \rangle$$

Second order chaos

In the Hilbert space $L^2[0, 1]$ of square-integrable functions with the norm

$$\|f\|_2 = \left(\int_0^1 |f(t)|^2 dt \right)^{1/2},$$

the Walsh functions form an orthonormal basis. By the Pythagorean, a.k.a. Parseval's, theorem

$$\left\| \langle x | \rho \rangle \right\|^2 = \sum_{d \in \mathcal{D}} |x_d|^2.$$

That is, every $f \in L^2[0, 1]$ admits a Rademacher chaos representation.

Random chaos

Geometric Brownian Motion is a functional of BM B_t :

$$G = e^{\alpha B(t) - \alpha^2 t/2}.$$

(it is a solution of the stochastic differential equation

$$dG = \alpha G dB)$$

Every square integrable functional of Brownian Motion,

*$F \in L^2(BM)$, admits a **random chaos representation**:*

$$F = \sum_d x_d \gamma^d,$$

where $\gamma = (\gamma_n)$ are i.i.d. standard normal random variables.

Spectral representation

Every square integrable process with stationary increments can be represented as a simple stochastic integral w.r.t. an orthogonally scattered process:

$$W(t) = \int_{-\infty}^{\infty} (1 - e^{it\omega}) Z(d\omega)$$

Example: Let B_t be a (real or complex) BM, $0 < H < 1$.

$$W_H(t) = \int_{\mathcal{S}} (1 - \exp it\omega) |\omega|^{-H-1/2} B(d\omega)$$

is called a Fractional BM, $FBM(H)$. $H = 1/2$ yields a BM.

FBM

The process can be perceived as a **fractional derivative** of BM:

$$W_H = D^{1/2-H} B :$$

(rather "fractional integral" for $H > 1/2$).

This heuristic concept can be defined and studied rigorously e.g. via Laurent Schwartz **distribution theory** or **stochastic calculus**.

When $H < 1/2$, FBM is more volatile or "chaotic" than BM.